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Common Fixed Points of Two Pairs of Selfmaps Satisfying $(E.A)$-property in b-metric Spaces Using a New Control Function

Research Article

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Abstract: In this paper, we apply the $(E.A)$-property to prove the existence and uniqueness of common fixed points of four selfmaps in the setting b-metric spaces using a new control function. We provide an example in support of our results. Our results generalize the fixed point results of Ozturk and Radenovic [11].

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1. Introduction and Preliminaries

In 1993, Stefan Czerwik [4] introduced the concept of b-metric spaces which is a generalization of metric space and generalized the Banach contraction principle in the context of complete b-metric spaces. Afterwards, many mathematicians studied fixed point theorems for single-valued and multi-valued mappings in b-metric spaces. In 2002, Aamari and Moutawakil [1] introduced the notion of property $(E.A)$. Different authors apply this concept to prove the existence of common fixed points (see [2], [9], [11] [12]). We now mention some well-known notations, definitions and primary known results in the literature that will be needed in the sequel.

Definition 1.1 ([4])**.** *Let X be a non-empty set. A function $d : X \times X \to [0, \infty)$ is said to be a b-metric if the following conditions are satisfied;*

(1). $0 \leq d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = 0$ *if and only if* $x = y$*,*

(2). $d(x, y) = d(y, x)$ *for all* $x, y \in X$*,*

(3). there exists $s \geq 1$ *such that* $d(x, z) \leq s\big[d(x, y) + d(y, z)\big]$ *for all* $x, y, z \in X$*.*

In this case, the pair (X, d) is called a b-metric space with coefficient s.

Every metric space is a b-metric space with $s = 1$. In general, every b-metric space is not a metric space.

Definition 1.2 ([3])**.** *Let (X, d) be a b-metric space.*

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(1). *A sequence* $\{x_n\}$ *in* X *is called b-convergent if there exists* $x \in X$ *such that* $d(x_n, x) \to 0$ *as* $n \to \infty$. *In this case, we write* $\lim_{n\to\infty} x_n = x$.

(2). *A sequence* $\{x_n\}$ *in* X *is called b-Cauchy if* $d(x_n, x_m) \to 0$ *as* $n, m \to \infty$.

(3). *The b-metric space* (X, d) *is b-complete if every b-Cauchy sequence in* X *is b-convergent.*

(4). *Let* $Y \subset X$. *Then* Y *is called b-closed if and only if for each sequence* $\{x_n\}$ *in* Y *which b-converges to an element* x, *we have* $x \in Y$.

Remark 1.3. *A b-metric need not be a continuous function. For more details, we refer [5].*

Lemma 1.4 ([5]). *Let* (X, d) *be a b- metric space with* $s \geq 1$.

(1). *If a sequence* $\{x_n\} \subset X$ *is a b- convergent sequence, then it admits a unique limit.*

(2). *Every b-convergent sequence in* X *is b-Cauchy.*

Definition 1.5 ([8]). *Let* f *and* g *be selfmaps on a metric space* (X, d). *If* $fx = gx = w$ *for some* $x \in X$, *then* x *is called a coincidence point of* f *and* g *and the set of all coincidence points of* f *and* g *is denoted by* $C(f, g)$, *and* w *is called point of coincidence of* f *and* g.

Definition 1.6 ([6]). *A pair* (f, g) *of selfmaps on a metric space* (X, d) *is said to be compatible if* $\lim_{n\to\infty} d(gfx_n, fgx_n) = 0$ *whenever* $\{x_n\}$ *is a sequence in* X *such that* $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ *for some* z *in* X.

Definition 1.7. *A pair* (f, g) *of selfmaps on a metric space* (X, d) *is said to be noncompatible if there exists at least one sequence* $\{x_n\}$ *in* X *such that* $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ *for some* z *in* X *but* $\lim_{n\to\infty} d(gfx_n, fgx_n)$ *is either non-zero or does not exist.*

Definition 1.8 ([1]). *A pair* (f, g) *of selfmaps on a metric space* (X, d) *is said to be satisfy* $(E.A)$*-property if there exists a sequence* $\{x_n\}$ *in* X *such that* $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ *for some* z *in* X.

Definition 1.9 ([10]). *A pair* (f, g) *of selfmaps on a b-metric space* (X, d) *is said to be satisfy b-*$(E.A)$*-property if there exists a sequence* $\{x_n\}$ *in* X *such that* $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ *for some* z *in* X.

