Common Fixed Points of g-quasicontractions for Faithly Compatible Mappings in 0-complete Partial Metric Spaces

Research Article

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Abstract: Common fixed point results are obtained in 0-complete partial metric spaces under various contractive conditions, including g-quasicontractions and mappings with a contractive iterate. In this way, several results obtained recently are generalized. In this paper we introduce a generalization of the concept of compatible mappings, and using that condition in partial ordered metric spaces, we obtain some new fixed point theorems under both contractive and noncontractive conditions, which may allow the existence of a common fixed point or the existence of multiple fixed or coincidence points. We also manifest that the new concept is a necessary condition for the existence of a common fixed point.

1.   Introduction and Preliminaries

Matthews [21] introduced the notion of a partial metric space as a part of the study of denotational semantics of dataflow networks.He showed that the Banach contraction mapping theorem can be generalized to the partial metric context for applications in program verification. Subsequently, several authors [10, 13, 15, 19, 20, 22, 23, 26, 28] derived fixed point theorems in partial metric spaces. See also the presentation by Bukatin et al. [11] where the motivation for introducing non-zero distance (i.e., the "distance" p where $p(x,x) \neq 0$) is explained, which is also leading to interesting research in foundations of topology. The following definitions and details can be seen, e.g., in [10, 11, 18, 21, 22, 27].

Definition 1.1. *A partial metric on a nonempty set X is a function $p : X \times X \to R^+$ such that for all $x, y, z \in X$:*

(p_1) $x = y \iff p(x,x) = p(x,y) = p(y,y),$

(p_2) $p(x,x) \leq p(x,y),$

(p_3) $p(x,y) = p(y,x),$

(p_4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z).$

The pair (X, p) is called a partial metric on X.

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It is clear that, if $p(x, y) = 0$, then from (p_1) and (p_2) $x = y$. But if $x = y, p(x, y)$ may not be 0.

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x, \epsilon) : x \in X, \epsilon > 0\}$, where $B_p(x, \epsilon) = \{y \in X : p(x, y) < p(x, x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$. A sequence $\{x_n\}$ in (X, p) converges to a point $x \in X$ (in the sense of τ_p) if $\lim_{n \to \infty} p(x, x_n) = p(x, x)$. This will be denoted as $x_n \to x$ as $(n \to \infty)$ or $\lim_{n \to \infty} x_n = x$. If $T : X \to X$ is continuous at $x \in X$ (with respect to τ_p), then for each sequence $\{x_n\}$ in X, we have $x_n \to x_0$ as $Tx_n \to Tx_0$.

Remark 1.2. *Clearly, a limit of a sequence in a partial metric space need not be unique. Moreover, the function $p(.,.)$ need not be continuous in the sense that $x_n \to x$ and $y_n \to y$ imply $p(x_n, y_n) \to p(x, y)$.*

If p is a partial metric on X, then the function $p^s : X \times X \to R^+$ given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y) \tag{1}$$

is a metric on X. Furthermore, $\lim_{n \to \infty} p^s(x_n, x) = 0$ if and only if $p(x, x) = \lim_{n \to \infty} p(x_n, x) = \lim_{n,m \to \infty} p(x_n, x_m)$.

Example 1.3. *A paradigmatic example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p(x, y) = \max\{x, y\}$ for all $x, y \in R^+$. The corresponding metric is $p^s(x, y) = 2 \max\{x, y\} - x - y = |x - y|$.*

Example 1.4. *If (X, d) is a metric space and $c \geq 0$ is arbitrary, then $p(x, y) = d(x, y) + c$ defines a partial metric on X and the corresponding metric is $p^s(x, y) = 2d(x, y)$.*

Other examples of partial metric spaces which are interesting from a computational point of view may be found in [16, 21].

