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The Henstock-Stieltjes Integral for n-dimensional Fuzzy-Number-Valued Functions

Research Article

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Abstract: In this paper, the concept of (FHS) integral for n-dimensional fuzzy-number-valued functions is presented, several necessary and sufficient conditions of integrability for n-dimensional fuzzy-number-valued functions are given by means of this concept.

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1. Introduction

Since the concept of fuzzy sets was firstly introduced by Zadeh in 1965 [23], it has been studied extensively from many different aspects of the theory and applications, such as fuzzy topology, fuzzy analysis, fuzzy decision making and fuzzy logic, information science and so on. It's well known that the concept of the Stieltjes integral for fuzzy-number-valued functions was originally introduced by Nanda [12] in 1989. Nonetheless, as Wu et al. [18] pointed out that the existence of supremum and infimum for a finite set of fuzzy numbers wasn't easy at first thought. That is, Nanda's concept of fuzzy Riemann-Stieltjes (FRS) integral in [12] was incorrect. In 1998, Wu [17] introduced the notion of (FRS) integral by means of the representation theorem of fuzzy-number-valued functions, whose membership function could be obtained by solving a nonlinear programming problem, but it's difficult to calculate and extend to the higher-dimensional space. In 2006, Ren et al. proposed the notion of two types of (FRS) integral for fuzzy-number-valued functions [14, 15] and showed that a continuous fuzzy-number-valued function was (FRS) integrable with respect to a real-valued increasing function. Gong et al. defined and discussed the (HS) integral for fuzzy-number-valued functions and proved two convergence theorems for sequences of the (FHS) integrable functions in 2012 [4]. In this paper, the (HS) integral for n-dimensional fuzzy-number-valued functions is defined and some basic properties of this integral are discussed. The paper is organized as follows, in Section 2, we shall review the relevant concepts and properties of fuzzy-number-valued functions in E^n. Section 3 is devoted to discussing (HS) integral for n-dimensional fuzzy-number-valued functions. In Section 4, we introduce some linearity properties of (HS) integrability for n-dimensional fuzzy-number-valued functions. Section 5 proposes the characterization theorems of (FHS) integral for n-dimensional fuzzy-number-valued functions. The final section provides Conclusions.

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2. Preliminaries

Definition 2.1 ([7, 9]). *Let $\delta : [a,b] \to \mathbb{R}^+$ be a positive real-valued function. $P = \{[x_{i-1}, x_i]; \xi_i\}$ is said to be a δ-fine division, if the following conditions are satisfied:*

(1). $a = x_0 < x_1 < x_2 < ... < x_n = b$;

(2). $\xi_i \in [x_{i-1}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))(i = 1, 2, \cdots, n)$.

For brevity, we write $P = \{[u,v]; \xi\}$, where $[u,v]$ denotes a typical interval in P and ξ is the associated point of $[u,v]$.

Definition 2.2 ([16]). *A fuzzy-number-valued function $\widetilde{F} : [a,b] \to E^n$ is said to be Henstock (H) integrable to $\widetilde{H} \in E^n$ if for every $\varepsilon > 0$, there is a function $\delta(t) > 0$ such that for any δ-fine division $P = \{[u,v]; \xi\}$ of $[a,b]$, we have*

$$D\big(\sum \widetilde{F}(\xi)(v-u), \widetilde{H}\big) < \varepsilon, \tag{1}$$

where the sum $\sum$ is understood to be over P and we write $(FH)\int_a^b \widetilde{F}(t)dt = \widetilde{H}$, and $\widetilde{F}(t) \in FH[a,b]$.

Definition 2.3 ([4]). *Let $\alpha : [a,b] \to \boldsymbol{R}$ be an increasing function. A fuzzy-number-valued function $\tilde{f}(t)$ is said to be fuzzy Henstock-Stieltjes (FHS) integrable with respect to α on $[a,b]$ if there exists a fuzzy number $\widetilde{H} \in E^1$ such that for every $\varepsilon > 0$, there is a function $\delta(t) > 0$ such that for any δ-fine division $P = \{[u_i, v_i]; \xi_i\}_{i=1}^n$ we have*

$$D\big(\sum_{i=1}^n \tilde{f}(\xi_i)[\alpha(v_i) - \alpha(u_i)], \widetilde{H}\big) < \varepsilon. \tag{2}$$

We write $(FHS)\int_a^b \tilde{f}(t)d\alpha = \widetilde{H}$, and $(\tilde{f}, \alpha) \in FHS[a,b]$.

