

Analytical Solution For Time-Fractional Diffusion Equation By Aboodh Decomposition Method

Research Article

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Abstract: In this paper, the derivation of exact solutions of time-fractional heat diffusion equations by employing the new integral transform, the Aboodh transform coupled to the Adomian decomposition method is proposed. The coupling is based on the Caputo fractional derivative definition solely chosen owing to its flexibility. The proposed method is in fact, very reliable as it gives exact solutions in little iteration without requiring either of discretization, perturbation or linearization among others. Some test examples are given with their graphical illustration.

Keywords: Fractional Diffusion Equation, Aboodh Transform, Decomposition Method.

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1. Introduction

The differential equations featuring fractional derivatives (FDEs) and partial differential equations featuring fractional derivatives (FPEs) are recently taking center stages among many researchers due to their frequent encounter industrially. Their numerous applications cover a wide range of processes ranging from science and engineering. Moreover, considerable effort has been invested in this regard trying to find robust and efficient numerical and analytical methods for tackling these equations of interest, see [1-14]. However, in this paper, the derivation of exact solutions of time-fractional heat diffusion equations by employing the Aboodh transform [15] coupled to the Adomian decomposition method [16] is proposed. The Aboodh integral transform is a newly introduced transform with numerous applications, see [17-20]. Furthermore, the coupling is based on the Caputo fractional derivative definition [21] solely chosen owing to its flexibility. The proposed method gives exact solutions in little number of iteration without requiring either of discretization, perturbation or linearization among others. The paper is organized as follows: In Section 2, we present the concept of the Aboodh transform and some of its properties. Section 3 presents some basics about the fractional calculus, Section 4 gives the methodology. In Section 5, we apply the methodology to solve some time-fractional heat diffusion equations, and finally, Section 6 gives the conclusion.

2. The Aboodh Transform

The Aboodh transform is a new integral transform similar to the Laplace transform and other integral transforms that are defined in the time domain $t \geq 0$, such as the Sumudu transform [22] Natural transform [23] and Elzaki transforms [24]

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among others. The Aboodh transform is defined for functions of exponential order. We consider functions in the set B defined by:

$$B = \{u : |u(t)| < Me^{k_j|t|}, \text{ if } t \in (-1)^j \times [0, \infty), j = 1, 2; (M, k_1, k_2 > 0)\}.$$

For a given function in the set B , the constant M must be finite number; k_1, k_2 may be finite or infinite. Then, the Aboodh integral transform denoted by the operator $\mathcal{A}(\cdot)$ is defined (by [15]) by the integral equation:

$$\mathcal{A}\{u(t)\} = \frac{1}{v} \int_0^\infty e^{-vt} u(t) dt, \quad t \geq 0, \quad v \in (k_1, k_2). \quad (1)$$

2.1. Linearity Property

If a and b are any constants and $u(t)$ and $w(t)$ are functions defined over the set B above, then

$$\mathcal{A}\{au(t) \pm bw(t)\} = a\mathcal{A}\{u(t)\} \pm b\mathcal{A}\{w(t)\}.$$

2.2. Aboodh Transform for Derivatives

For any given function $u(t)$ defined over the set B , the Aboodh transform of the m^{th} derivative of $u(t)$ ($m \in \mathbb{N}$) is given by

$$\mathcal{A}\{u^m(t)\} = v^m \mathcal{A}\{u(t)\} - \sum_{k=0}^{m-1} \frac{u^k(0)}{v^{2-m+k}}. \quad (2)$$

3. Fractional Calculus and Some Definitions

In this section, we give some preliminary definitions of fractional calculus theory which will be used later on as follows:

3.1. Caputo Fractional Derivative

The Caputo derivative of a casual function $u(t)$ ($u(t) = 0, t < 0$) with $\alpha > 0$ is defined by [21]

$$C_0 D_0^\alpha u(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^\infty (t-s)^{m-\alpha-1} u^m(s) ds, \quad (m-1 < \alpha \leq m). \quad (3)$$

3.2. Aboodh Transform For Caputo Fractional Derivative

We define the Aboodh transform for Caputo fractional derivative by virtue of equation (2) as

$$\mathcal{A}\{u^\alpha(t)\} = v^\alpha \mathcal{A}\{u(t)\} - \sum_{k=0}^{n-1} \frac{u^k(0)}{v^{2-\alpha+k}}, \quad (m-1 < \alpha \leq m). \quad (4)$$

