International Journal of Mathematics And its Applications
Volume 5, Issue 1–B (2017), 165–169.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Estimates on Initial Coefficients of Certain Subclasses of bi-univalent Functions Associated with the Class $\mathcal{P}_m(\beta)$

Research Article

Prem Pratap Vyas[1] and Shashi Kant[1]*

Department of Mathematics, Government Dungar College, Bikaner, India.

Abstract: In this paper, we introduce and investigate certain new subclasses of the function class Σ of bi-univalent function defined in the open unit disk, which are associated with the class $\mathcal{P}_m(\beta)$. We find estimates on the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in these subclasses. Several known and new consequences of these results are also pointed out in the form of corollaries.

MSC: 30C45.

Keywords: Analytic and Univalent functions, Bi-Univalent functions, Coefficient estimates.

1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ that have the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{U}) \tag{1}$$

and let $\mathcal{S}$ be the class of all functions from $\mathcal{A}$ which are univalent in $\mathbb{U}$. The Koebe-one quarter theorem [6] states that the image of $\mathbb{U}$ under every function f from $\mathcal{S}$ contains a disk of radius $\frac{1}{4}$. Thus, every such univalent function has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z \ \ (z \in \mathbb{U}) \ \text{ and } \ f(f^{-1}(w)) = w \ \ (|w| < r_0(f),\ r_0(f) \geq 1/4).$$

In fact,the inverse function f^{-1} is given by

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + ... = g(w). \tag{2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$ and let Σ denotes the class of bi-univalent functions defined in the unit disk $\mathbb{U}$. The class Σ of bi-univalent function was first investigated by Lewin [9] and it was shown that $|a_2| < 1.15$. Brannan and Clunie [2] improved Lewin's result and conjectured that $|a_2| \leq \sqrt{2}$. Later, Netanyahu [10] showed that $max|a_2| = \frac{4}{3}$. Subsequently, Brannan and Taha [3] also introduced certain subclasses of bi-univalent class Σ and obtained estimate for there initial coefficients. Many researchers [1, 4, 5, 12, 14, 15] have recently introduced and investigated several interesting subclass of the bi-univalent function class Σ and they have found non-sharp estimates on the first two Taylor-MacLaurin coefficient $|a_2|$ and $|a_3|$.

* E-mail: drskant.2007@yahoo.com

Definition 1.1 ([11]). *Let $\mathcal{P}_m(\beta)$ with $m \geq 2$ and $0 \leq \beta < 1$ denote the class of univalent analytic functions p, normalized with $p(0)=1$ and satisfying*

$$\int_0^{2\pi} \Big|\frac{Rep(z)-\beta}{1-\beta}\Big| d\theta \leq n\pi,$$

where $z = re^{\iota\theta}$. For $\beta = 0$, we denote $\mathcal{P}_m = \mathcal{P}_m(0)$, hence the class $\mathcal{P}_m$ represents the class of functions $p(z)$, analytic in $\mathbb{U}$, normalized with $p(0)=1$ and having the representation

$$p(z) = \int_0^{2\pi} \frac{1-ze^{\iota t}}{1+ze^{\iota t}} d\mu(t),$$

where μ is a real-valued function with bounded variation which satisfies

$$\int_0^{2\pi} d\mu(t) = 2\pi \quad \text{and} \quad \int_0^{2\pi} |d\mu(t)| \leq m, \quad m \geq 2.$$

Note that $\mathcal{P} = \mathcal{P}_2$ is the well known class of caratheodory function. i.e. the normalized functions with positive real part in the open unit disc $\mathbb{U}$. Motivated by the earlier work of Bulboaca *et al.* [4], Altinkaya *et al.* [1], Goswami *et al.* [8], Peng *et al.* [13], Deniz [5], we introduce here certain new subclasses of the function class Σ of complex order $\gamma \in \mathbb{C}\backslash\{0\}$ associated with the class $\mathcal{P}_m(\beta)$ and find estimates on the coefficients $|a_2|$ and $|a_3|$ for the function that belong to these new subclasses.

Definition 1.2. *For $\alpha \geq 1, \gamma \geq 0$ and $0 \leq \beta < 1$, a function $f \in \Sigma$ given by (1) is said to be in class $\mathcal{R}_\Sigma(\tau, \alpha, \gamma; \beta)$, if the following two conditions are satisfied:*

$$\begin{aligned} 1 + \frac{1}{\tau}[(1-\alpha)\frac{f(z)}{z} + \alpha f^{'}(z) + \gamma z f^{''}(z) - 1] \in \mathcal{P}_m(\beta) \ \ \text{and} \\ 1 + \frac{1}{\tau}[(1-\alpha)\frac{g(w)}{w} + \alpha g^{'}(w) + \gamma w g^{''}(w) - 1] \in \mathcal{P}_m(\beta), \end{aligned} \tag{3}$$

where $\tau \in \mathbb{C}\backslash\{0\}$, the function $g = f^{-1}$ is given by (2), and $z, w \in \mathbb{U}$.