The symbol $P_k(R^n)$ denotes the family of all nonempty compact convex subsets of R^n, define the addition and scalar multiplication in $P_k(R^n)$ as following, for $A, B \in P_k(R^n)$, $a \in R$,

$$A + B = \{x + y \mid x \in A,\ y \in B\}, \qquad aA = \{ax \mid x \in A\}.$$

For every $A, B \in P_k(R^n)$, define the Hausdorff metric of A and B by the equation [19]

$$d(A,B) = \max\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|b-a\|\}. \tag{3}$$

Definition 2.4 ([5, 20]). *X is a Banach space. Let $z : [a,b] \to X$ be a vector-valued function. $z(t)$ is said to be abstract (H) integrable to z_0 on $[a,b]$ if for every $\varepsilon > 0$ there is a function $\delta(\xi) > 0$ such that for any δ-fine division $P = \{[u,v]; \xi\}$ of $[a,b]$ we have*

$$\big\|\sum z(\xi)(v-u) - z_0\big\| < \varepsilon, \tag{4}$$

where the sum $\sum$ is understood to be over P and $\|\cdot\|$ stands for the norm of X. We write $(VH)\int_a^b z(t)dt = z_0$ and $z(t) \in VH[a,b]$. Here (VH) stands for the (H) integral for vector-valued functions.

Definition 2.5 ([6]). *For $A \in P_k(R^n)$, $x \in S^{n-1}$, define the support function of A as $\sigma(x,A) = \sup_{y\in A}\langle y, x\rangle$, where S^{n-1} is the unit sphere of R^n, i.e., $S^{n-1} = \{x \in R^n : \|x\| = 1\}$, $\langle\cdot,\cdot\rangle$ is the inner product in R^n.*

Definition 2.6 ([19]). *E^n is said to be a fuzzy number space if $E^n = \{u :\ R^n \to [0,1] : u$ satisfies (1)-(4) below$\}$:*

(1). u is normal, i.e., there exists an $x_0 \in R^n$ such that $u(x_0) = 1$;

(2). u is a convex fuzzy set, i.e., $u(rx + (1-r)y) \geqslant \min(u(x), u(y))$, $x, y \in R^n$, $r \in [0,1]$;

(3). u is upper semi-continuous;

(4). $[u]^0 = \overline{\{x \in R^n : u(x) > 0\}}$ is compact, for $0 < r \leq 1$, denote $[u]^r = \{x : x \in R^n$ and $u(x) \geqslant r\}$, $[u]^0 = \overline{\bigcup_{r\in(0,1]}[u]^r}$.

Lemma 2.7 ([19]). *If $u, v \in E^n$, $k \in R$, for any $r \in [0,1]$, we have*

$$[u+v]^r = [u]^r + [v]^r,\ [ku]^r = k[u]^r. \tag{5}$$

Lemma 2.8 ((fuzzy number representation theorem) [8, 13, 19]). *If $u \in E^n$, then*

(1). $[u]^r$ is a nonempty compact convex subset of R^n, then $[u]^r \in P_k(R^n)$ for any $r \in [0,1]$;

(2). $[u]^{r_2} \subseteq [u]^{r_1}$, whenever $0 \leq r_1 \leq r_2 \leq 1$;

(3). If $\{r_m\}$ is a nondecreasing sequence converging to $r \in (0,1]$, then $\bigcap\limits_{m=1}^{\infty}[u]^{r_m} = [u]^r$.

Conversely, if $\{[A]^r \subseteq R^n : r \in [0,1]\}$ satisfying the above (1)-(3), then there exists a unique $u \in E^n$ such that $[u]^r = [A]^r$, $r \in (0,1]$, $[u]^0 = \overline{\bigcup_{r\in(0,1]}[u]^r} \subseteq A^0$.

Lemma 2.9 ([2, 19]). *Given $u, v \in E^n$ the distance $D : E^n \times E^n \to [0, +\infty)$ between u and v is defined by the equation $D(u,v) = \sup\limits_{r\in[0,1]} d([u]^r, [v]^r)$, then*

(1). (E^n, D) is a complete metric space;

(2). $D(u+w, v+w) = D(u,v)$;

(3). $D(u+v, w+e) \leqslant D(u,w) + D(v,e)$;

(4). $D(ku, kv) = |k|D(u,v), k \in R$;

(5). $D(u+v, \tilde{0}) \leqslant D(u, \tilde{0}) + D(v, \tilde{0})$;

(6). $D(u+v, w) \leqslant D(u,w) + D(v, \tilde{0})$.

Where $u, v, w, e, \widetilde{0} \in E^n$, $\widetilde{0} = \mathcal{X}_{(\{0\})}$.