3.3. Mittag-Leffler Function

The one parameter Mittag-Leffler function is given by [25]

$$E_\alpha(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(1+\alpha m)}, \quad \alpha > 0, z \in \mathbb{C}. \quad (5)$$

3.4. Gamma Function Relation With Factorial

If $m (> -1)$ is a real number, then

$$\Gamma(m+1) = m!.$$

4. Aboodh Decomposition Method

This section deals with the presentation of the Aboodh decomposition method. In presenting the method, we consider the one-dimensional time-fractional diffusion equation under external force [10];

$$u_t^\alpha(x, t) = ku_{xx}(x, t) - \partial_x\{F(x)u(x, t)\}, \quad 0 < \alpha \leq 1, \quad k > 0, \quad (6)$$

with the initial condition

$$u(x, 0) = g(x), \quad x \in [a, b], \quad 0 < t \leq T, \quad (7)$$

where u_t^α is the Caputo derivative of order α , and $F(x)$ is the external force.

By taking the Aboodh transform of equation (6) in t , subject to the prescribed initial condition given in equation (7), we obtain

$$\mathcal{A}\{u(x, t)\} = \frac{g(x)}{v^2} + \frac{1}{v^\alpha} \mathcal{A}\{ku_{xx}(x, t) - \partial_x\{F(x)u(x, t)\}\}. \quad (8)$$

We now take the inverse Aboodh transform of equation (8), yielding

$$u(x, t) = g(x) + \mathcal{A}^{-1}\left\{\frac{1}{v^\alpha} \mathcal{A}\{ku_{xx}(x, t) - \partial_x\{F(x)u(x, t)\}\}\right\}. \quad (9)$$

Now, from equation (9), we assume the unknown function $u(x, t)$ to have the series solution

$$u(x, t) = \sum_{m=0}^{\infty} u_m(x, t). \quad (10)$$

Thus, equation (9) becomes

$$\sum_{m=0}^{\infty} u_m(x, t) = g(x) + \sum_{m=0}^{\infty} \mathcal{A}^{-1}\left\{\frac{1}{v^\alpha} \mathcal{A}\{ku_{m_{xx}}(x, t) - \partial_x\{F(x)u_m(x, t)\}\}\right\}. \quad (11)$$

Thus we identify $u_0(x, t)$ with the initial condition term, i.e., $u(x, 0)$, and the rest of the components $u_m(x, t)$ are determined recursively by the formula:

$$\begin{cases} u_0(x, t) = g(x) \\ u_{m+1}(x, t) = \mathcal{A}^{-1}\left\{\frac{1}{v^\alpha} \mathcal{A}\{ku_{m_{xx}}(x, t) - \partial_x\{F(x)u_m(x, t)\}\}\right\}, \quad m \geq 0. \end{cases} \quad (12)$$

5. Applications and Results

In this section, we apply the proposed method to two different time-fractional heat diffusion equations and later illustrated the solutions graphically using Mathematica software as follows:

5.1. Example One

Consider the time-fractional diffusion equation [Yan *et al* [9]]

$$u_t^\alpha(x, t) = \mu u_{xx}(x, t), \quad (13)$$

where μ is a constant, with the initial condition

$$u(x, 0) = e^x. \quad (14)$$

Then, on taking the Aboodh transform of both sides of equation (13) subject to the initial condition given in equation (14), we obtain

$$\mathcal{A}\{u(x, t)\} = \frac{e^x}{v^2} + \frac{1}{v^\alpha} \mathcal{A}\{\mu u_{xx}\}. \quad (15)$$

Taking the inverse Aboodh Transform of equation (15), we get

$$u(x, t) = e^x + \mathcal{A}^{-1}\left\{\frac{1}{v^\alpha} \mathcal{A}\{\mu u_{xx}\}\right\}. \quad (16)$$

Now, from equation (16), we assume the unknown function $u(x, t)$ to have the series solution

$$u(x, t) = \sum_{m=0}^{\infty} u_m(x, t). \quad (17)$$

Thus, equation (16) becomes

$$\sum_{m=0}^{\infty} u_m(x, t) = e^x + \mathcal{A}^{-1}\left\{\frac{1}{v^\alpha} \mathcal{A}\left\{\mu \sum_{m=0}^{\infty} u_{m,xx}(x, t)\right\}\right\}. \quad (18)$$

