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Open-Type Quadrature Methods with Equispaced Nodes and a Maximal Polynomial Degree of Exactness

Research Article

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Abstract: In this paper we develop Open-Type Quadrature Method. If the interval of definite integral can divided a number of equal subinterval then We are using the nodes of Quadrature Method as mid-point of each interval. We are comparing mid point method to Other Quadrature methods. Also we are developing the composite formula and estimated errors.

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1. Introduction

With the advent of the modern high speed electronic digital computer, the Numerical Integration have been successfully applied to study problems in Mathematics, Engineering, Computer Science and Physical Science. Numerical integration is the study of how the approximate numerical value of a definite integral can be found. It is helpful for the following cases:

—• Many integrals cant be evaluated analytically or dont possess a closed form solution.

—• Closed form solution exists, but numerical evaluation of the answer can be bothersome.

—• The integrand f(x) is not known explicitly, but a set of data points is given for this integrand.

—• The integrand f(x) may be known only at certain points, such as obtained by sampling.

Numerical integration of a function of a single variable is called Quadrature, which represents the area under the curve f(x) bounded by the ordinates x_0, x_n and x-axis. The numerical integration of a multiple integral is sometimes described as Cubature. Numerical integration problems go back at least to Greek antiquity when e.g. the area of a circle was obtained by successively increasing the number of sides of an inscribed polygon. In the seventeenth century, the invention of calculus originated a new development of the subject leading to the basic numerical integration rules. In the following centuries, the field became more sophisticated and, with the introduction of computers in the recent past, many classical and new algorithms had been implemented leading to very fast and accurate results. An extensive research work has already been done by many researchers in the field of numerical integration. M. Concepcion Ausin [1] compared different numerical integration producers and discussed about more advanced numerical integration procedures. Gordon K. Smith [2] gave an analytic analysis on numerical integration and provided a reference list of 33 articles and books dealing with that topic. Rajesh Kumar Sinha [3] worked to evaluate an integrable polynomial discarding Taylor Series. Gerry Sozio [4] analyzed a

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detailed summary of various techniques of numerical integration. J. Oliver [5] discussed the various processes of evaluation of definite integrals using higher-order formulae. Otherwise, every numerical analysis book contains a chapter on numerical integration. The formulae of numerical integrations are described in the books of S.S. Sastry[6] , R.L. Burden [7], J.H. Mathews [8] and many other authors.

The purpose of this paper is quadrature methods for approximate calculation of definite integrals

$$I = \int_a^b f(x)dx \tag{1}$$

where $f(x)$ is integrable, in the Riemann sense on [a b]. The limit of the integration may be finite. Numerical integration is always carried out by mechanical quadrature and its basic scheme is as follows:

$$\int_a^b f(x) = \sum_{i=0}^{n-1} A_i f_i, \tag{2}$$

where $f_i = f(x_i)$, $A_i > 0, i = 0, 1, 2, ...n-1$ and $x_i \in$[a b] $i = 0, 1, 2..., n-1$. are called **Coefficients(Weights)** and **nodes** for Numerical Quadrature, respectively. Once the coefficients and nodes are set down, the scheme (1) can be determined.

2. Preliminaries

Definition 2.1 (Order of Numerical Integration)**.** *Order of accuracy, or precision, of a Quadrature formula is the largest positive integer n such that the formula is exact for x^k, for each $k = 0, 1, \ldots, n$.*

Definition 2.2. *The Integration (1) is approximated by a finite linear combination of value of $f(x)$ in the form (2). The error of approximation of (2) is given as*

$$R_n = \frac{C}{(m+1)!} f^{(m+1)}(\xi), \tag{3}$$

where $\xi =$(a b), $m \geq n$ is order of (2) and error constant of (2) is

$$C = \int_a^b x^{m+1} - \sum_{i=0}^{n-1} A_i x_i^{m+1} \tag{4}$$

Definition 2.3 (Open or Closed type Integration Method)**.** *The Quadrature method (2)of (1) is called Open Type method If the nodes $x_i \in (a\ b)$, $\forall i = 0, 1, ..., n-1$. and is called Closed Type method if the nodes $x_0 = a$, and $x_{n-1} = b$.*

3. Mid-Point Quadrature Method

Consider the integral in the form (2) for each $i = 0, 1, 2..., n-1$. Now we dividing the interval [a b] into $n \in \mathbf{N}$ equal sub interval and take the nodes $x's$ are equispaced points such that $x_i = a + (h/2) + ih \in$[a b], i=0,1,2,...n-1, where $h = (b-a)/(n)$. So that $a = x_0 - h/2$ and $b = x_n + h/2$. so this method has n unknown $A's$ and making this method exact for $f(x) = 1, x, x^2, ..., x^{n-1}$. Then the error constant is (4) for error (3). this integration method is called Mid Point Integration method or M_n-Integration Method. Now following case arise.