Definition 1.5. *Let (X, p) be a partial metric space. Then:*

(1) *A sequence $\{x_n\}$ in (X, p) is called a Cauchy sequence if $\lim_{n,m \to \infty} p(x_n, x_m)$ exists (and is finite).*

(2) *The space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m \to \infty} p(x_n, x_m)$.*

(3) *A sequence $\{x_n\}$ in (X, p) is called 0-Cauchy if $\lim_{n,m \to \infty} p(x_n, x_m) = 0$. The space (X, p) is said to be 0-complete if every 0-Cauchy sequence in X converges (in τ_p) to a point $x \in X$ such that $p(x, x) = 0$.*

Lemma 1.6. *Let (X, p) be a partial metric space.*

(a) *$\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s).*

(b) *The space (X, p) is complete if and only if the metric space (X, p^s) is complete.*

(c) *Every 0-Cauchy sequence in (X, p) is Cauchy in (X, p^s).*

(d) *If (X, p) is complete, then it is 0-complete.*

The converse assertions of (c) and (d) do not hold as the following easy example shows.

Example 1.7. *The space $X = [0, \infty) \cap Q$ with the partial metric $p(x, y) = \max\{x, y\}$ is 0-complete, but is not complete (since $p^s(x, y) = |x - y|$ and (X, p^s) is not complete). Moreover, the sequence $\{x_n\}$ with $x_n = 1$ for each $n \in N$ is a Cauchy sequence in (X, p), but it is not a 0-Cauchy sequence.*

Recall that Romaguera proved in [[24], Theorem 2.3] that a partial metric space (X, p) is 0-complete if and only if every p^s-Caristi mapping on X has a fixed point. It is easy to see that every closed subset of a 0-complete partial metric space is 0-complete. Let (X, p) be a partial metric space and $f, g : X \to X$ be two self maps. When constructing various contractive conditions, usually one of the following sets is used:

$$M_{f,g}^5(x, y) = \{p(gx, gy), p(gx, fx), p(gy, fy), p(gx, fy), p(gy, fx)\},$$
$$M_{f,g}^4(x, y) = \{p(gx, gy), p(gx, fx), p(gy, fy), \frac{1}{2}(p(gx, fy) + p(gy, fx))\},$$
$$M_{f,g}^3(x, y) = \{p(gx, gy), \frac{1}{2}(p(gx, fx) + p(gy, fy)), \frac{1}{2}(p(gx, fy) + p(gy, fx))\}.$$

Then, the contractive condition takes the form

$$p(fx, fy) \leq \lambda \max M_{f,g}^i(x, y), \tag{2}$$

where $\lambda \in [0, 1)$. Mappings f satisfying 2 with $i = 5$ for all $x, y \in X$ (in metric case) are usually called g-quasi contractions (see Ćirić [12] and Das and Naik [14]). (Common) fixed point results in partial metric spaces using conditions of mentioned type in the case $i = 3, 4$ were obtained in various papers.We prove a common fixed point theorem for g-quasi contractions in 0-complete spaces that contains as special cases several other results. A partial metric extension of Sehgal-Guseman result for mappings having a contractive iterate is obtained. Finally, we deduce a partial metric version of (common) fixed point theorem under the condition [23], [25] of B. E. Rhoades. Examples are provided when these results can be applied and neither corresponding metric results nor the results with the standard completeness assumption of the underlying partial metric space can.

2.   Quasi Contractions in Partial Metric Spaces

Definition 2.1. *A pair of self-maps (A, S) of a partial metric space (X, p) is said to be*

(i) *compatible if and only if $\lim_{n \to \infty} p(A(S(y_n)), S(A(y_n))) = 0$, whenever $\{y_n\}$ is a sequence in X such that $\lim_{n \to \infty} A(y_n) = \lim_{n \to \infty} S(y_n) = t$ for some t in X;*

(ii) *noncompatible if there exists a sequence $\{y_n\}$ in X such that $\lim_{n \to \infty} A(y_n) = \lim_{n \to \infty} S(y_n) = t$ for some t in X, but $\lim_{n \to \infty} p(A(S(y_n)), S(A(y_n)))$ is either non-zero or non-existent;*