Let S^{n-1} be the unit sphere of R^n, i.e., $S^{n-1} = \{x \in R^n |\ \|x\| = 1\}$, here $\|\cdot\|$ denote the standard norm of the Euclidean space R^n, $\langle\cdot,\cdot\rangle$ be the inner product in R^n, i.e., $\langle x, y\rangle = \sum\limits_{i=1}^{n} x_i y_i$ for any $x = (x_1, x_2, \cdots, x_n) \in R^n$ and $y = (y_1, y_2, \cdots, y_n) \in R^n$. Then for any $r \in [0,1]$ and $x \in S^{n-1}$, the support function of u is defined by

$$u^*(r,x) = \sup_{a\in[u]^r} \langle a, x\rangle. \tag{6}$$

Lemma 2.10 ([1, 11, 19, 22]). *Suppose $u \in E^n$, then*

(1). $u^*(r, x+y) \leq u^*(r,x) + u^*(r,y)$;

(2). $u^*(r,x) \leq \sup_{a\in[u]^r} \| a \|$, i.e., $u^*(r,x)$ is bounded on S^{n-1} for each fixed $r \in [0,1]$;

(3). $u^*(r,x)$ is nonincreasing and left continuous in $r \in [0,1]$, right continuous at $r = 0$, for each fixed $x \in S^{n-1}$;

(4). $u^*(r,x)$ *is Lipschitz continuous in* x

$$|u^*(r,x) - u^*(r,y)| \le (\sup_{a\in[u]^r} \|a\|)\|x-y\|;$$

(5). if $u, v \in E^n$, $r \in [0,1]$, *then*

$$d([u]^r,[v]^r) = \sup_{x\in S^{n-1}} |u^*(r,x) - v^*(r,x)|;$$

(6). $(u+v)^*(r,x) = u^*(r,x) + v^*(r,x)$;

(7). $(ku)^*(r,x) = ku^*(r,x)$, $k \ge 0$;

(8). $-u^*(r,-x) \le u^*(r,x)$.

Lemma 2.11 ([21]). *Let* $A_r \in P_k(R^n), \{A_{r_m}\} \subset P_k(R^n)$, r_m *is nondecreasing convergent to* r, $A_{r_m} \supset A_{r_{m+1}} \supset A_r$ $(m = 1,2,\cdots)$, *if* $\sigma(x, A_{r_m})$ *convergent to* $\sigma(x, A_r)$ *for any* $x \in S^{n-1}$, *then* $A_r = \bigcap\limits_{m=1}^{\infty} A_{r_m}$.

Lemma 2.12 ([3]). *If* $A, B \in P_k(R^n)$, *then* $d(A,B) = \sup\limits_{x\in S^{n-1}} \big|\sigma(x,A) - \sigma(x,B)\big|$.

3. The Henstock-Stieltjes Integral for n-dimensional Fuzzy-Number-Valued Functions

In this section, we shall give the definition of the Henstock-Stieltjes (HS) integral for n-dimensional fuzzy-number-valued functions and investigate some of their properties.

Definition 3.1. *Let* $\alpha : [a,b] \to \boldsymbol{R}$ *be an increasing function. A fuzzy-number-valued function* $\widetilde{F} : [a,b] \to E^n$ *is said to be fuzzy Henstock-Stieltjes* (FHS) *integrable with respect to* α *on* $[a,b]$, *if there exists* $\widetilde{A} \in E^n$, *for every* $\varepsilon > 0$, *there is a function* $\delta(t) > 0$, *such that for any* δ*-fine division* $P = \{[u,v],\xi\}$ *of* $[a,b]$, *we have*

$$D\Big(\sum_{(P)} \widetilde{F}(\xi)[\alpha(v) - \alpha(u)], \widetilde{A}\ \Big) < \varepsilon. \tag{7}$$

We write $(FHS)\int\limits_a^b \widetilde{F}(t)d\alpha = \widetilde{A}$.

Remark 3.2. *Let* $\alpha : [a,b] \to \boldsymbol{R}$ *be an increasing function. If a fuzzy-number-valued function* $\widetilde{F} : [a,b] \to E^n$ *is* (FHS) *integrable with respect to* α *on* $[a,b]$, *then there exists a unique integral value.*

Theorem 3.3. *Let* $\alpha : [a,b] \to \boldsymbol{R}$ *be an increasing function. A fuzzy-number-valued function* $\widetilde{F} : [a,b] \to E^n$ *is* (FHS) *integrable with respect to* α *on* $[a,b]$ *if and only if* $F^*(t)(r,x)$ *is* (RHS) *integrable with respect to* α *on* $[a,b]$ *uniformly for any* $r \in [0,1]$ *and* $x \in S^{n-1}$, *we have*

$$\left((FHS)\int_a^b \widetilde{F}(t)d\alpha\right)^*(r,x) = (RHS)\int_a^b F^*(t)(r,x)d\alpha. \tag{8}$$

Uniformly for any $r \in [0,1]$.

Proof. Since the real valued function $F^*(t)(r,x)$ is (RHS) integrable with respect to α on $[a,b]$ uniformly for any $r\in[0,1]$ and $x\in S^{n-1}$, then there is $a(r,x)\in R$, for every $\varepsilon>0$, there is a function $\delta(t)>0$ (independently to r), such that for any δ-fine division $P=\{[u,v],\xi\}$ of $[a,b]$, we have

$$\Big|\sum_{(P)} F^*(\xi)(r,x)[\alpha(v)-\alpha(u)]-a(r,x)\Big|<\varepsilon. \tag{9}$$

Uniformly for any $r\in[0,1]$.