One point formula: Take $n = 1$ in (2), we get $I = \int_a^b f(x)dx = A_0 f_0$, where $h = b-a$,$x_0 = a + h/2 = a + (b-a)/2 = (a+b)/2$. The method has one unknown A_0. Making the method exact for $f(x) = 1$, we get

$$\int_a^b 1dx = A_0 \Rightarrow A_0 = (b-a).$$

Hence, the method is given by

$$\int_a^b f(x)dx = (b-a)f\left(\frac{a+b}{2}\right). \tag{5}$$

which is same as mid-point formula and it's called $M_1 - rule$. The error constant is

$$C = \int_a^b x^2 dx - (b-a)(a+b)^2/4 = \frac{1}{12}h^3$$

the error is

$$R_1 = \frac{C}{2!}f^{(2)}(\xi) = \frac{h^3}{24}f^{(2)}(\xi) = \frac{(b-a)^3}{24}f^{(2)}(\xi)$$

where $\xi \in [ab]$ and

Two point formula: Take $n = 2$ in(2), we get,

$$I = \int_a^b f(x)dx = A_0 f_0 + A_1 f_1$$

where $h = (b-a)/2$, $x_0 = a + h/2$, $x_1 = a + 3h/2$. The method has two unknowns A_0, A_1. Making the method exact for $f(x) = 1$, x, we get

$$A_0 + A_1 = b - a, A_0 x_0 + A_1 x_1 = \frac{b^2 - a^2}{2}.$$

Solving for A_0 A_1 we get,$A_0 = A_1 = h$, there fore the method given by

$$\int_a^b f(x) = h\left(f_0 + f_1\right). \tag{6}$$

This formula is called $M_2 - rule$. The error constant is

$$C = \int_a^b x^2 - h(x_0^2 + x_1^2) = \frac{b^3 - a^3}{3} - h(x^2 + (x+h)^2) = \frac{h^3}{6}.$$
$$R_2 = \frac{C}{2!}f^{(2)}(\xi) = \frac{h^3}{12}f^{(2)}(\xi) = \frac{(b-a)^3}{96}f^{(2)}(\xi), \text{ where } \xi = [ab].$$

Tree point formula: Take $n = 3$ in(2), we get,

$$I = \int_a^b f(x) = A_0 f_0 + A_1 f_1 + A_2 f_2.$$

where $x_0 = a + h/2$, $x_1 = a + 3h/2$ $x_2 = a + 5h/2$. The method has two unknowns A_0, A_1 A_2. Making the method exact for $f(x) = 1, x, x^2$, we get

$$A_0 + A_1 + A_2 = b - a, A_0 x_0 + A_1 x_1 + A_2 x_2 = \frac{b^2 - a^2}{2}, A_0 x_0^2 + A_1 x_1^2 + A_2 x_2^2 = \frac{b^3 - a^3}{3}.$$

Solving for A_0 A_1 and A_2, we get$A_0 = A_2 = \frac{9h}{8}$ and $A_1 = \frac{3h}{4}$. This method is given by

$$\int_a^b f(x) = \frac{3h}{8}\left(3f_0 + 2f_1 + 3f_3\right). \tag{7}$$

This rule is called $M_3 - rule$. The error constant is

$$C = \int_a^b x^3 - \frac{3h}{8}\left(3x_0^3 + 2x_1^3 + 3x_2^3\right) = 0$$

It means this method is exact for order 3. Again find C for $n = 4$.

$$C = \int_a^b x^4 - \frac{3h}{8}\left(3x_0^4 + 2x_1^4 + 3x_2^4\right) = \frac{63h^5}{80}$$
$$R_3 = \frac{C}{4!}f^{(4)}(\xi) = \frac{21h^5}{640}f^{(4)}(\xi) = \frac{7(b-a)^5}{51840}f^{(4)}(\xi), \text{ where } \xi = [ab].$$