(iii) *weakly compatible if the pair commutes on the set of coincidence points (a point $x \in X$ is called a coincidence point of the pair (A, S) if $A(x) = S(x)$, i.e., $A(S(x)) = S(A(x))$ whenever $A(x) = S(x)$ for some $x \in X$,*

(iv) *occasionally weakly compatible if there exists a coincidence point x in X such that $A(x) = S(x)$ implies $A(S(x)) = S(A(x))$,*

(v) *conditionally commuting if the pair commutes on a nonempty subset of the set of coincidence points whenever the set of coincidences is nonempty;*

(vi) *subcompatible if there exists a sequence $\{y_n\}$ in X such that $\lim_{n \to \infty} A(y_n) = \lim_{n \to \infty} S(y_n) = t \in X$ and $\lim_{n \to \infty} p(A(S(y_n)), S(A(y_n))) = 0$,*

(vii) *conditionally compatible if and only if whenever the set of sequences $\{y_n\}$ satisfying $\lim_{n \to \infty} A(y_n) = \lim_{n \to \infty} S(y_n)$ is nonempty, there exists a sequence $\{z_n\}$ such that $\lim_{n \to \infty} A(z_n) = \lim_{n \to \infty} S(z_n) = t$ and $\lim_{n \to \infty} p(A(S(y_n)), S(A(y_n))) = 0$.*

It may be observed that compatibility is independent of the notion of conditional compatibility, and in the setting of a unique common fixed point (or unique point of coincidence), conditional compatibility does not reduce to the class of compatibility. The following examples illustrate these facts.

Example 2.2. *Let $X = [0, \infty)$ with the partial metric p. Define mappings $A, S : X \to X$ by $A(x) = x$, $\forall x$ and $S(x) = 2x$, $\forall x$. Then it can be verified that f and g are compatible but not conditionally compatible.*

Example 2.3. *Let $X = [1, 8]$ with the partial metric p. Define mappings $A, S : X \to X$ as follows:*

$$A(x) = 2 \quad if \quad x \leq 2, \qquad A(x) = 5 \quad if \quad x > 2,$$

$$S(x) = 6 - 2x \quad if \quad x \leq 2, \qquad S(x) = 8 \quad if \quad x > 2.$$

In this example A and S are conditionally compatible but not compatible. To see this, we can consider the constant sequence $z_n = 2$, then $\lim_{n \to \infty} A(z_n) = 2$, $\lim_{n \to \infty} S(z_n) = 2$, $\lim_{n \to \infty} A(Sz_n) = 2$, $\lim_{n \to \infty} S(A(z_n)) = 2$ and $\lim_{n \to \infty} p(A(S(z_n)), S(A(z_n))) = 0$. Again, if we consider the sequence $y_n = 2 - \frac{1}{n}$, then $\lim_{n \to \infty} A(y_n) = 2$, $\lim_{n \to \infty} S(y_n) = \lim_{n \to \infty} (2 + \frac{1}{n})$, $\lim_{n \to \infty} A(S(y_n)) = 5$, $\lim_{n \to \infty} S(A(y_n)) = 2$ and $\lim_{n \to \infty} p(A(S(z_n)), S(A(z_n))) = 5$. Thus f and g are conditionally compatible but not compatible.

It may also be observed that conditional compatibility need not imply commutativity at the coincidence points. The following example illustrates this fact.