We can proof the set $\{y\in R^n|\langle y,x\rangle\le (RHS)\int_a^b F^*(t)(r,x)\mathrm{d}\alpha,\ x\in S^{n-1}\}=\{y\in R^n|\langle y,x\rangle\le a(r,x),\ x\in S^{n-1}\}$ satisfy the condition of (the fuzzy number representation theorem) Lemma 2.8 for any $r\in[0,1]$, that is, it determines a unique of fuzzy number $\widetilde{A}\in E^n$.

(a) The set $\{y\in R^n|\langle y,x\rangle\le a(r,x),\ x\in S^{n-1}\}$ is nonempty set. Let $b,c\in\{y\in R^n|\langle y,x\rangle\le a(r,x),\ x\in S^{n-1}\}$, then for any $x\in S^{n-1}$, we have

$$\langle b,x\rangle\le a(r,x),\qquad \langle c,x\rangle\le a(r,x). \tag{10}$$

So for any $s\in[0,1]$, $\langle sb+(1-s)c,x\rangle=s\langle b,x\rangle+(1-s)\langle c,x\rangle\le a(r,x)$, such that $\{y\in R^n|\langle y,x\rangle\le a(r,x),\ x\in S^{n-1}\}$ is a convex subsets of R^n. Let $\{a_m\}_{m=1}^{\infty}\subset\{y\in R^n|\langle y,x\rangle\le a(r,x),\ x\in S^{n-1}\}$, then for each $x\in S^{n-1}$, $\langle a_m,x\rangle\le a(r,x)=(RHS)\int_a^b F^*(t)(r,x)\mathrm{d}\alpha\le(RHS)\int_a^b F^*(t)(0,x)\mathrm{d}\alpha$, i.e. $\langle a_m,x\rangle$ is bounded. So for any $x\in S^{n-1}$, $\langle a_m,x\rangle$ there exist convergent subsequence $\langle a_{m_k},x\rangle$. Let $\lim_{k\to\infty}\langle a_{m_k},x\rangle=b_x$, then $b_x\le a(r,x)$, apparently,

$$\langle\lim_{k\to\infty}a_{m_k},x\rangle=\lim_{k\to\infty}\langle a_{m_k},x\rangle=b_x\le a(r,x), \tag{11}$$

i.e. $\lim_{k\to\infty}a_{m_k}\in\{y\in R^n|\langle y,x\rangle\le a(r,x),\ x\in S^{n-1}\}$. So, $\{y\in R^n|\langle y,x\rangle\le a(r,x),\ x\in S^{n-1}\}$ is a compact set on R^n.

(b) Since $\widetilde{F}\in E^n$ and by using Lemma 2.8 (2), for any $r_1,r_2\in[0,1]$, $F_{r_1}(t)\supset F_{r_2}(t)$, whenever $0\le r_1\le r_2\le 1$. Then for any $x\in S^{n-1}$, we have

$$F^*(t)(r_1,x)=\sup_{b\in F_{r_1}(t)}\langle b,x\rangle\ge\sup_{c\in F_{r_2}(t)}\langle c,x\rangle=F^*(t)(r_2,x),$$

so, $(RHS)\int_a^b F^*(t)(r_1,x)\mathrm{d}\alpha\ge(RHS)\int_a^b F^*(t)(r_2,x)\mathrm{d}\alpha$, i.e. $a(r_1,x)\ge a(r_2,x)$. Thus

$$\{y\in R^n|\langle y,x\rangle\le a(r_1,x),\ x\in S^{n-1}\}\supset\{y\in R^n|\langle y,x\rangle\le a(r_2,x),\ x\in S^{n-1}\}.$$

(c) Since $\widetilde{F}\in E^n$ and by using Lemma 2.10 (3), if a positive sequence $\{r_m\}$ is nondecreasing convergent to r, for any $r\in(0,1]$, we have

$$\lim_{m\to\infty}F^*(t)(r_m,x)=F^*(t)(r,x). \tag{12}$$

By using Lemma 2.10 (2), for any $x\in S^{n-1}$,

$$|F^*(t)(r_m,x)|\le\sup_{a\in F_0(t)}\|a\|. \tag{13}$$

By the (RH) integral-dominated convergence theorem [9, 10], for any $x\in S^{n-1}$, we have

$$\lim_{m\to\infty}(RHS)\int_a^b F^*(t)(r_m,x)\mathrm{d}\alpha=(RHS)\int_a^b F^*(t)(r,x)\mathrm{d}\alpha, \tag{14}$$

i.e. $\lim_{m\to\infty} a(r_m,x)=a(r,x)$. Let $M_{r_m}=\{y\in R^n|\langle y,x\rangle\le a(r_m,x),\ x\in S^{n-1}\}$, $M_r=\{y\in R^n|\langle y,x\rangle\le a(r,x),\ x\in S^{n-1}\}$. Combining (a), (b) we have M_{r_m} and M_r is nonempty compact convex sets on R^n and there $M_{r_m}\supset M_{r_{m+1}}\supset M_r$ $(m=1,2,\cdots)$. Then by M_{r_m} and M_r is a compact, obviously we have