Following this process we get the table bellow.

n	Formula with $h=(b-a)/n$	Error	Order
1	hf_0	$\frac{(b-a)^3}{24}f^{(2)}(\xi)$	2
2	$h\left(f_0+f_1\right)$	$\frac{(b-a)^3}{96}f^{(2)}(\xi)$	2
3	$\frac{3h}{8}\left(3f_0+2f_1+3f_2\right)$	$\frac{7(b-a)^5}{51840}f^{(4)}(\xi)$	4
4	$\frac{h}{12}\left(13f_0+11f_1+11f_2+13f_3\right)$	$\frac{103(b-a)^5}{1474560}f^{(4)}(\xi)$	4
5	$\frac{5h}{1152}\left[275f_0+100f_1+402f_2+100f_3+275f_4\right]$	$\frac{223(b-a)^7}{604800000}f^{(6)}(\xi)$	6
6	$\frac{3h}{640}\left[247f_0+139f_1+254f_2+254f_3+139f_4+247f_5\right]$	$\frac{1111(b-a)^7}{5016453120}f^{(6)}(\xi)$	6

where $f_i=f(x_i)$,$x_0=a+h/2$ and $x_i=x_0+ih$ $i=1,2,...,n-1$.

4. Composite Formulas

To avoid the use of higher order methods and still obtain accurate results, we use the composite integration methods. We divide the interval [a, b] into a number of subintervals and evaluate the integral in each subinterval by a particular method.If we divide the interval [a b] into $n=cN$, where $c,N\in\mathbf{N}$ equal subinterval. Then

$$\int_a^b f(x)dx=\underbrace{\left(\int_{a=c_0}^{c_1}+\int_{c_1}^{c_2}+...+\int_{c_{N-1}}^{b=c_N}\right)}_{\text{N- integrations}}f(x)dx \tag{8}$$

where c_i, i =1,2,..,N-1 are end points of each interval, respectively. Now following case arise.

Composite M_1-rule Take c=1, that is number of sub interval is $n=N$ and $h=(b-a)/N$. Apply One point formula for each integration in above integration (8), we get

$$\int_a^b f(x)dx=hf_0+hf_1+....+hf_{N-1}=h\sum_{i=0}^{N-1}f_i \tag{9}$$

where $x_i=x_0+ih, i=1,2,..,N-1$ and $x_0=a+h/2$. The error of this integration is

$$R_N=\frac{-h^3}{12}[f^{(2)}(\xi_1)+f^{(2)}(\xi_2)+...+f^{(2)}(\xi_N)]$$

where $c_i<\xi<c_{i+1}$, $i=0,1,...,N-1$. If $f^{(2)}$ is constant for all x in[a b], then

$$\mid R_N\mid\leqslant\frac{Nh^4}{12}f^{(2)}(\zeta)=\frac{(b-a)^3}{24N^2}f^{(2)}(\zeta)$$

where $f^{(2)}(\zeta)=MAX_{a\leqslant x\leqslant b}\mid f^{(2)}(x)\mid$, $a<\zeta<b$.

Composite M_2-rule Take c=2, that is number of sub interval is $n=2N$ and $h=(b-a)/2N$. Apply Two point formula for each integration in above integration (8), we get

$$\int_a^b f(x)dx=h(f_0+f_1)+h(f_2+f_3+...+h(f_{2N-2}+f_{2N-1})=h\sum_{i=0}^{2N}f_i. \tag{10}$$

This is same as Composite one point formula. where $x_i = x_0 + ih, i = 1, 2, .., 2N - 1$ and $x_0 = a + h/2$. The error is

$$| R_{2N} | \leqslant \frac{(b-a)^3}{768N^2} f^{(2)}(\zeta)$$

where $f^{(2)}(\zeta) = MAX_{a \leqslant x \leqslant b} | f^{(2)}(x) |$, $a < \zeta < b$.