Example 2.4. *Let $X = [0, \infty)$ and partial metric p on X. Define A and $S : X \to X$ by $A(x) = x^2$ and $S(x) = x + 6$ if $x \in [0, 9] \cup (16, \infty)$, $x + 72$ if $x \in (9, 16]$. In this example A and S are conditionally compatible, but they do not commute their only coincidence point $x = 3$. To see this, let us consider the sequence $y_n = 3 + \frac{1}{n}$, then $\lim_{n \to \infty} A(y_n) = 9 = \lim_{n \to \infty} S(y_n)$ and $\lim_{n \to \infty} p(A(S(z_n)), S(A(z_n))) = 0$. Thus f and g are conditionally compatible. On the other hand, we have $A(x) = S(x)$ iff $x = 3$ and $A(S(3)) = A(9) = 81$, $S(A(3)) = S(9) = 15$. Then $A(3) = 9 = S(3)$, but $A(S(3)) \neq S(A(3))$.*

In this paper we define the notion of conditionally compatible maps in a slightly different manner as follows.

Definition 2.5. *Two self-mappings A and S of a partial metric p will be called to be faintly compatible iff A and S are conditionally compatible and A and S commute on a nonempty subset of coincidence points whenever the set of coincidences is nonempty.*

If A and S are compatible, then they are obviously faintly compatible, but the converse is not true in general.

Example 2.6. *Let $X = [3, 6]$ and p be the partial metric on X. Define self-mappings A and S on X as follows:*

$$A(x) = 3 \qquad if \qquad x = 3 \quad or \quad x > 5,$$

$$A(x) = x + 1 \quad if \quad 3 < x \leq 5,$$

$$S(3) = 3, \qquad S(x) = \frac{x + 7}{3} \quad if \quad 3 < x \leq 5,$$

$$S(x) = \frac{x + 1}{2} \qquad if \quad x > 5.$$

In this example A and S are faintly compatible but not compatible. To see this, if we consider the constant sequence $\{y_n = 3\}$, then A and S are faintly compatible. On the other hand, if we choose the sequence $\{x_n = 5 + \frac{1}{n}\}$ then $\lim_{n \to \infty} A(x_n) = 3 = \lim_{n \to \infty} S(x_n)$, and $\lim_{n \to \infty} d(A(S(x_n)), S(A(x_n))) \neq 0$. Thus A and S are faintly compatible, but they are not compatible. It is also relevant to mention here that faint compatibility and noncompatibility are independent concepts. To see this, we can consider the following examples.

Example 2.7. *Let $X = [2, 10]$ and p be the partial metric on X. Define self-mappings A and S on X as follows:*

$$A(x) = 8 \qquad if \qquad 2 \leq x \leq 5,$$

$$A(x) = 2 \quad if \quad x > 5,$$

$$S(x) = 2 \quad if \quad 2 \leq x < 5,$$

$$S(x) = x - 3 \qquad if \qquad x \geq 5.$$

In this example A and S are noncompatible, but not faintly compatible. To see this, let us consider the sequence $x_n = 5 + \frac{1}{n}$, then $\lim_{n \to \infty} A(x_n) = 2$, $\lim_{n \to \infty} S(x_n) = 2$, but $\lim_{n \to \infty} d(A(S(x_n)), S(A(x_n))) = 6$. Thus A and S are noncompatible, but not faintly compatible.

Example 2.8. *Let $X = [1, \infty)$ and p be the partial metric on X. Define $A, S : X \to X$ by $A(x) = x^2$ and $S(x) = 3x - 2$ for all $x \in [1, \infty)$.*

In this example A and S are faintly compatible, but not noncompatible. Examples 2.7 and 2.8 clearly show that faint compatibility and noncompatibility are independent of each other. If A and S are weakly compatible, then they are also faintly compatible, but the converse is not true in general. It is worth mentioning here that if f and g are owc, then they are also faintly compatible, but the converse is not true in general.

Example 2.9. *Let $X = [1, \infty)$ and p be the partial metric on X. Define $A, S : X \to X$ by $A(x) = x$ and $S(x) = x + 1$ for all $x \in [1, \infty)$. Then it can be verified that A and S are trivially faintly compatible but not owc.*

It may be pointed out that the notion of owc implies commutativity at some coincidence points, but it does not help in establishing the existence of coincidence points, whereas the new notion is useful in establishing the existence of coincidence points.