Definition 1.10 ([7]). *A pair* (f, g) *of selfmaps on a set* X *is said to be weakly compatible if* $fgx = gfx$ *whenever* $fx = gx$ *for any* x *in* X.

We denote $\Psi = \{\psi : [0, \infty) \to [1, \infty) |\ \psi$ is continuous, nondecreasing on $[0, \infty)$ and $\psi(t) = 1$ if and only if $t = 0$ $\}$.

Example 1.11. *The following functions* $\psi : [0, \infty) \to [1, \infty)$ *are elements of* Ψ. *For* $t \in [0, \infty)$ *and* $a \in (1, \infty)$

(1). $\psi(t) = t + 1$,

(2). $\psi(t) = a^t$,

(3). $\psi(t) = a^{\sqrt{t}}$.

Very recently, Ozturk and Radenovic [11] obtained the following result in b-metric spaces.

Theorem 1.12 ([11]). *Let (X, d) be a b-metric space with coefficient $s > 1$ and $f, g, S, T : X \to X$ be selfmappings of X with $fX \subset TX$ and $gX \subset SX$ such that*

$$s^{\epsilon} d(fx, gy) \leq M_s(x, y)) \text{ for all } x, y \in X, \tag{1}$$

where $\epsilon > 1$ is a constant and
$M(x, y) = max\{d(Sx, Ty), d(Sx, fx), d(Ty, gy), \frac{d(Sx,gy)+d(Ty,fx)}{2s})\}$. Suppose one of the pairs (f, S) and (g, T) satisfy the b-$(E.A)$-property and that of one of the subspaces fX, gX, SX and TX is b-closed in X. Then the pairs (f, S) and (g, T) have a point of coincidence in X. Moreover, if the pairs (f, S) and (g, T) are weakly compatible, then f, g, S and T have a unique common fixed point.

In Section 2, we prove our main results in which we study the existence of common fixed points of two pairs of selfmaps satisfying b-$(E.A)$-property in b-metric spaces. In Section 3, we provide corollaries and an example in support of our results. Our results generalize the results of Ozturk and Radenovic [11].

2. Main Results

Proposition 2.1. *Let (X, d) be a b-metric space with coefficient $s \geq 1$. Let $f, g, S, T : X \to X$ be selfmaps of X with $fX \subset TX$ and $gX \subset SX$. Assume that there exist $\psi \in \Psi$ and $k \in [0, 1)$ such that*

$$\psi(sd(fx, gy)) \leq (\psi(M_s(x, y)))^k \text{ for all } x, y \in X, \tag{2}$$

where $M_s(x, y) = max\{d(Sx, Ty), d(Sx, fx), d(Ty, gy), \frac{d(Sx,gy)+d(Ty,fx)}{2s}\}$. Suppose that the pairs (f, S) and (g, T) are weakly compatible. Then $F(f, S) \neq \emptyset$ if and only if $F(g, T) \neq \emptyset$, where $F(f, S)$ and $F(g, T)$ are the set of all common fixed points of the pairs (f, S) and (g, T) respectively. In this case, if $q \in F(f, S)$ then $q \in F(g, T)$ and q is the unique common fixed point of f, g, S and T. Similarly, if $q \in F(g, T)$ then $q \in F(f, S)$ and q is the unique common fixed point of f, g, S and T.

Proof. First we assume that $F(f, S) \neq \emptyset$. Let $q \in F(f, S)$, then $q = fq = Sq$. Now, we show that $q \in F(g, T)$. Since $fX \subset TX$ there exists $r \in X$ such that $q = fq = Tr$, then we have $Sq = fq = Tr = q$. We now show that $gr = q$. Suppose that $gr \neq q$. From (2) we have

$$\psi(sd(q, gr)) = \psi(sd(fq, gr)) \leq (\psi(M_s(q, r)))^k, \tag{3}$$

where

$$\begin{aligned} M_s(q, r) &= \max\{d(Sq, Tr), d(Sq, fq), d(Tr, gr), \frac{d(Sq, gr) + d(Tr, fq)}{2s}\} \\ &= \max\{d(q, q), d(q, q), d(q, gr), \frac{d(q, gr) + d(q, q)}{2s}\} \\ &= \max\{0, 0, d(q, gr), \frac{d(q, gr)}{2s}\} \\ &= d(q, gr). \end{aligned} \tag{4}$$

