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n-almost Finitely Copresented Modules

Research Article

Abdul-Qawe Kaed[1]*

1 Department of Mathematics, Faculty of applied Science, University of Thamar.

Abstract: In [3] the notin of almost finitely copresented modules is introduced and studied. In this paper, we introduce and study a notion of n-almost finitely copresented R-modules.

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1. Introduction

Throughout this paper R means a commutative ring with an identity element and all modules are unital R-modules. In [4] the notion of almost finitely cogenerated module is introduced and studied, such that an R-module M is called almost finitely cogenerated, if it is not finitely cogenerated but all its factors are finitely cogenerated. Recall that an R-module M is called finitely cogenerated if for every family $\{M_i\}_{i\in I}$ of submodules of M with $\cap_{i\in I} M_i = 0$, there is a finite subset $J \subset I$ such that $\cap_{i\in J} N_i = 0$. In [3] the notion of almost finitely copresented module is introduced and studied, such that an R-module M is called almost finitely copresented if there is an exact sequence of the form $0 \longrightarrow M \longrightarrow M_0 \longrightarrow M_1$, where M_0 and M_1 are almost finitely cogenerated modules. In this paper we introduce and study a notion of n-almost finitely copresented R-modules, such that we define it as the following : For a ring R and a positive integer n, an R-module N is called n-almost finitely copresented modules if there is an exact sequence of R-modules of the form $0 \to N \to L_0 \to L_1 \to \cdots \to L_n$, where, for $i = 0, ..., n$, L_i is an injective and almost finitely cogenerated. N is called an almost infinitely copresented modules, if it is n-almost finitely copresented modules for every positive integer n. And if $m \leq n$ for every positive integer n, then n-almost finitely copresented modules is m-almost finitely copresented modules. The proposition 2.3 shows that N is 0-almost finitely copresented module is an almost finitely cogenerated module. Also the proposition 2.4 shows that N is 1-almost finitely copresented module if and only if it is almost finitely copresented module. Finally the main result is Theorem 2.5 which studies a behavior of this notion on short exact sequences.

2. n-almost Finitely Copresented Modules

Definition 2.1. *For a ring R and a positive integer n, an R-module N is called n-almost finitely copresented modules if there is an exact sequence of R-modules of the form $0 \to N \to L_0 \to L_1 \to \cdots \to L_n$, such that L_i is an injective and almost*

* *E-mail: Abdul-Qawe_dabouan@yahoo.com*

finitely cogenerated modules and $i = 0, ..., n$*.*

Remark 2.2.

1. *N is called an almost infinitely copresented modules, if it is n-almost finitely copresented modules for every positive integer n.*

2. *If $m \leq n$ for every positive integer n, then n-almost finitely copresented modules is m-almost finitely copresented modules.*

3. *If L is injective and almost finitely copresented R-modules, then it is almost infinitely copresented modules associated to the short exact sequence $0 \to L = L \to 0$.*

The following propositions shows that 0-almost finitely copresented module is an almost finitely cogenerated module.

Proposition 2.3. *For a ring R, an R-module N is 0-almost finitely copresented modules then it is an almost finitely cogenerated module.*

Proof. $\Rightarrow$) Suppose that N is 0-almost finitely copresented modules then, by definition 2.1, there is an exact sequence of R-modules of the form $0 \to N \to I_0$, such that I_0 is an injective and almost finitely cogenerated modules. Therefore, N is an almost finitely cogenerated module as a submodule of I_0 see 2.2 in [3] and see [4]. □

The converse of above proposition is not true see the example in [3] which shows that $\mathbb{Z}$ is an almost finitely cogenerated but it is not an almost finitely copresented. The following proposition shows that 1-copresented modules is equivalent that an almost finitely copresented.

Proposition 2.4. *For a ring R, an R-module N is 1-almost finitely copresented module if and only if it is almost finitely copresented module.*

Proof. $\Rightarrow$) Suppose that N is 1-almost finitely copresented module, Then there exists an exact sequence of the form $0 \to N \to E_0 \to E_1$ such that E_0 and E_1 are injective and almost finitely cogenerated, then N is an almost finitely copresented by defintion 2.1 in [3] .