Theorem 2.10. *Let f and g be noncompatible faintly compatible self-mappings of a partial metric space (X, p) also $f(X) \subseteq g(X)$. If there exists $\lambda \in [0, 1)$ such that the condition*

$$p(fx, fy) \leq \lambda \max M_{f,g}^5(x, y) \tag{3}$$

holds for all $x, y \in X$, where

$$M_{f,g}^5(x, y) = \max\{p(gx, gy), p(gx, fx), p(gy, fy), p(gx, fy), p(gy, fx)\},$$

If either f or g is continuous, then f and g have a unique common fixed point u such that $p(u, u) = 0 = p(fu, gu)$.

Proof. For arbitrary $x_0 \in X$, and using that $f(X) \subseteq g(X)$, choose a Jungck sequence $\{y_n\}$ in X by $y_n = fx_n = gx_{n+1}$, $n \in \mathbb{N}_0$. Denote by $O_{f,g}(x_0, n) = \{y_0, y_1, \ldots, y_n\}$ the nth orbit of x_0 and by $O_{f,g}(x_0, \infty) = \{y_0, y_1, y_2, \ldots\}$ its orbit. Also, denote by $diam A = \sup\{p(x, y) | x, y \in A\}$ the diameter of a nonempty set $A \subseteq X$. Note that $diam A = 0$ implies that A is a singleton, but the converse is not true. If $p(y_n, y_{n+1}) = 0$ for some $n \in N_0$, then it is easy to prove (using properties (p_2) and (p_4) of the partial metric, and the contractive condition 3 that $p(y_{n+1}, y_{n+2}) = 0$, i.e., $y_n = y_{n+1} = y_{n+2} = \ldots$. Hence, in this case, $\{y_n\}$ is a 0-Cauchy sequence in (X, p). Suppose now that $p(y_n, y_{n+1}) > 0$ for each $n \in N_0$.

Claim 1.

$$diam\, O_{f,g}(x_0, \infty) \leq \frac{1}{1 - \lambda} p(fx_0, fx_1) = \frac{1}{1 - \lambda} p(y_0, y_1).$$

Indeed, let $1 \leq i, j \leq n$. Then

$$
\begin{aligned}
p(y_i, y_j) &= p(fx_i, fx_j) \\
&\leq \lambda \max\{p(gx_i, gx_j), p(gx_i, fx_i), p(gx_j, fx_j), p(gx_i, fx_j), p(gx_j, fx_i)\} \\
&\leq \lambda \max\{p(y_{i-1}, y_{j-1}), p(y_{i-1}, y_i), p(y_{j-1}, y_j), p(y_i, y_{j-1}), p(y_{j-1}, y_i)\}
\end{aligned}
\tag{4}
$$

Since the points $y_{i-1}, y_i, y_{j-1}, y_j$ belong to the set $O_{f,g}(x_0, n)$, it follows that

$$
p(y_i, y_j) \leq \lambda \ diam \ O_{f,g}(x_0, n) < diam \ O_{f,g}(x_0, n).
$$

Hence, there exists $k \leq n$ such that $diam \ O_{f,g}(x_0, n) = p(y_0, y_k)$. Since, by (p_4),

$$
\begin{aligned}
p(y_0, y_k) &\leq p(y_0, y_0) + p(y_1, y_k) - p(y_1, y_1) \\
&\leq p(y_0, y_1) + p(y_1, y_k),
\end{aligned}
$$

we have
$$
diam \ O_{f,g}(x_0, n) \leq p(y_0, y_1) + \lambda \ diam \ O_{f,g}(x_0, n),
$$

i.e., $diam \ O_{f,g}(x_0, n) \leq \frac{1}{1-\lambda} p(y_0, y_1)$. Taking the supremum in this inequality, the proof of Claim 1 is obtained.