Composite $M_3 - rule$ Take c=3, that is number of sub interval is $n = 3N$ and $h = (b-a)/3N$. Apply three point formula for each integration in above integration (8), we get

$$\int_a^b f(x) = \frac{3h}{8}(3f_0 + 2f_1 + 3f_2) + \frac{3h}{8}(3f_3 + 2f_4 + 3f_5) + ... + \frac{3h}{8}(3f_{3N-3} + 2f_{3N-2} + 3f_{3N-1})$$
$$\int_a^b f(x) = \frac{3h}{8}(3(f_0 + f_2 + f_3 + f_5 + ... + f_{3N-3} + f_{3N-1}) + 2(f_1 + f_4 + ... + f_{3N-2})). \tag{11}$$

where $x_i = x_0 + ih, i = 1, 2, .., 3N - 1$ and $x_0 = a + h/2$. The error is

$$| R_{3N} | \leqslant \frac{7(b-a)^5}{12597120N^4} f^{(4)}(\zeta)$$

where $f^{(4)}(\zeta) = MAX_{a \leqslant x \leqslant b} | f^{(4)}(x) |$, $a < \zeta < b$.

Composite $M_4 - rule$ Take c=4, that is number of sub interval is $n = 4N$ and $h = (b-a)/4N$. Apply three point formula for each integration in above integration (8), we get

$$\int_a^b f(x) = \frac{h}{12}(13f_0 + 11f_1 + 11f_2 + 13f_3) + \frac{h}{12}(13f_4 + 11f_5 + 11f_6 + 13f_7) + ...+$$
$$\frac{h}{12}(13f_{4N-4} + 11f_{4N-3} + 11f_{4N-2} + 13f_{4N-1})$$
$$\int_a^b f(x) = \frac{h}{12}(13(f_0 + f_3 + f_4 + f_7 + ... + f_{4N-4} + f_{4N-1}) + 11(f_1 + f_2 + f_5 + f_6 + ... + f_{4N-3} + f_{4N-2})). \tag{12}$$

where $x_i = x_0 + ih, i = 1, 2, .., 4N - 1$ and $x_0 = a + h/2$. The error is

$$| R_{4N} | \leqslant \frac{103(b-a)^5}{377487360N^4} f^{(4)}(\zeta)$$

where $f^{(4)}(\zeta) = MAX_{a \leqslant x \leqslant b} | f^{(6)}(x) |$, $a < \zeta < b$.

5. Comparing $M_3 - rule$ to Others Three Points Formula

The interval of formula (7) can change to [-1 1], we get

$$I = \frac{1}{4}\left(3f(\frac{-2}{3}) + 2f(0) + 3f(\frac{2}{3})\right) + \frac{7}{1620} f^{(4)}(\xi).$$

The below table is three points formulas in the interval [-1 1] and $-1 < \xi < 1$.

Name of formula	Formula	Error	order
$M_3 - rule$	$I = \frac{1}{4}\left(3f(\frac{-2}{3}) + 2f(0) + 3f(\frac{2}{3})\right)$	$\frac{7}{1620}f^{(4)}(\xi)$	4
Simpson's 1/3 rule	$\frac{1}{3}(f(-1) + 4f(0) + f(1))$	$\frac{-1}{90}f^{(4)}(\xi)$	4
Open newton-cotes	$\frac{2}{3}\left(2f(\frac{-1}{2}) - f(0) + 2f(\frac{1}{2})\right)$	$\frac{7}{720}f^{(4)}(\xi)$	4
Quasi-Monte Carlo	$\frac{2}{3}\left(f(\frac{-1}{\sqrt{2}}) + f(0) + f(\frac{1}{\sqrt{2}})\right)$	$\frac{1}{360}f^{(4)}(\xi)$	4

From above table we know, The error of $M_3 - rule$ is smaller then three point Newton's cotes (open or Closed) formula. Comparing with Simpson 1-3^{rd} rule it's give 275 percentage accuracy value.

6. Problems

Problem 6.1. *Evaluate*

$$I = \int_{-1}^{1} \frac{e^{-x}}{1+x^2} dx$$

By three points formula. The exact value is 1.795521283.

solution Here $f(x) = \dfrac{e^{-x}}{1+x^2}$. The solution of I by using three points formula is given below.

Name of formula	Solution	|Error| $\simeq$
$M_3 - rule$	1.77790541	$1.762 \cdot 10^{-2}$
Simpson's 1/3 rule	1.847693545	$5.2172 \cdot 10^{-2}$
Open newton-cotes	1.7389353933	$5.659 \cdot 10^{-2}$
Quasi-Monte Carlo	1.7871927433	$0.833 \cdot 10^{-2}$
Gauss-Legendre	1.86501225889	$6.949 \cdot 10^{-2}$

Problem 6.2. *Evaluate*

$$\int_{0}^{1} \frac{sin(1+x)e^{x^2}}{1+x} dx.$$

Compare with exact value 0.913631.