Thus we identify $u_0(x, t)$ with the initial condition term, i.e., ue^x , and the rest of the components $u_m(x, t)$ are determined recursively by the formula:

$$\begin{cases} u_0(x, t) = e^x, & m = 0 \\ u_{m+1}(x, t) = \mathcal{A}^{-1}\left\{\frac{1}{v^\alpha} \mathcal{A}\{\mu u_{m,xx}\}\right\}, & m \geq 0. \end{cases} \quad (19)$$

We now obtain some few terms from equation (19) as follows

$$u_0(x, t) = e^x, \quad (20)$$

$$\begin{aligned} u_1(x, t) &= \mathcal{A}^{-1}\left\{\frac{1}{v^\alpha} \mathcal{A}\{\mu u_{0,xx}\}\right\}, \\ &= \mathcal{A}^{-1}\left\{\frac{1}{v^\alpha} \mathcal{A}\{\mu e^x\}\right\}, \\ &= \mathcal{A}^{-1}\left\{\frac{1}{v^{\alpha+2}} \mu e^x\right\}, \\ &= \frac{t^\alpha}{\alpha!} \mu e^x, \end{aligned} \quad (21)$$

$$\begin{aligned} u_2(x, t) &= \mathcal{A}^{-1}\left\{\frac{1}{v^\alpha} \mathcal{A}\{\mu u_{1,xx}\}\right\}, \\ &= \mathcal{A}^{-1}\left\{\frac{1}{v^\alpha} \mathcal{A}\left\{\mu^2 \frac{t^\alpha}{\alpha!} e^x\right\}\right\}, \\ &= \mathcal{A}^{-1}\left\{\frac{1}{v^{2\alpha+2}} \mu^2 e^x\right\}, \\ &= \frac{t^{2\alpha}}{(2\alpha!)} \mu^2 e^x, \end{aligned} \quad (22)$$

$$\begin{aligned}
 u_3(x, t) &= \mathcal{A}^{-1}\left\{\frac{1}{v^\alpha}\mathcal{A}\{\mu u_{2xx}\}\right\}, \\
 &= \mathcal{A}^{-1}\left\{\frac{1}{v^\alpha}\mathcal{A}\left\{\mu^3\frac{t^{2\alpha}}{(2\alpha!)}e^x\right\}\right\}, \\
 &= \mathcal{A}^{-1}\left\{\frac{1}{v^{3\alpha+2}}\mu^3e^x\right\}, \\
 &= \frac{t^{3\alpha}}{(3\alpha!)}\mu^3e^x,
 \end{aligned}
 \tag{23}$$

and so on. We therefore sum up the above iterations to get

$$u(x, t) = \sum_{m=0}^{\infty} u_m(x, t) = e^x \left(1 + \mu\frac{t^\alpha}{\alpha!} + \mu^2\frac{t^{2\alpha}}{(2\alpha!)} + \mu^3\frac{t^{3\alpha}}{(3\alpha!)} + \dots\right),
 \tag{24}$$

which leads to the exact solution

$$u(x, t) = e^x \sum_{m=0}^{\infty} \frac{\mu^m t^{m\alpha}}{\Gamma(1 + m\alpha)} = e^x E_\alpha(\mu t^\alpha).
 \tag{25}$$

The graph of the solution of equation (25) is shown in Figure 1a and 1b respectively;