$$\sigma(x,M_{r_m})=\sup_{y\in M_{r_m}}\langle y,x\rangle=a(r_m,x),\qquad \sigma(x,M_r)=\sup_{y\in M_r}\langle y,x\rangle=a(r,x),$$

it is clear M_{r_m} and M_r satisfy the condition of the Lemma 2.11, therefore we have $\bigcap_{m=1}^{\infty} M_{r_m}=M_r$, such that

$$\bigcap_{m=1}^{\infty}\{y\in R^n|\langle y,x\rangle\le a(r_m,x),\ x\in S^{n-1}\}=\{y\in R^n|\langle y,x\rangle\le a(r,x),\ x\in S^{n-1}\}.$$

Summary, the set $\{y\in R^n|\langle y,x\rangle\le a(r,x),x\in S^{n-1}\}$ it determines a unique of fuzzy number, denoted by $\widetilde{A}$, it is clear $A^*(r,x)=a(r,x)$. Then by Lemma 2.9 and Lemma 2.10 (5), (6), (7) it follows that

$$\begin{aligned}
D\Big(\sum_{(P)}\widetilde{F}(\xi)[\alpha(v)-\alpha(u)],\widetilde{A}\Big) &= \sup_{r\in[0,1]} d\Big(\Big[\sum_{(P)}\widetilde{F}(\xi)[\alpha(v)-\alpha(u)]\Big]^r,[\widetilde{A}]^r\Big)\\
&= \sup_{r\in[0,1]}\sup_{x\in S^{n-1}}\Big|\Big(\sum_{(P)}\widetilde{F}(\xi)[\alpha(v)-\alpha(u)]\Big)^*(r,x)-A^*(r,x)\Big|\\
&= \sup_{r\in[0,1]}\sup_{x\in S^{n-1}}\Big|\sum_{(P)}F^*(\xi)(r,x)[\alpha(v)-\alpha(u)]-a(r,x)\Big|\\
&= \sup_{x\in S^{n-1}}\sup_{r\in[0,1]}\Big|\sum_{(P)}F^*(\xi)(r,x)[\alpha(v)-\alpha(u)]-a(r,x)\Big|\le\varepsilon. \qquad (15)
\end{aligned}$$

□

Theorem 3.4. *Let $\alpha:[a,b]\to\boldsymbol{R}$ be an increasing function such that $\alpha\in C^1[a,b]$ and a fuzzy-number-valued function $\widetilde{F}(t)=\widetilde{0}$ almost everywhere on $[a,b]$, then $(\widetilde{F},\alpha)\in FHS[a,b]$ and $\int_a^b\widetilde{F}(t)d\alpha=\widetilde{0}$.*

Proof. If $\widetilde{F}:[a,b]\to E^n$ is (FHS) integrable with respect to α on $[a,b]$, then $F^*(t)(r,x)$ is (RHS) integrable with respect to α on $[a,b]$ uniformly for any $r\in[0,1]$ and $x\in S^{n-1}$. Since $\alpha\in C^1[a,b]$, there exists a number $G>0$ such that $|\alpha'(t)|\le G$, for all $t\in[a,b]$. By the mean value theorem, there exists $\xi_i\in[u_i,v_i]$ such that

$$\alpha(v_i)-\alpha(u_i)=\alpha'(\xi_i)(v_i-u_i)\le G(v_i-u_i). \qquad (16)$$

Let $E=\{t|\widetilde{F}(t)\ne\widetilde{0}\}$ and for each positive integer n, set $E=\bigcup E_n\subset[a,b]$, where $E_n=\{t|n-1\le|F^*(t)(r,x)|<n\}$, $n=1,2,\cdots$ for every $\varepsilon>0$ and a positive integer n, choose an open set W_n such that $E_n\subset W_n$ and $\mu(W_n)<\frac{\epsilon}{nG2^n}$. Define $\delta(t)$ on $[a,b]$ by

$$\delta(t)=\begin{cases}1, & t\in[a,b]\backslash E,\\ \delta(t), & \text{such that } (t-\delta(t),t+\delta(t))\subset W_n,\ t\in E_n,\ n=1,2,\cdots\end{cases}$$

For any δ-fine division $P=\{[u_i,v_i],\xi_i\}$ and $r\in[0,1]$, $x\in S^{n-1}$, we have

$$\begin{aligned}
\Big|\sum F^*(\xi)(r,x)[\alpha(v)-\alpha(u)]\Big| &= \sum_{n=1}^{\infty}\Big|\sum_{\xi_{ni}\in E_n}F^*(\xi_{ni})(r,x)[\alpha(v_{ni})-\alpha(u_{ni})]\Big|\\
&= \sum_{n=1}^{\infty}\Big|\sum_{\xi_{ni}\in E_n}F^*(\xi_{ni})(r,x)\alpha'(\xi'_{ni})(v_{ni}-u_{ni})\Big|\\
&< \sum_{n=1}^{\infty} nG\mu(W_n)\\
&< \sum_{n=1}^{\infty}\varepsilon 2^{-n}\\
&= \varepsilon.
\end{aligned}$$