Definition 1.3. *For $0 \leq \lambda \leq 1$ and $0 \leq \beta < 1$, a function $f \in \Sigma$ given by (1) is said to be in class $\mathcal{S}_\Sigma(\lambda, \tau; \beta)$,if the following two conditions are satisfied:*

$$\begin{aligned} 1 + \frac{1}{\tau}\Big[\frac{zf^{'}(z) + \lambda z^2 f''(z)}{\lambda z f'(z) + (1-\lambda) f(z)} - 1\Big] \in \mathcal{P}_m(\beta) \ \ \text{and} \\ 1 + \frac{1}{\tau}\Big[\frac{wg^{'}(w) + \lambda w^2 g''(w)}{\lambda w g'(w) + (1-\lambda) g(w)} - 1\Big] \in \mathcal{P}_m(\beta), \end{aligned} \tag{4}$$

where $\tau \in \mathbb{C}\backslash\{0\}$, the function $g = f^{-1}$ is given by (2), and $z, w \in \mathbb{U}$.

In order to derive our main results, we shall need the following lemma:

Lemma 1.4. *Let the function $\phi(z) = 1 + h_1 z + h_2 z^2 + \ldots; z \in \mathbb{U}$ such that $\phi \in \mathcal{P}_m(\beta)$ then, $|h_n| \leq m(1-\beta)$; $n \geq 1$.*

2. Main Results

We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions belonging to the class $\mathcal{R}_\Sigma(\tau, \alpha, \gamma; \beta)$. Supposing that the functions $p, q \in \mathcal{P}_m(\beta)$, with

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k \quad (z \in \mathbb{U}), \quad \text{and} \tag{5}$$

$$q(z) = 1 + \sum_{k=1}^{\infty} q_k z^k \quad (z \in \mathbb{U}). \tag{6}$$

From Lemma 1.4 it follows that

$$|p_k| \le m(1-\beta), \quad \text{and} \tag{7}$$

$$|q_k| \le m(1-\beta) \quad \textit{for all} \;\; k \ge 1. \tag{8}$$

Theorem 2.1. *If $f \in \mathcal{R}_\Sigma(\tau, \alpha, \gamma; \beta)$, then*

$$|a_2| \le \;\; min\left\{ \sqrt{\frac{m|\tau|(1-\beta)}{1+2\alpha+6\gamma}}, \frac{m|\tau|(1-\beta)}{1+\alpha+2\gamma} \right\} \;\; and \tag{9}$$

$$|a_3| \le \;\; \frac{m|\tau|(1-\beta)}{1+2\alpha+6\gamma}. \tag{10}$$

Proof. Since $f \in \mathcal{R}_\Sigma(\tau, \alpha, \gamma; \beta)$, from the definition relations (3) it follows that

$$1 + \frac{1}{\tau}[(1-\alpha)\frac{f(z)}{z} + \alpha f^{'}(z) + \gamma z f^{''}(z) - 1] =$$
$$1 + \frac{1}{\tau}[(1+\alpha+2\gamma)a_2 z + (1+2\alpha+6\gamma)a_3 z^2 + ...] = p(z) \tag{11}$$

and

$$1 + \frac{1}{\tau}[(1-\alpha)\frac{g(w)}{w} + \alpha g^{'}(w) + \gamma w g^{''}(w) - 1] =$$
$$1 + \frac{1}{\tau}[-(1+\alpha+2\gamma)a_2 w + (1+2\alpha+6\gamma)(2a_2^2 - a_3)w^2 + ...] = q(w) \tag{12}$$

where $p, q \in \mathcal{P}_m(\beta)$, and are of the form (5) and (6), respectively. Now equating the coefficients of z and z^2 in (11), we get

$$\frac{1}{\tau}(1+\alpha+2\gamma)a_2 = p_1 \tag{13}$$

$$\frac{1}{\tau}(1+2\alpha+6\gamma)a_3 = p_2. \tag{14}$$

Similarly from (12), we have

$$-\frac{1}{\tau}(1+\alpha+2\gamma)a_2 = q_1 \tag{15}$$

$$\frac{1}{\tau}(1+2\alpha+6\gamma)(2a_2^2 - a_3) = q_2. \tag{16}$$

From (13) and (15), it follows that

$$a_2 = \frac{\tau p_1}{1+\alpha+2\gamma} = \frac{-\tau q_1}{1+\alpha+2\gamma} \tag{17}$$

and (14), (16) yields

$$a_2^2 = \frac{\tau(p_2+q_2)}{2(1+2\alpha+6\gamma)}. \tag{18}$$

Now, (17) and (18) gives the bound on $|a_2|$ as asserted in (9), by applying Lemma 1.4. Further computation (13) to (18) leads to

$$a_3 = \frac{\tau p_2}{(1+2\alpha+6\gamma)}, \tag{19}$$

thus we obtain the bound on $|a_3|$ as asserted in (10), by applying Lemma 1.4. □