$\Leftarrow$) Suppose that N is an almost finitely copresented, then there exists an exact sequence $0 \to N \to E_0 \to E_1$ where, E_0 and E_1 are almost finitely cogenerated, therefore N is 1-copresented. □

The main result is Theorem 2.5 which studies a behavior of this notion on short exact sequences and is an extension for 3.1.3 in [2].

Theorem 2.5. *Let R be a ring and let $0 \to K \to L \to M \to 0$ be a short exact sequence of R-modules. Then, for a positive integer n, we have:*

1. *If K and M are n-almost finitely copresented, then L is n-almost finitely copresented.*

2. *If M is $(n–1)$-almost finitely copresented and L is n-almost finitely copresented, then K is n-almost finitely copresented.*

3. *If K is $(n+1)$-almost finitely copresented and L is n-almost finitely copresented, then M is n-almost finitely copresented.*

4. *If $L = K \oplus M$, then L is n-almost finitely copresented if and only if K and M are n-almost finitely copresented.*

Proof. 1. Since K and M are n-almost finitely copresented, then there are exact sequences of R-modules $\to K \to K_0 \to K_1 \to \cdots \to K_n$ and $0 \to M \to M_0 \to M_1 \to \cdots \to M_n$ where, for $i = 0, ..., n$, K_i and M_i are injective and almost finitely cogenerated. From ([2], Lemma 2.4), we get the following commutative diagram of R-modules with exact sequences:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K & \to & L & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K_0 & \to & K_0 \oplus M_0 & \to & M_0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K_1 & \to & K_1 \oplus M_1 & \to & M_1 & \to & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K_n & \to & K_n \oplus M_n & \to & M_n & \to & 0
\end{array}
$$

We get the exact sequence $0 \to K \to K_0 \oplus M_0 \to K_1 \oplus M_1 \to \cdots \to K_n \oplus M_n$. where $K_i \oplus M_i$ are injective and almost finitely cogenerated, for $i = 0, ..., n$ see 2.4 in [3], then we deduce that L is n-almost finitely cogenerated modules.

2. Suppose that M is $(n-1)$-almost finitely copresented and L is n-almost finitely copresented. That is implies that there is an exact sequence of R-modules $0 \to L \to L_0 \to L_1 \to \cdots \to L_n$, where for $i = 0, ..., n$, L_i are injective and almost finitely cogenerated. Then, we get the following exact sequences

$$0 \to L \to L_0 \to T \to 0 \quad \text{and} \quad 0 \to T \to L_1 \to L_2 \to \cdots \to L_n,$$

where $T = L_0/L$. Then T is $(n-1)$-almost finitely copresented. Consider the pushout diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & K & \to & L & \to & M & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & K & \to & L_0 & \to & F & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & T & = & T & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Since M and T are $(n-1)$-almost finitely copresented, then by (1), F is $(n-1)$-almost finitely copresented and we get an exact sequence of R-modules $0 \to F \to F_0 \to F_1 \to \cdots \to F_{n-1}$, where each F_i is injective and almost finitely cogenerated. We combine this sequence with the sequence $0 \to K \to L_0 \to F \to 0$, we get the following commutative diagram

$$
\begin{array}{ccccccccccccc}
0 & \longrightarrow & K & \longrightarrow & L_0 & \longrightarrow & & \longrightarrow & F_0 & \longrightarrow & F_1 & \longrightarrow \cdots \longrightarrow & F_{n-1} \\
 & & & & & \searrow & & \nearrow & & & & & \\
 & & & & & & F & & & & & & \\
 & & & & & \nearrow & & \searrow & & & & & \\
 & & & & 0 & & & & 0 & & & &
\end{array}
$$

So we get this sequence exact $0 \to K \to L_0 \to F_0 \to F_1 \to \cdots \to F_{n-1}$, Hence, K is n-almost finitely copresented.