Uniformly for any $r\in[0,1]$. □

Theorem 3.5. *Let $\alpha : [a,b] \to \boldsymbol{R}$ be an increasing function. A fuzzy-number-valued function $\widetilde{F} : [a,b] \to E^n$ is (FHS) integrable with respect to α on $[a,b]$ if and only if for every $\varepsilon > 0$, there is a function $\delta(t) > 0$, such that for any two δ-fine divisions $P_1 = \{[u_1, v_1], \xi_1\}$ and $P_2 = \{[u_2, v_2], \xi_2\}$, we have*

$$D\big(\sum_{(P_1)} \widetilde{F}(\xi_1)[\alpha(v_1) - \alpha(u_1)], \sum_{(P_2)} \widetilde{F}(\xi_2)[\alpha(v_2) - \alpha(u_2)]\big) < \varepsilon. \tag{17}$$

Proof. If $\widetilde{F} : [a,b] \to E^n$ is (FHS) integrable with respect to α on $[a,b]$, then $F^*(t)(r,x)$ is (RHS) integrable with respect to α on $[a,b]$ uniformly for any $r \in [0,1]$ and $x \in S^{n-1}$. Suppose first that $\widetilde{F}(t)$ is (FHS) integrable with respect to α on $[a,b]$ and $\varepsilon > 0$, there is a function $\delta(t) > 0$ on $[a,b]$ such that

$$\big|\sum_{(P_1)} F^*(\xi_1)(r,x)[\alpha(v_1)-\alpha(u_1)]-(RHS)\int_a^b F^*(t_1)(r,x)\mathrm{d}\alpha\big| < \frac{\varepsilon}{2}, \quad \big|\sum_{(P_2)} F^*(\xi_2)(r,x)[\alpha(v_2)-\alpha(u_2)]-(RHS)\int_a^b F^*(t_2)(r,x)\mathrm{d}\alpha\big| < \frac{\varepsilon}{2}.$$

Whenever P_1 and P_2 are two δ-fine divisions of $[a,b]$. Then

$$\begin{aligned}
&\big|\sum_{(P_1)} F^*(\xi_1)(r,x)[\alpha(v_1) - \alpha(u_1)] - \sum_{(P_2)} F^*(\xi_2)(r,x)[\alpha(v_2) - \alpha(u_2)]\big| \\
&\qquad \le \big|\sum_{(P_1)} F^*(\xi_1)(r,x)[\alpha(v_1) - \alpha(u_1)] - (RHS)\int_a^b F^*(t_1)(r,x)\mathrm{d}\alpha\big| \\
&\qquad + \big|\sum_{(P_2)} F^*(\xi_2)(r,x)[\alpha(v_2) - \alpha(u_2)] - (RHS)\int_a^b F^*(t_2)(r,x)\mathrm{d}\alpha\big| \\
&\qquad < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&\qquad = \varepsilon.
\end{aligned}$$

Hence, the Cauchy criterion is satisfied.

Conversely, suppose that the Cauchy criterion is satisfied. For each positive integer n, choose $\delta_n(t) > 0$ on $[a,b]$ such that

$$\big|\sum_{(P_1)} F^*(\xi_1)(r,x)[\alpha(v_1) - \alpha(u_1)] - \sum_{(P_2)} F^*(\xi_2)(r,x)[\alpha(v_2) - \alpha(u_2)]\big| < \frac{1}{n}. \tag{18}$$

Whenever P_1 and P_2 are two δ-fine divisions of $[a,b]$. We may assume that the sequence $\{\delta_n(t)\}$ is non-increasing. For every n, let P_n be a δ_n-fine division of $[a,b]$. The sequence $\{F^*(\xi_n)(r,x)[\alpha(v_n) - \alpha(u_n)]\}$ is a Cauchy sequence since

$$m > n \ge N \text{ implies } \big|\sum_{(P_m)} F^*(\xi_m)(r,x)[\alpha(v_m) - \alpha(u_m)] - \sum_{(P_n)} F^*(\xi_n)(r,x)[\alpha(v_n) - \alpha(u_n)]\big| < \frac{1}{N}.$$

Let M be the limit of this sequence and $\varepsilon > 0$. Choose a positive integer N such that

$$\frac{1}{N} < \frac{\varepsilon}{2} \text{ and } \big|\sum_{(P_n)} F^*(\xi_n)(r,x)[\alpha(v_n) - \alpha(u_n)] - M\big| < \frac{\varepsilon}{2} \text{ for all } n \le N.$$