Claim 2. Let $m > n \geq 1$. Then

$$
p(y_m, y_n) \leq \frac{\lambda^n}{1 - \lambda} p(y_0, y_1).
\tag{5}
$$

Similarly as in (4), we have that

$$
p(y_m, y_n) \leq \lambda \max\{p(y_{m-1}, y_{n-1}), p(y_{m-1}, y_m), p(y_{n-1}, y_n), p(y_m, y_{n-1}), p(y_{n-1}, y_m)\}.
$$

Since $y_{m-1}, y_m, y_{n-1}, y_n \in O_{f,g}(x_{n-1}, m - n + 1)$, we have

$$
p(y_m, y_n) \leq \lambda O_{f,g}(x_{n-1}, m - n + 1) = \lambda p(y_{n-1}, y_{k_1})
\tag{6}
$$

for some $k_1 \leq m$. Now, similarly,

$$
\begin{aligned}
p(y_{n-1}, y_{k_1}) &\leq \lambda \max\{p(y_{n-2}, y_{k_1-1}), p(y_{n-2}, y_{n-1}), p(y_{k_1-1}, y_{k_1}), p(y_{n-2}, y_{k_1}), p(y_{n-1}, y_{k_1-1})\} \\
&\leq \lambda \ diam \ O_{f,g}(x_{n-2}; m - n + 2),
\end{aligned}
$$

which, together with 6, gives
$$
p(y_m, y_n) \leq \lambda p(y_{n-2}, y_{k_2})
$$

for some $k_2 \leq m$. Continuing the process, we obtain that

$$
\begin{aligned}
p(y_m, y_n) &\leq \lambda^{n-1} \ diam \ O_{f,g}(x_1, m - 1) = \lambda^{n-1} p(y_1, y_{k_{n-1}}) \\
&\leq \lambda^{n-1}. \lambda \ diam \ O_{f,g}(x_0, m) \leq \frac{\lambda^n}{1 - \lambda} p(y_0, y_1)
\end{aligned}
$$

and Claim 2 is proved.

Noncompatibility of f and g implies that there exists some sequence $\{x_n\}$ in X such that $f(x_n) \to u$ and $g(x_n) \to u$ for some $u \in X$, but $\lim_{n \to \infty} p(f(g(x_n)), g(f(x_n))) \neq 0$ or non existent. Since f and g are faintly compatible and $\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} g(x_n) = u$, there exists a sequence $\{z_n\}$ in X satisfying $\lim_{n \to \infty} f(z_n) = \lim_{n \to \infty} g(z_n) = t$ such that $\lim_{n \to \infty} p(f(g(z_n)), g(f(z_n))) = 0$. Further, since f is continuous, then $\lim_{n \to \infty} f(f(z_n)) = f(t)$ and $\lim_{n \to \infty} f(g(z_n)) = f(t)$. The last three limits together imply $\lim_{n \to \infty} g(f(z_n)) = f(t)$. Since $f(X) \subseteq g(X)$ implies that $f(t) = g(w)$ for some $w \in X$ and $f(f(z_n)) \to g(w)$, $g(f(z_n)) \to g(w)$. Also, using (3), we get $p(f(w), f(f(z_n))) \leq \lambda p(g(w), g(f(z_n)))$. On letting $n \to \infty$, we get $f(w) = g(w)$. Thus w is a coincidence point of f and g. Further, faintly compatibility implies $f(g(w)) = g(f(w))$, and hence $f(g(w)) = g(f(w)) = f(f(w)) = g(g(w))$. If $f(w) \neq f(f(w))$, then using (3) we get $p(f(w), f(f(w))) \leq \lambda M_{f,g}^5 = \lambda p(f(w), f(f(w)))$, a contradiction. Hence $f(w)$ is a common fixed point of f and g. The same conclusion is obtained when g is assumed to be continuous since the continuity of g implies the continuity of f. The uniqueness of the common fixed point theorem is an easy consequence of the condition (3). □

We now give an example to illustrate Theorem 2.10.