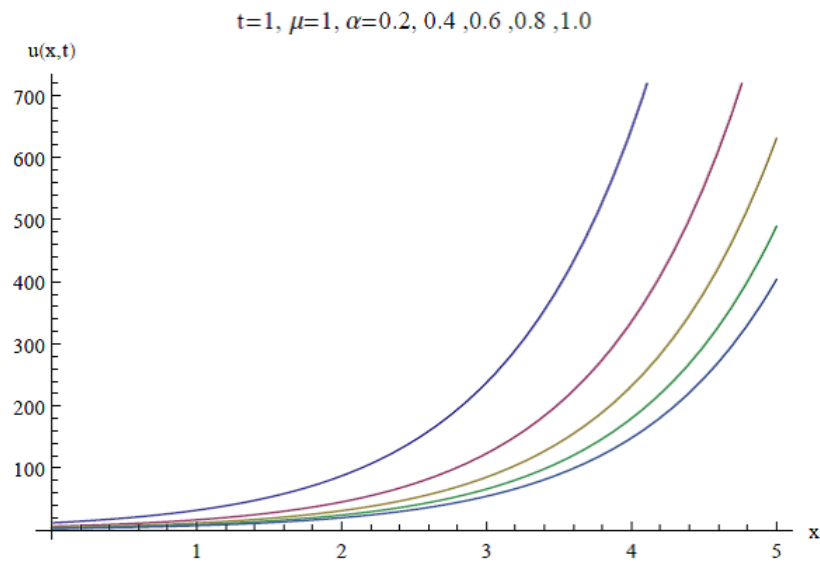


Figure 1a: Solution of equation (25) at $\mu = 1$ with various α 's

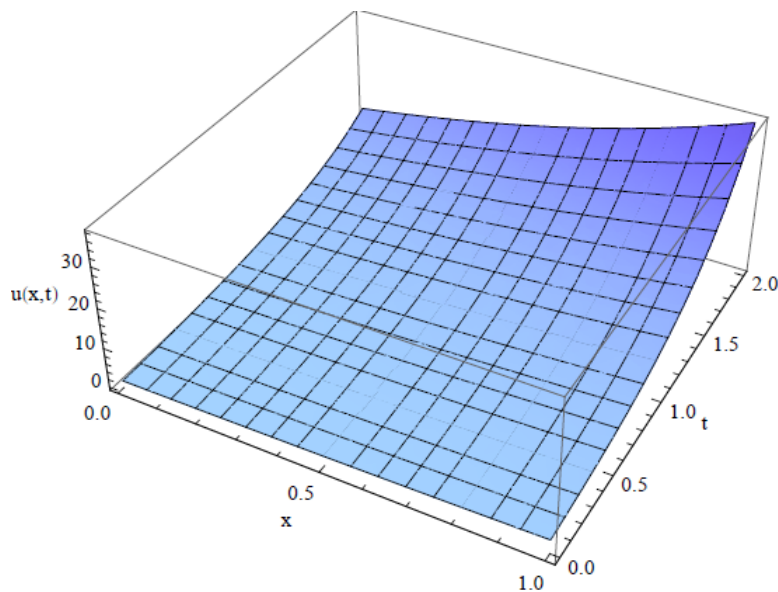


Figure 1b: 3D Solution of equation(25) at $\mu = 1$

5.2. Example Two

Consider the time-fractional diffusion equation [Ray and Bera [10]]

$$u_t^{\frac{1}{2}}(x, t) = u_{xx} + \partial_x \{xu(x, t)\}, \quad (26)$$

with the initial condition

$$u(x, 0) = x. \quad (27)$$

Proceeding as discussed, we get the recursive relation:

$$\begin{cases} u_0(x, t) = x, & m = 0 \\ u_{m+1}(x, t) = \mathcal{A}^{-1} \left\{ \frac{1}{v^{\frac{1}{2}}} \mathcal{A} \{u_{m,xx} + \partial_x \{xu_m\}\} \right\}, & m \geq 0. \end{cases} \quad (28)$$

Thus, we get the first few terms as

$$u_0(x, t) = x, \quad (29)$$

$$\begin{aligned} u_1(x, t) &= \mathcal{A}^{-1} \left\{ \frac{1}{v^{\frac{1}{2}}} \mathcal{A} \{u_{0,xx} + \partial_x \{xu_0\}\} \right\}, \\ &= \mathcal{A}^{-1} \left\{ \frac{1}{v^{\frac{1}{2}}} \mathcal{A} \{\partial_x \{x^2\}\} \right\}, \\ &= \mathcal{A}^{-1} \left\{ \frac{1}{v^{\frac{1}{2}}} \mathcal{A} \{2x\} \right\}, \\ &= \mathcal{A}^{-1} \left\{ \frac{2x}{v^{\frac{3}{2}}} \right\}, \\ &= 2x \frac{t^{\frac{1}{2}}}{(\frac{1}{2})!}, \end{aligned} \quad (30)$$