Now, from (3) using (4) we have

$$\psi(sd(q, gr)) = \psi(sd(fq, gr)) \leq (\psi(M_s(q, r)))^k = (\psi(d(q, gr)))^k < \psi(d(q, gr)),$$

a contradiction. Hence $gr = q$. Therefore $gr = Tr = q$. Since g and T are weakly compatible, we have $gq = Tq$. We now show that $gq = q$. Suppose $gq \neq q$. From (2) we have

$$\psi(sd(q, gq)) = \psi(sd(fq, gq)) \leq (\psi(M_s(q, q)))^k, \tag{5}$$

where

$$\begin{aligned} M_s(q,q) &= \max\{d(Sq,Tq), d(Sq,fq), d(Tq,gq), \frac{d(Sq,gq)+d(Tq,fq)}{2s}\} \\ &= \max\{d(q,gq), d(q,q), d(gq,gq), \frac{d(q,gq)+d(gq,q)}{2s}\} \\ &= \max\{d(q,gq), 0, 0, \frac{d(q,gq)}{s}\} \\ &= d(q,gq). \end{aligned} \tag{6}$$

From (5) and using (6), we have

$$\psi(sd(q,gq)) = \psi(sd(fq,gq)) \leq (\psi(M_s(q,q)))^k = (\psi(d(q,gq)))^k < \psi(d(q,gq)),$$

a contradiction. Hence $gq = q$. Therefore $Tq = gq = q$ and hence $F(g, T) \neq \emptyset$.

Conversely, we assume that $F(g, T) \neq \emptyset$. Let $u \in F(g, T)$ i.e., $gu = Tu = u$. On using similar steps as above we can show that $u \in F(f, S)$ and hence $F(f, S) \neq \emptyset$. We now show that f, g, S and T have a unique common fixed point. Let u and q be common fixed points of f, g, S and T. Suppose that $u \neq q$. From (2), we have

$$\psi(sd(u, q)) = \psi(sd(fu, gq)) \leq (\psi(M_s(u, q)))^k \tag{7}$$

where

$$\begin{aligned} M_s(u,q) &= \max\{d(Su,Tq), d(Su,fu), d(Tq,gq), \frac{d(Su,gq)+d(Tq,fu)}{2}\} \\ &= \max\{d(u,q), d(u,u), d(q,q), \frac{d(u,q)+d(q,u)}{2s}\} \\ &= \max\{d(u,q), 0, 0, \frac{d(u,q)}{s}\} \\ &= d(u,q). \end{aligned} \tag{8}$$

From (7) and using (8), we have

$$\psi(sd(u,q)) = \psi(sd(fu,gq)) \leq (\psi(M_s(u,q)))^k = (\psi(d(u,q)))^k < \psi(d(u,q)),$$

a contradiction. Hence $u = q$. Therefore S, f, g and T have a unique common fixed point. □

The main results of this paper is the following.

Theorem 2.2. *Let (X, d) be a b-metric space with coefficient $s \geq 1$. Let $f, g, S, T : X \to X$ be selfmaps of X with $fX \subset TX$ and $gX \subset SX$. Assume that there exist $\psi \in \Psi$ and $k \in [0, 1)$ such that*

$$\psi(sd(fx, gy)) \leq (\psi(M_s(x, y)))^k \text{ for all } x, y \in X, \tag{9}$$

where $M_s(x, y) = max\{d(Sx, Ty), d(Sx, fx), d(Ty, gy), \frac{d(Sx,gy)+d(Ty,fx)}{2s}\}$. Suppose that one of the pairs (f, S) and (g, T) satisfies the b-$(E.A)$-property and that one of the subspaces fX, gX, SX and TX is b-closed in X. Then the pairs (f, S) and (g, T) have a point of coincidence in X. Moreover, if the pairs (f, S) and (g, T) are weakly compatible, then f, g, S and T have a unique common fixed point.

Proof. We first assume that the pair (f, S) satisfies the b-(E. A)-property. So there exists a sequence $\{x_n\}$ in X satisfying

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = q \text{ for some } q \in X. \tag{10}$$

As $fX \subset TX$, there exists a sequence $\{y_n\}$ in X such that $fx_n = Ty_n$, and hence

$$\lim_{n\to\infty} Ty_n = q. \tag{11}$$

Now, we show that $\lim_{n\to\infty} gy_n = q$. Suppose that $\limsup\limits_{n\to\infty} d(fx_n, gy_n) > 0$. From (9), we have

$$\psi(sd(fx_n, gy_n)) \le (\psi(M_s(x_n, y_n)))^k, \tag{12}$$

where

$$\begin{aligned} M_s(x_n, y_n) &= \max\{d(Sx_n, Ty_n), d(Sx_n, fx_n), d(Ty_n, gy_n), \frac{d(Sx_n, gy_n) + d(Ty_n, fx_n)}{2s}\} \\ &= \max\{d(Sx_n, fx_n), d(Sx_n, fx_n), d(fx_n, gy_n), \frac{d(Sx_n, gy_n) + d(fx_n, fx_n)}{2s}\} \\ &\le \max\{d(Sx_n, fx_n), d(fx_n, gy_n), \frac{s[d(Sx_n, fy_n) + d(fx_n, gy_n)]}{2s}\}. \end{aligned}$$