Let P be a δ_N-fine division of $[a,b]$ and compute

$$\begin{aligned}
\big|\sum_{(P)} F^*(\xi)(r,x)[\alpha(v) - \alpha(u)] - M\big| &\le \big|\sum_{(P)} F^*(\xi)(r,x)[\alpha(v) - \alpha(u)] - \sum_{(P_N)} F^*(\xi_N)(r,x)[\alpha(v_N) - \alpha(u_N)]\big| \\
&+ \big|\sum_{(P_N)} F^*(\xi_N)(r,x)[\alpha(v_N) - \alpha(u_N)] - M\big| \\
&< \tfrac{1}{N} + \tfrac{\varepsilon}{2} \\
&< \varepsilon.
\end{aligned}$$

□

Theorem 3.6. *Let $\alpha : [a,b] \to \boldsymbol{R}$ be an increasing function. Let $\widetilde{F} : [a,b] \to E^n$ be a fuzzy-number-valued function. If $\widetilde{F}(t)$ is (FHS) integrable with respect to α on $[a,b]$, then $\widetilde{F}(t)$ is (FHS) integrable with respect to α on every subinterval of $[a,b]$.*

Proof. If $\widetilde{F} : [a,b] \to E^n$ is (FHS) integrable with respect to α on $[a,b]$, then $F^*(t)(r,x)$ is (RHS) integrable with respect to α on $[a,b]$ uniformly for any $r \in [0,1]$ and $x \in S^{n-1}$. Let $[c,d] \subseteq [a,b]$ and let $\varepsilon > 0$. Choose a positive function δ on $[a,b]$ such that

$$\Big|\sum_{(P_1)} F^*(\xi_1)(r,x)[\alpha(v_1) - \alpha(u_1)] - \sum_{(P_2)} F^*(\xi_2)(r,x)[\alpha(v_2) - \alpha(u_2)]\Big| < \varepsilon. \tag{19}$$

Whenever P_1 and P_2 are two δ-fine divisions of $[a,b]$. Fix δ-fine division P_a of $[a,c]$ and P_b of $[d,b]$. Let P_1' and P_2' be δ-fine divisions of $[c,d]$ and define $P_1 = P_a \cup P_1' \cup P_b$ and $P_2 = P_a \cup P_2' \cup P_b$. Then P_1 and P_2 are two δ-fine divisions of $[a,b]$ and

$$\begin{aligned}&\Big|\sum_{(P_1')} F^*(\xi_1')(r,x)[\alpha(v_1') - \alpha(u_1')] - \sum_{(P_2')} F^*(\xi_2')(r,x)[\alpha(v_2') - \alpha(u_2')]\Big| \\ &\qquad = \Big|\sum_{(P_1)} F^*(\xi_1)(r,x)[\alpha(v_1) - \alpha(u_1)] - \sum_{(P_2)} F^*(\xi_2)(r,x)[\alpha(v_2) - \alpha(u_2)]\Big| < \varepsilon. \end{aligned}\tag{20}$$

□

4. Linearity Properties of (HS) Integrability for n-dimensional Fuzzy-Number-Valued Functions

In this section, we introduce some linearity properties of (HS) integrability for n-dimensional fuzzy-number-valued functions.

Definition 4.1. *Let $\alpha : [a,b] \to \boldsymbol{R}$ be an increasin function. Let $\widetilde{F}, \widetilde{G} : [a,b] \to E^n$ be a fuzzy-number-valued functions. We say that $\widetilde{F}(t) = \widetilde{G}(t)$ almost everywhere on $[a,b]$, if $F^*(t)(r,x) - G^*(t)(r,x) = 0$ almost everywhere on $[a,b]$ for any $r \in [0,1]$ and $x \in S^{n-1}$.*

Theorem 4.2. *Let $\alpha : [a,b] \to \boldsymbol{R}$ be an increasing function. Let $\widetilde{F} : [a,b] \to E^n$ and let $c \in (a,b)$. If $\widetilde{F}(t)$ is (FHS) integrable with respect to α on each of the intervals $[a,c]$ and $[c,b]$, then $\widetilde{F}(t)$ is (FHS) integrable with respect to α on $[a,b]$ and*

$$(FHS)\int_a^b \widetilde{F}(t)d\alpha = (FHS)\int_a^c \widetilde{F}(t)d\alpha + (FHS)\int_c^b \widetilde{F}(t)d\alpha. \tag{21}$$

Proof. If $\widetilde{F} : [a,b] \to E^n$ is (FHS) integrable with respect to α on $[a,b]$, then $F^*(t)(r,x)$ is (RHS) integrable with respect to α on $[a,b]$ uniformly for any $r \in [0,1]$ and $x \in S^{n-1}$. Let $\varepsilon > 0$. By hypothesis, there exists a positive function δ_1 on $[a,c]$ such that

$$\Big|\sum_{(P)} F^*(\xi)(r,x)[\alpha(v) - \alpha(u)] - \int_a^c F^*(t)(r,x)\mathrm{d}\alpha\Big| < \frac{\varepsilon}{2}. \tag{22}$$