For $\alpha = 1$, $\gamma = 0$ and $\alpha = 1$, $\gamma = 1$ the above Theorem 2.1 reduces in the following corollaries respectively:

Corollary 2.2. *If* $1+\frac{1}{\tau}[f^{'}(z)-1] \in \mathcal{P}_m(\beta)$ *and* $1+\frac{1}{\tau}[g^{'}(w)-1] \in \mathcal{P}_m(\beta)$, *then* $|a_2| \leq min\left\{\sqrt{\frac{m|\tau|(1-\beta)}{3}}, \frac{m|\tau|(1-\beta)}{2}\right\}$ *and* $|a_3| \leq \frac{m|\tau|(1-\beta)}{3}$.

Corollary 2.3. *If* $1+\frac{1}{\tau}[f^{'}(z)+zf^{''}(z)-1] \in \mathcal{P}_m(\beta)$ *and* $1+\frac{1}{\tau}[g^{'}(w)+wg^{''}(w)-1] \in \mathcal{P}_m(\beta)$, *then* $|a_2| \leq min\left\{\frac{\sqrt{m|\tau|(1-\beta)}}{3}, \frac{m|\tau|(1-\beta)}{4}\right\}$ *and* $|a_3| \leq \frac{m|\tau|(1-\beta)}{9}$.

Remark 2.1.

(1). Putting $\tau=1, \gamma=0$ *and m=2 in Theorem 2.1, we obtain improvement result corresponding to the result given in Theorem 3.2 by Frasin and Aouf [7].*

(2). Putting $\tau=1$, *and m=2 in corollary 2.1, we get result given in Theorem 2 by Srivastava et. al [14].*

Theorem 2.4. *If* $f \in \mathcal{S}_\Sigma(\lambda, \tau; \beta)$, *then*

$$|a_2| \leq \; min\left\{\sqrt{\frac{m|\tau|(1-\beta)}{1+2\lambda-\lambda^2}}, \frac{m|\tau|(1-\beta)}{1+\lambda}\right\} \quad and \tag{20}$$

$$|a_3| \leq \frac{m|\tau|(1-\beta)}{1+2\lambda-\lambda^2}. \tag{21}$$

Proof. Since $f \in \mathcal{S}_\Sigma(\lambda, \tau; \beta)$, from the definition relations (4) it follows that

$$1+\frac{1}{\tau}\Big[\frac{zf^{'}(z)+\lambda z^2 f''(z)}{\lambda z f'(z)+(1-\lambda)f(z)}-1\Big] =$$

$$1+\frac{1}{\tau}[(1+\lambda)a_2 z+\{2(1+2\lambda)a_3-(1+\lambda)^2 a_2^2\}z^2-...] = p(z) \tag{22}$$

and

$$1+\frac{1}{\tau}\Big[\frac{wg^{'}(w)+\lambda w^2 g''(w)}{\lambda w g'(w)+(1-\lambda)g(w)}-1\Big] =$$

$$1+\frac{1}{\tau}[-(1+\lambda)a_2 w+\{2(1+2\lambda)(2a_2^2-a_3)-(1+\lambda)^2 a_2^2\}w^2+...] = q(w) \tag{23}$$

where $p, q \in \mathcal{P}_m(\beta)$, and are of the form (5) and (6), respectively. Now equating the coefficient of z and z^2 in (2.18), we get

$$\frac{1}{\tau}(1+\lambda)a_2 = p_1 \tag{24}$$

$$\frac{1}{\tau}\{2(1+2\lambda)a_3-(1+\lambda)^2 a_2^2\} = p_2. \tag{25}$$

Similarly for (23), gives

$$\frac{1}{\tau}\{-(1+\lambda)a_2\} = q_1 \tag{26}$$

and

$$\frac{1}{\tau}\{2(1+2\lambda)(2a_2^2-a_3)-(1+\lambda)^2 a_2^2\} = q_2. \tag{27}$$

From (24) and (26), it follows that

$$a_2 = \frac{\tau p_1}{1+\lambda} = \frac{-\tau q_1}{1+\lambda} \tag{28}$$

and (25), (27) yields

$$a_2^2 = \frac{(p_2+q_2)\tau}{2(1+2\lambda-\lambda^2)}. \tag{29}$$

we readily get the estimate given in (20) by applying Lemma 1.1. Now further computation (24) to (29) leads to

$$a_3 = \frac{\tau}{4(1+\lambda)}[(3+6\lambda-\lambda^2)p_2+(1+2\lambda-\lambda^2)q_2] \tag{30}$$

and thus we obtain the bound on $|a_3|$ as asserted in (21), by applying Lemma 1.1. □