On taking limit supremum as $n \to \infty$ in the above inequality we have

$$\begin{aligned} \limsup_{n\to\infty} M_s(x_n, y_n) &= \max\{0, 0, \limsup_{n\to\infty} d(fx_n, gy_n), \frac{\limsup\limits_{n\to\infty} d(fx_n, gy_n)}{2}\} \\ &= \limsup_{n\to\infty} d(fx_n, gy_n). \end{aligned} \tag{13}$$

On taking limits supremum as $n \to \infty$ in (12) and using (13), we have

$$\begin{aligned} \psi(s \limsup_{n\to\infty} d(fx_n, gy_n)) = \limsup_{n\to\infty} \psi(sd(fx_n, gy_n)) &\le \limsup_{n\to\infty} (\psi(M_s(x_n, y_n)))^k \\ &= (\psi(\limsup_{n\to\infty} d(fx_n, gy_n)))^k < \psi(\limsup_{n\to\infty} d(fx_n, gy_n)), \end{aligned}$$

a contradiction. Hence $\limsup\limits_{n\to\infty}(fx_n, gy_n) = 0$, which implies that $\lim\limits_{n\to\infty}(fx_n, gy_n) = 0$. Now, we have

$$d(q, gy_n) \le s[d(q, fx_n) + d(fx_n, gy_n)]. \tag{14}$$

On taking limits as $n \to \infty$ in (14), we have

$$0 \le \lim_{n\to\infty} d(gy_n, q) \le s \lim_{n\to\infty} [d(q, fx_n) + d(fx_n, gy_n)] = 0. \tag{15}$$

Therefore $\lim\limits_{n\to\infty} d(q, gy_n) = 0$.

<u>*Case*</u> (i) : Assume that TX is a b-closed subset of X.

In this case $q \in TX$ and hence we can choose $r \in X$ such that $Tr = q$. Now we show that $gr = q$. Now, we have

$$d(q, gr) \le s[d(q, fx_n) + d(fx_n, gr)]. \tag{16}$$

On taking limit supremum as $n \to \infty$ in (16), we have

$$d(q, gr) \le s \limsup_{n\to\infty} d(fx_n, gr). \tag{17}$$

Suppose $d(q, gr) > 0$. From (9), we have

$$\psi(sd(fx_n, gr)) \leq (\psi(M_s(x_n, r)))^k, \tag{18}$$

where

$$\begin{aligned} M_s(x_n, r) &= \max\{d(Sx_n, Tr), d(Sx_n, fx_n), d(Tr, gr), \frac{d(Sx_n, gr) + d(Tr, fx_n)}{2s}\} \\ &\leq \max\{d(Sx_n, q), d(Sx_n, fx_n), d(q, gr), \frac{s[d(Sx_n, q) + d(q, gr)] + d(q, fx_n)}{2s}\}. \end{aligned}$$

On taking limit supremum as $n \to \infty$ in the above inequality we have

$$\limsup_{n\to\infty} M_s(x_n, r) \leq \max\{0, 0, d(q, gr), \frac{d(q, gr)}{2}\} = d(q, gr). \tag{19}$$

On taking limits supremum as $n \to \infty$ in (18) and using (17) and (19), we have

$$\begin{aligned} \psi(d(q, gr)) \leq \psi(s \limsup_{n\to\infty} d(fx_n, gr)) &= \limsup_{n\to\infty} \psi(sd(fx_n, gr)) \\ &\leq \limsup_{n\to\infty} (\psi(M_s(x_n, r)))^k = (\psi(\limsup_{n\to\infty} M_s(x_n, r)))^k \\ &\leq (\psi(d(q, gr)))^k < \psi(d(q, gr)), \end{aligned}$$

a contradiction. Hence $d(q, gr) = 0$. Therefore $gr = q$, i.e., $gr = Tr = q$ and hence r is a coincidence point of g and T.
Since $q = gr$ and $gX \subset SX$, we have $q \in SX$ and hence there exists $z \in X$ such that $Sz = q = gr$.
Now, we show that $Sz = fz$. Suppose $Sz \neq fz$. By (9), we have