$$\begin{aligned} u_2(x, t) &= \mathcal{A}^{-1} \left\{ \frac{1}{v^{\frac{1}{2}}} \mathcal{A} \{u_{1,xx} + \partial_x \{xu_1\}\} \right\}, \\ &= \mathcal{A}^{-1} \left\{ \frac{1}{v^{\frac{1}{2}}} \mathcal{A} \{\partial_x \{2x^2 \frac{t^{\frac{1}{2}}}{(\frac{1}{2})!}\}\} \right\}, \\ &= \mathcal{A}^{-1} \left\{ \frac{1}{v^{\frac{1}{2}}} \mathcal{A} \{4x \frac{t^{\frac{1}{2}}}{(\frac{1}{2})!}\} \right\}, \\ &= \mathcal{A}^{-1} \left\{ \frac{4x}{v^{\frac{5}{2}}} \right\}, \\ &= -4xt, \end{aligned} \quad (31)$$

$$\begin{aligned} u_3(x, t) &= \mathcal{A}^{-1} \left\{ \frac{1}{v^{\frac{1}{2}}} \mathcal{A} \{u_{2,xx} + \partial_x \{xu_2\}\} \right\}, \\ &= \mathcal{A}^{-1} \left\{ \frac{1}{v^{\frac{1}{2}}} \mathcal{A} \{\partial_x \{4x^2 t\}\} \right\}, \\ &= \mathcal{A}^{-1} \left\{ \frac{1}{v^{\frac{1}{2}}} \mathcal{A} \{8xt\} \right\}, \\ &= \mathcal{A}^{-1} \left\{ \frac{8x}{v^{\frac{7}{2}}} \right\}, \\ &= 8x \frac{t^{\frac{3}{2}}}{(\frac{3}{2})!}, \end{aligned} \quad (32)$$

and so on. Thus, on summing up, we obtain

$$u(x, t) = \sum_{m=0}^{\infty} u_m(x, t) = x \left(1 + \frac{2t^{\frac{1}{2}}}{\frac{1}{2}!} + 4t + \frac{8t^{\frac{3}{2}}}{\frac{3}{2}!} + \dots \right), \quad (33)$$

which leads to the exact solution

$$u(x, t) = \sum_{m=0}^{\infty} \frac{2^m x t^{\frac{m}{2}}}{\Gamma(1 + \frac{m}{2})} = x E_{\frac{1}{2}}(2t^{\frac{1}{2}}). \quad (34)$$

The graph of the solution of equation (34) is shown in Figure 2;

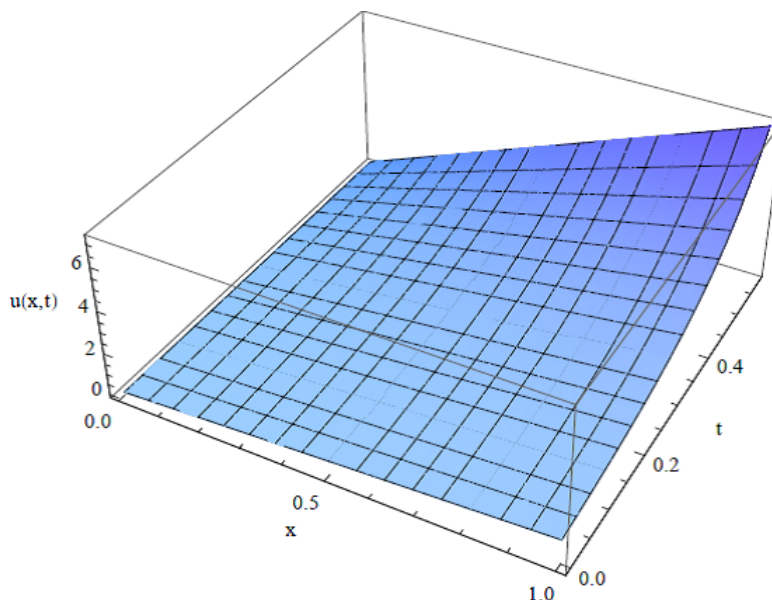


Figure 2: 3D Solution of equation(34) at $\mu = 1$

6. Conclusion

In conclusion, the Aboodh decomposition method for finding exact solutions of time-fractional heat diffusion equations by employing the new integral transform, the Aboodh transform coupled to the Adomian decomposition method is proposed. The coupling is based on the Caputo fractional derivative definition solely chosen owing to its flexibility. More, the method is applied to some test examples yielding exact solutions in little number of iterations without requiring either of discretization, perturbation or linearization among others. Thus, the presented method can be used in solving various fractional models as it gives solution in form of rapid convergent series with easily computable components.

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