3. We have that K is $(n+1)$-almost finitely copresented, then there is an exact sequence of R-modules

$$0 \to K \to H_0 \to H_1 \to \cdots \to H_n \to H_{n+1},$$

where each H_i is injective and almost finitely cogenerated. Thus we get the two exact sequences $0 \to K \to H_0 \to T \to 0$ and $0 \to T \to H_1 \to H_2 \to \cdots \to H_n \to H_{n+1}$ where $T = H_0/K$. Then, T is n-almost finitely copresented. Consider the pushout diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & K & \to & M & \to & L & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & H_0 & \to & D & \to & L & \to & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & T & = & T & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

Since M and T are n-almost finitely copresented, D is n-almost finitely copresented by (1). And since H_0 is injective, the middle horizontal sequence splits and so $D = H_0 \oplus L$. Thus we get the following short exact sequence $0 \to M \to D = H_0 \oplus M \to H_0 \to 0$. Since D is n-almost finitely copresented, that is implies that an exact sequence of R-modules $0 \to D \to D_0 \to D_1 \to \cdots \to D_n$, such that D_i is injective and almost finitely cogenerated. This gives a short exact sequence $0 \to D \to D_0 \to T \to 0$ such that $T = D_0/D$ is $(n-1)$-almost finitely copresented. Then we have the following pushout diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & M & \to & D & \to & H_0 & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & M & \to & D_0 & \to & E & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & T & = & T & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

Being an almost finitely copresented and injective R-modules, H_0 is infinitely copresented. Then, by the right vertical exact sequence and (1), E is $(n-1)$-copresented. Then there is an exact sequence of R-modules $0 \to E \to E_0 \to E_1 \to \cdots \to E_{n-1}$ where each E_i is injective and almost finitely cogenerated. Combining this sequence with $0 \to M \to D_0 \to E \to 0$ to get the following exact sequence:

$$0 \to M \to D_0 \to E_0 \to E_1 \to \cdots \to E_{n-1}$$

Therefore, M is n-almost finitely copresented R-modules.

4. suppose that K and M are n-almost finitely copresented. From (1) we get the following short exact sequence $0 \to K \to L = K \oplus M \to M \to 0$, hence L is n-almost finitely copresented.

Conversely, suppose that $L = K \oplus M$ is n-almost finitely copresented. Then by 2.1 and 2.4 L is almost finitely copresented and by theorem 2.5 and lemma 2.4 in [3], also are K and M. Then there are two short exact sequences $0 \to K \to H_0 \to H_1 \to 0$ and $0 \to M \to T_0 \to T_1 \to 0$ where H_0, H_1, T_0, T_1 are injective and almost finitely cogenerated. We add these sequences such that we get a short exact sequence

$$0 \to L = K \oplus M \to H_0 \oplus T_0 \to H_1 \oplus T_1 \to 0$$

Then by lemma 2.4 in [3] $H_0 \oplus T_0$ is injective and almost finitely cogenerated. By 2.2 and applying (3), then $H_1 \oplus T_1$ is $(n-1)$-almost finitely cogenerated and also H_1 and T_1 . Therefore, applying (2) to the above two short exact sequences, we get that K and M are n-almost finitely cogenerated. □

Corollary 2.6. *Let R be a ring and let $0 \to H \to L_0 \to L_1 \to \cdots \to L_n \to T \to 0$ be an exact sequence of R-modules, where n is a positive integer and, for $i = 0, ..., n$, L_i is $(m-(i+1))$-almost finitely copresented for a positive integer $m \geq n$. Then, H is m-almost finitely copresented if and only if T is $(m-n-1)$-almost finitely copresented.*

Proof. We decompose the sequence $0 \to H \to L_0 \to L_1 \to \cdots \to L_n \to T \to 0$ into short exact sequences as follows:

$$0 \to T_i \to L_i \to T_{i+1} \to 0, \quad \text{for} \quad i = 0, ..., n$$

such that $T_0 = H$ and $T_{n+1} = T$, and by applying recursively theorem 2.5 to each of these sequences we obtain the desired result. □

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