$$\psi(sd(fz, q)) = \psi(sd(fz, gr)) \leq (\psi(M_s(z, r)))^k, \tag{20}$$

where

$$\begin{aligned} M_s(z, r) &= \max\{d(Sz, Tr), d(Sz, fz), d(Tr, gr), \frac{d(Sz, gr) + d(Tr, fz)}{2s}\} \\ &= \max\{0, d(q, fz), 0, \frac{d(q, fz)}{2s}\} = d(fz.q). \end{aligned} \tag{21}$$

From (20) and using (21), we have

$$\psi(sd(fz, q)) = \psi(d(fz, gr)) \leq (\psi(M_s(z, r)))^k = (\psi(d(fz, q)))^k < \psi(d(fz, q)),$$

a contradiction. Hence $fz = Sz = q$, so that z is a coincidence point of f and S. Since the pairs (f, S) and (g, T) are weakly compatible, we have $fq = Sq$ and $Tq = gq$ so that q is also a coincidence point of (f, S) and (g, T). Now, we show that q is a common fixed point of f, g, S and T. Suppose $fq \neq q$. From (9), we have

$$\psi(sd(fq, q)) = \psi(sd(fq, gr)) \leq (\psi(M_s(q, r)))^k, \tag{22}$$

where

$$\begin{aligned} M_s(q, r) =& \max\{d(Sq, Tr), d(Sq, fq), d(Tr, gr), \frac{d(Sq, gr) + d(Tr, fq)}{2s}\} \\ =& \max\{d(fq, q), 0, 0, \frac{d(q, fq)}{s}\} \\ =& d(fq, q). \end{aligned} \tag{23}$$

From (22) and using (23), we have

$$\psi(sd(fq,q)) = \psi(sd(fq,gr)) \leq (\psi(M_s(q,r)))^k = (\psi(d(fq,q)))^k < \psi(d(fq,q)),$$

a contradiction. Hence $fq = q$. Therefore $Sq = fq = q$, so that q is common fixed point of f and S and hence $F(f,S) \neq \emptyset$. By Proposition 2.1, we have $F(g,T) \neq \emptyset$ and $q \in F(g,T)$ and q is the unique common fixed point of f, g, S and T.

Case (ii) : Suppose fX is b-closed.

In this case, we have $q \in fX$ and since $fX \subset TX$, we choose $r \in X$ such that $q = Tr$. Hence the proof follows as in Case (i).

Case (iii) : SX is b-closed.

We follow the argument similar to the case (i), and get the conclusion.

Case (iv) : Suppose gX is b-closed.

As in case (ii), we get the conclusion.

For the case of (g,T) satisfies the b-$(E.A)$-property, we follow the argument similar to the case (f,S) satisfies the b-$(E.A)$-property. This complete the proof of the theorem. □

3. Corollaries and Examples

Corollary 3.1. *Let (X,d) be a b-metric space with coefficient $s \geq 1$. Let $f, g, S, T : X \to X$ be selfmaps of X with $fX \subset TX$ and $gX \subset SX$ such that*

$$sd(fx,gy)) \leq kM_s(x,y)) \text{ for all } x, y \in X, \tag{24}$$

where $M_s(x,y) = max\{d(Sx,Ty), d(Sx,fx), d(Ty,gy), \frac{d(Sx,gy)+d(Ty,fx)}{2s}\}$ and $0 \leq k < 1$. Suppose that one of the pairs (f,S) and (g,T) satisfies the b-$(E.A)$-property and that of one of the subspaces fX, gX, SX and TX is b-closed in X. Then the pair (f,S) and (g,T) have a point of coincidence in X. Moreover, if the pairs (f,S) and (g,T) are weakly compatible, then f, g, S and T have a unique common fixed point.