Solution Here $f(x) = \dfrac{sin(1+x)e^{x^2}}{1+x}$, $a = 0$ and $b = 1$.
If $n = 1$ then $h = 1$, and $x_0 = a + h/2 = 0 + 1/2 = 1/2$.

If $n = 2$ then $h = 1/2$ $x_0 = 0 + h/2 = 1/4$, and $x_1 = h/2 + h = 1/4 + 1/2 = 3/4$.

If $n = 3$ then $h = 1/3$, $x_0 = h/3 = 1/6$, $x_1 = h/2 + h = 3/6$ and $x_2 = h/2 + 2h = 5/6$.

If $n = 4$ then $h = 1/4$ $x_0 = 1/8$, $x_1 = 3/8$, $x_2 = 5/8$, and $x_3 = 7/8$.

If $n = 5$ then $h = 1/5$v $x_0 = 1/10$, $x_1 = 3/10$, $x_2 = 5/10$, $x_3 = 7/10$ and $x_4 = 9/10$

If $n = 6$ then $h = 1/6$ $x_0 = 1/12$, $x_1 = 3/12$, $x_2 = 5/12$,$x_3 = 7/12$, $x_4 = 9/12$ and $x_5 = 11/12$.

the value is given below.

Name of rule Formula	Value	Error
$M_1 - rule$	0.853873	0.059758
$M_2 - rule$	0.897490	0.016141
$M_3 - rule$	0.912915	0.000716
$M_4 - rule$	0.913260	0.000371
$M_5 - rule$	0.913629	0.000002
$M_6 - rule$	0.913630	0.0000001

Problem 6.3. *Evaluate*

$$\int_0^1 \frac{1}{1+x}dx.$$

By using Composite $M_3 - rule$, take $n = 3, 6$ and 12. Compare with exact value ln(2)=0.69314718.

Solution Here $f(x) = \dfrac{1}{1+x}$. Let I_n and $E(I_n)$ be represent the value obtained by composite three points rule using n nodes and error of I_n, respectively. The composite $M_3 - rule$ is

$$I_{n=3N} = \frac{3h}{8}\left(3(f_0 + f_2 + f_3 + f_5 + ... + f_{3N-2} + f_{3N-1}) + 2(f_1 + f_4 + ... + f_{3N-2})\right)$$

When $n = 3$ or N=1, we have $h = 1/3$ and.

$$\therefore I_3 = \frac{1}{8}\left(3f_0 + 2f_1 + 3f_2\right) = 0.69264069.$$

When $n = 6$ or N=2, we have $h = 1/6$ and

$$I_6 = \frac{1}{16}\left(3(f_0 + f_2 + f_3 + f_5) + 2(f_1 + f_4)\right)) = 0.69310558.$$

When $n = 12$ or N=4, we have $h = 1/12$ and

$$I_{12} = \frac{1}{32}\left(3(f_0 + f_2 + f_3 + f_5 + f_6 + f_8 + f_9 + f_{11}) + 2(f_1 + f_4 + f_7 + f_{10})\right)) = 0.69314432.$$

The errors $E(I_3) = 0.00050649$, $E(I_6) = 0.0000416$ and $E(I_{12}) = 0.00000286$

7. Conclusion

We develop this method for easy to solve definite integral of finite interval. The purpose of this method is the nodes of composite method have been taken as midpoints and it's give good accuracy more then Closed or Open Type Newton-cotes rules. If n is the number of sub intervals then the number of nodes in closed Newton cotes formula is $n + 1$ and in open type Newton Cotes formula is $n - 1$. there is no open type Newtons cotes formula for $n = 1$. But in Mid Point formula the number of notes is equal to the number of subintervals. so there exist a formula for any value of n. Suppose in Simpson 1/3 rule, three nodes and two equal subintervals, in $M_3 - rule$ three nodes and three equal subintervals. Hence the error in this method is small(i.e the value h is small compare with Newton cotes formula). Hence we researched about the nodes, there are no fixed nodes to give exact value of integration for all integrable functions $f(x)$. We are researching about mid point nodes, this method is give stable for all functions $f(x)$. So many persons used composite Simpson's rule, because the nodes of composite Simpson's rule are equispaced points. So this method is better then Simpson's (1/3-rule or 3/8-rule).

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