Example 2.11. *Let $X = [0, 20]$ and let p be the partial metric on X. Define $f, g : X \to X$ as follows:*

$$fx = \frac{x}{10} \ if \ x < 10,$$
$$fx = \frac{20 - x}{10} \ if \ x \geq 10,$$
$$g0 = 0 \ and \ gx = 20 - x \ if \ x > 0.$$

Then f and g satisfy all the conditions of Theorem 2.10 and have a unique common fixed point at $x = 0$. It can be verified in this example that f and g satisfy the condition (3) with $\lambda = \frac{1}{10}$. Furthermore, f and g are faintly compatible. Also, f and g are noncompatible. To see that, let us consider an increasing sequence $\{x_n\} \in X = [2, 20]$ such that $x_n \to 20$. Then $f(x_n) \to 0$, $g(x_n) \to 0$, $f(g(x_n)) \to 0$ and $g(f(x_n)) \to 20$ as $n \to \infty$. Therefore, f and g are noncompatible.

It is well known that the strict contractive condition $p(f(x), f(y)) < M_{f,g}^5$ does not ensure the existence of common fixed points unless the space taken to be compact or some sequence of iterates is assumed to be a 0-Cauchy sequence. The next theorem illustrates the applicability of faintly compatible mappings satisfying the strict contractive condition.

Theorem 2.12. *Let f and g be noncompatible faintly compatible self-mappings of a partial metric space (X, p) also $f(X) \subseteq g(X)$ such that the condition*

$$p(fx, fy) < M_{f,g}^5(x, y) \tag{7}$$

holds for all $x, y \in X$, where

$$M_{f,g}^5(x, y) = \max\{p(gx, gy), p(gx, fx), p(gy, fy), p(gx, fy), p(gy, fx)\}.$$

If either f or g is continuous, then f and g have a unique common fixed point u such that $p(u, u) = 0 = p(fu, gu)$.

Proof. Noncompatibility of f and g implies that there exists some sequence $\{x_n\}$ in X such that $f(x_n) \to u$ and $g(x_n) \to u$ for some $u \in X$, but $\lim_{n \to \infty} p(f(g(x_n)), g(f(x_n))) \neq 0$ or nonexistent. Since f and g are faintly compatible and $\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} g(x_n) = u$, there exists a sequence $\{z_n\}$ in X satisfying $\lim_{n \to \infty} f(z_n) = \lim_{n \to \infty} g(z_n) = t$ such that

$$\lim_{n \to \infty} p(f(g(z_n)), g(f(z_n))) = 0.$$

Further, since f is continuous, then $\lim_{n \to \infty} f(f(z_n)) = f(t)$ and $\lim_{n \to \infty} f(g(z_n)) = f(t)$. The last three limits together imply $\lim_{n \to \infty} g(f(z_n)) = f(t)$. Since $f(X) \subseteq g(X)$ implies that $f(t) = g(w)$ for some $w \in X$ and $f(f(z_n)) \to g(w)$, $g(f(z_n)) \to g(w)$. Also, using $f(X) \subseteq g(X)$ and (7), we get $p(f(w), f(f(z_n))) < p(g(w), g(f(z_n)))$. On letting $n \to \infty$, we get $f(w) = g(w)$. Thus w is a coincidence point of f and g. Further, faintly compatibility implies $f(g(w)) = g(f(w))$, and hence $f(g(w)) = g(f(w)) = f(f(w)) = g(g(w))$. If $f(w) \neq f(f(w))$, then using (7) we get $p(f(w), f(f(w))) < M_{f,g}^5 = p(f(w), f(f(w)))$, a contradiction. Hence $f(w)$ is a common fixed point of f and g. The same conclusion is obtained when g is assumed to be continuous since the continuity of g implies the continuity of f. The uniqueness of the common fixed point theorem is an easy consequence of the condition $f(X) \subseteq g(X)$. $\square$

We now show that the notion of faint compatibility is also useful in studying fixed points of mappings satisfying Lipschitz-type conditions.