For $\lambda = 0$ and $\lambda = 1$, the above Theorem 2.2 reduces in the following corollaries respectively:

Corollary 2.5. *If* $1 + \frac{1}{\tau}\left[\frac{zf'(z)}{f(z)} - 1\right] \in \mathcal{P}_m(\beta)$ *and* $1 + \frac{1}{\tau}\left[\frac{wg'(w)}{w(w)} - 1\right] \in \mathcal{P}_m(\beta)$, *then* $|a_2| \leq min\left\{\sqrt{m|\tau|(1-\beta)},\ m|\tau|(1-\beta)\right\}$ *and* $|a_3| \leq m|\tau|(1-\beta)$.

Corollary 2.6. *If* $1+\frac{1}{\tau}\left[1+\frac{zf''(z)}{f'(z)}\right] \in \mathcal{P}_m(\beta)$ *and* $1+\frac{1}{\tau}\left[1+\frac{wg''(w)}{g'(w)}\right] \in \mathcal{P}_m(\beta)$, *then* $|a_2| \leq min\left\{\sqrt{\frac{m|\tau|(1-\beta)}{2}}, \frac{m|\tau|(1-\beta)}{2}\right\}$ *and* $|a_3| \leq \frac{m|\tau|(1-\beta)}{2}$.

Remark 2.7.

(1). Putting $\tau = 1$ *and m=2 in corollary 2.3, we obtain the improvement of result corresponding result given by Brannan and Taha [3].*

(2). Putting $\tau = (1-\delta)e^{-\iota\lambda}cos\lambda$, *m=2 and* $\beta = 0$ *in corollary 2.3, we get the result for bi-λ-spirallike univalent function of order* δ.

References

[1] S.Altinkaya and S.Yalcin, *Coefficient problem for certain subclasses of bi-univalent functions defined by convolution*, Mathematica Moravica, 20(2)(2016), 15-21.

[2] D.A.Brannan and J.G.Clunie, *Aspects of Contemporary Complex Analysis*, Academic Press London, (1980).

[3] D.A.Brannan and T.S.Taha, *On some classes of bi-univalent functions*, Studia Univ. Babes-Bolyai Math., 31(2)(1986), 70-77.

[4] T.Bulboaca and G.Murugusundaramoorthy, *Estimate for initial MacLaurin coefficients of certain subclasses of bi-univalent functions of complex order associated with the Hohlov operater*, arXiv:1607.08285v1 [math.CV], (2016).

[5] E.Deniz, *Certain subclasses of bi-univalent functions satisfying subordinate conditions*, J. Class. Anal., 2(1)(2013), 49-60.

[6] P.L.Duren, *Univalent functions, Grundlehren der mathematischen Wissenschaften* , 259, SpringerVerlag, New York, Berlin, Heidelbery and Tokyo, (1983).

[7] B.A.Frasin and M.K.Aouf, *New subclasses of bi-univalent functions*, Appl. Math. Lett., 24(2011), 1569-1573.

[8] P.Goswami, B.S.Alkahtani and T.Bulboaca, *Estimate for initial MacLaurin coefficients of certain sub-classes of bi-univalent functions*, arXiv:1503.04644v1 [math.CV], (2015).

[9] M.Lewin, *On a coefficient problem for bi-univalent functions*, Proc. Amer. Math. Soc., 18(1967), 63-68.

[10] E.Netanyahu, *The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in* $|z| < 1$, Arch. Ration. Mech. Anal., 32(1969), 100-112.

[11] K.Padmanabhan and R.Parvatham, *Properties of a class of functions with bounded boundary rotation*, Ann. Polon. Math., 31(1975), 311-323.

[12] T.Panigarhi and G.Murugusundaramoorthy, *Coefficient bounds for bi-univalent functions analytic functions associated with Hohlov operator*, Proc. Jangjeon Math. Soc., 16(1)(2013), 91-100.

[13] G.Peng, G.Murugusundaramoorthy and T.Janani, *Coefficient estimate of bi-univalent functions of complex order associated with the Hohlov operator*, J. Complex Anal., 2014(2014), Article ID 693908.

[14] H.M.Srivastava, A.K.Mishra and P.Gochhayat, *Certain subclasses of analytic and bi-univalent functions*, Appl. Math. Lett., 23(2010), 1188-1192.

[15] H.M.Srivastava, G.Murugusundaramoorthy and N.Magesh, *Certain subclasses of bi-univalent functions associated with the Hohlov operator*, Global Journal of Mathematical Analysis, 1(2)(2013), 67-73.