Proof. The result follows from Theorem 2.2 by choosing $\psi(t) = e^t$ for all $t \geq 0$. □

Remark 3.2. *Since the inequality (1) is a spacial case of inequality (9) with $\psi(t) = e^t, t \geq 0$ and $k = \frac{\mathbf{s}}{\mathbf{s}^{\epsilon}}$, the conclusion of Theorem 1.12 follows from Theorem 2.2. Hence Theorem 1.12 is a corollary to Theorem 2.2*

Corollary 3.3. *Let (X,d) be a b-metric space with coefficient $s \geq 1$. Let $f, T : X \to X$ be selfmaps of X with $fX \subset TX$ and $gX \subset SX$ such that*

$$\psi(sd(fx,fy)) \leq (\psi(M_s(x,y)))^k \text{ for all } x, y \in X, \tag{25}$$

where $M_s(x,y) = max\{d(Tx,Ty), d(Tx,fx), d(Ty,fy), \frac{d(Tx,fy)+d(Ty,fx)}{2s}\}$ and $k \in [0,1)$. Suppose that the pair (f,T) satisfies the b-$(E.A)$-property and that of one of the subspaces fX and TX is b-closed in X. Then the pair (f,T) has a point of coincidence in X. Moreover, if the pair (f,T) is weakly compatible, then f and T have a unique common fixed point.

Proof. The result follows from Theorem 2.2 by choosing $f \equiv g$ and $S \equiv T$. □

Example 3.4. *Let $X = [0,\infty)$ with the usual metric. We define f, g, S and T on X by*

$$fx = \begin{cases} \frac{x}{3} & \text{if } x \in [0,3] \\ 1 & \text{if } \in (3,\infty), \end{cases} \qquad gx = \begin{cases} \frac{x}{5} & \text{if } x \in [0,3] \\ 1 & \text{if } \in (3,\infty), \end{cases}$$

$$Sx = \begin{cases} 5x & \text{if } x \in [0,3] \\ 3 & \text{if } x \in (2,\infty), \end{cases} \qquad Tx = \begin{cases} 3x & \text{if } x \in [0,3] \\ 3 & \text{if } x \in (3,\infty). \end{cases}$$

Since $x = 0$ is the only coincidence point of the pairs (f,S) and (g,T) and $fS(0) = Sf(0)$ and $gT(0) = Tg(0)$ and hence the pairs (f,S) and (g,T) are weakly compatible. We choose a sequence x_n with $x_n = \frac{1}{n}, n = 1,2,3,\ldots$ with $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = 0$, hence the pair (f,S) satisfies the b-$(E.A)$-property. We now verify the inequality (9) with $\psi(t) = e^t$, $t \geq 0$ and $k = \frac{1}{2}$. Since $\psi(t) = e^t$ and $s = 1$, f,g,S and T satisfy the inequality (9) if and only if f,g,S and T satisfy the following inequality:

$$|fx - fy| \;\leq\; kM_s(x,y) = \frac{1}{2}M_s(x,y). \tag{26}$$

We have the following possible cases.

<u>Case</u> (i): $x,y \in [0,3]$.

In this case, $f(x) = \frac{x}{3}$, $gy = \frac{y}{5}$, $Sx = 5x$ and $Ty = 3y$, and hence $d(Sx,Ty) = |5x - 3y|$. Now, we have

$$d(fx,gy) = |\frac{x}{3} - \frac{y}{5}| = \frac{1}{15}|5x - 3y| \leq \frac{1}{2}|5x - 3y| = \frac{1}{2}|Sx - Ty| \leq \frac{1}{2}M_s(x,y)$$

<u>Case</u> (ii): $x,y \in (3,\infty)$.

In this case, since $f(x) = gy = 1$, the inequality (9) holds trivially.

<u>Case</u> (iii): $x \in [0,3]$, $y \in (3,\infty)$.

In this case, $f(x) = \frac{x}{3}$, $S(x) = 5x$, $gy = 1$ and $Ty = 3$ and hence $d(Ty,gy) = |3 - 1| = 2$. Now, we have $d(fx,gy) = |\frac{x}{3} - 1| \leq 1 = \frac{1}{2}d(Ty,gy) \leq \frac{1}{2}M_s(x,y)$.

<u>Case</u> (iv): $x \in (3,\infty), y \in [0,3]$.

In this case, $fx = 1$, $Sx = 3$, $gy = \frac{y}{5}$ and $Ty = 3y$ and hence $d(fx,Sx) = |3-1| = 2$. Now, we have $d(fx,gy) = |\frac{y}{5} - 1| \leq 1 = \frac{1}{2}d(fx,Sx) \leq \frac{1}{2}M_s(x,y)$.

Hence from all the above cases f,g,S and T satisfy the inequality (9). Therefore f,g,S and T satisfy all the hypotheses of Theorem 2.2 and $x = 0$ is the unique common fixed point of f,g,S and T. Here we observe that Theorem 1.12 is not applicable, since $s = 1$. Hence Remark 3.2 suggests that Theorem 2.2 is a generalization of Theorem 1.12.

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