Theorem 2.13. *Let f and g be noncompatible faintly compatible self-mappings of a partial metric space (X, p) also $f(X) \subseteq g(X)$. If there exists $\lambda > 0$ such that the condition*

$$p(fx, fy) \leq \lambda p(gx, gy) \tag{8}$$

$$p(fx, f(fx)) \neq \max\{p(fx, g(fx)), p(g(fx), f(fx))\} \tag{9}$$

holds for all $x, y \in X$. If either f or g is continuous, then f and g have a unique common fixed point u such that $p(u, u) = 0 = p(fu, gu)$.

Proof. Noncompatibility of f and g implies that there exists some sequence $\{x_n\}$ in X such that $f(x_n) \to u$ and $g(x_n) \to u$ for some $u \in X$, but $\lim_{n \to \infty} p(f(g(x_n)), g(f(x_n))) \neq 0$ or nonexistent. Since f and g are faintly compatible and $\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} g(x_n) = u$, there exists a sequence $\{z_n\}$ in X satisfying $\lim_{n \to \infty} f(z_n) = \lim_{n \to \infty} g(z_n) = t$ such that $\lim_{n \to \infty} p(f(g(z_n)), g(f(z_n))) = 0$. Further, since f is continuous, then $\lim_{n \to \infty} f(f(z_n)) = f(t)$ and $\lim_{n \to \infty} f(g(z_n)) = f(t)$. The last three limits together imply $\lim_{n \to \infty} g(f(z_n)) = f(t)$. Since $f(X) \subseteq g(X)$ implies that $f(t) = g(w)$ for some $w \in X$ and $f(f(z_n)) \to g(w)$, $g(f(z_n)) \to g(w)$. Also, using $f(X) \subseteq g(X)$ and (8), we get $p(f(w), f(f(z_n))) \leq \lambda p(g(w), g(f(z_n)))$. On letting $n \to \infty$, we get $f(w) = g(w)$. Thus w is a coincidence point of f and g. Further, faintly compatibility implies $f(g(w)) = g(f(w))$, and hence $f(g(w)) = g(f(w)) = f(f(w)) = g(g(w))$. If $f(w) \neq f(f(w))$, then using (9) we get $p(f(w), f(f(w))) \leq \lambda p(f(w), f(f(w)))$, a contradiction. Hence $f(w) = g(w)$ is a common fixed point of f and g. The same conclusion is obtained when g is assumed to be continuous since the continuity of g implies the continuity of f. The uniqueness of the common fixed point theorem is an easy consequence of the condition $f(X) \subseteq g(X)$. $\square$

Example 2.14. *Let $X = [0, 1]$ with the partial metric p. Define mappings $f, g : X \to X$ by*

$$f(x) = \frac{1}{2} - \mid \frac{1}{2} - x \mid \quad and \ \ g(x) = \frac{2(1-x)}{3}.$$

Then f and g satisfy all the conditions of Theorem 2.13 and have a common fixed point $x = \frac{2}{5}$ and a coincidence point $x = 1$ at which f and g do not commute. It may be verified in this example that f and g satisfy the condition (8) for $\lambda = \frac{3}{2}$ together with the condition (9). The mappings f and g are faintly compatible (take a constant sequence $y_n = \frac{2}{5}$) and they commute at the coincidence point at $x = \frac{2}{5}$. Moreover, f and g are noncompatible (consider a sequence $x_n = 1 - \frac{1}{n}$).

In Example 2.14, f and g do not commute at the coincidence point $x = 1$, hence they do not satisfy the condition of weakly compatible mappings. The next example also illustrates Theorem 2.13.

Example 2.15. *Let $X = [0, 1]$ and let p be the partial metric on X. Define self-mappings f on X as in the above Example 2.11 and $S : X \to X$ by*

$$S(x) = \frac{2}{3}$$

fractional part of $(1 - x)$. Then f and g satisfy all the conditions of Theorem 2.13 and have two common fixed points $x = \frac{2}{5}$ and $x = 1$.

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