Whenever P is δ_1-fine division of $[a,c]$, and there exists a positive function δ_2 on $[c,b]$ such that

$$\Big|\sum_{(P)} F^*(\xi)(r,x)[\alpha(v) - \alpha(u)] - \int_c^b F^*(t)(r,x)\mathrm{d}\alpha\Big| < \frac{\varepsilon}{2}. \tag{23}$$

Whenever P is δ_2-fine division of $[c,b]$. Define δ on $[a,b]$ by

$$\delta(t) = \begin{cases} \min\{\delta_1(t), c-t\}, & \text{if } a \le t < c; \\ \min\{\delta_1(c), \delta_2(c)\}, & \text{if } t = c; \\ \min\{\delta_2(t), t-c\}, & \text{if } c < t \le b. \end{cases}$$

Let P be δ-fine division of $[a,b]$ that each division occurs only once. Note that P must be of the form $P_a \cup (c,[u,v]) \cup P_b$ where δ-fine division P_a are less than c and δ-fine division P_b are grater than c. Let $P_1 = P_a \cup (c,[u,c])$ and let $P_2 = P_b \cup (c,[c,v])$. Then P_1 is δ_1-fine division of $[a,c]$ and P_2 is δ_2-fine division of $[c,b]$. Since

$$\Big|\sum_{(P)} F^*(\xi)(r,x)[\alpha(v)-\alpha(u)]\Big| = \Big|\sum_{(P_1)} F^*(\xi_1)(r,x)[\alpha(v_1)-\alpha(u_1)]\Big| + \Big|\sum_{(P_2)} F^*(\xi_2)(r,x)[\alpha(v_2)-\alpha(u_2)]\Big|. \tag{24}$$

We obtain

$$\begin{aligned}\Big|\sum_{(P)} F^*(\xi)(r,x)[\alpha(v)-\alpha(u)] - \int_a^c F^*(t)(r,x)\mathrm{d}\alpha - \int_c^b F^*(t)(r,x)\mathrm{d}\alpha\Big| &\le \Big|\sum_{(P_1)} F^*(\xi_1)(r,x)[\alpha(v_1)-\alpha(u_1)] \\ - \int_a^c F^*(t_1)(r,x)\mathrm{d}\alpha\Big| + \Big|\sum_{(P_2)} F^*(\xi_2)(r,x)[\alpha(v_2)-\alpha(u_2)] - \int_c^b F^*(t_2)(r,x)\mathrm{d}\alpha\Big| &< \varepsilon. \end{aligned}\tag{25}$$

□

Theorem 4.3. *Let $\alpha : [a,b] \to \boldsymbol{R}$ be an increasing function. Let $\widetilde{F}, \widetilde{G} : [a,b] \to E^n$ be (FHS) integrable with respect to α on $[a,b]$, then*

$$(FHS)\int_a^b \big(\widetilde{F}(t) + \widetilde{G}(t)\big)\,d\alpha = (FHS)\int_a^b \widetilde{F}(t)\,d\alpha + (FHS)\int_a^b \widetilde{G}(t)\,d\alpha. \tag{26}$$

Proof. If $\widetilde{F}, \widetilde{G}$ are (FHS) integrable with respect to α on $[a,b]$, then $F^*(t)(r,x)$, $G^*(t)(r,x)$ are (RHS) integrable with respect to α on $[a,b]$ uniformly for any $r \in [0,1]$ and $x \in S^{n-1}$. Let $(RHS)\int_a^b F^*(t)(r,x)\mathrm{d}\alpha = A^*(r,x)$ and $(RHS)\int_a^b G^*(t)(r,x)\mathrm{d}\alpha = B^*(r,x)$. Then for every $\varepsilon > 0$, there is a function $\delta(t) > 0$, such that for any δ_1-fine division $P_1 = \{[u_1,v_1],\xi_1\}$ of $[a,b]$, we have

$$\Big|\sum_{(P_1)} F^*(\xi_1)(r,x)[\alpha(v_1)-\alpha(u_1)] - A^*(r,x)\Big| < \frac{\varepsilon}{2}. \tag{27}$$

And for any δ_2-fine division $P_2 = \{[u_2,v_2],\xi_2\}$ of $[a,b]$, we have

$$\Big|\sum_{(P_2)} G^*(\xi_2)[\alpha(v_2)-\alpha(u_2)] - B^*(r,x)\Big| < \frac{\varepsilon}{2}. \tag{28}$$

Define δ on $[a,b]$ by $\delta(t) = \min\{\delta(t_1),\delta(t_2)\}$. Let P be a δ-fine division of $[a,b]$. Then

$$\begin{aligned}\Big|\sum_{(P)} \big(F^*(\xi)+G^*(\xi)\big)[\alpha(v)-\alpha(u)] - \big(A^*(r,x)+B^*(r,x)\big)\Big| &\le \Big|\sum_{(P_1)} F^*(\xi_1)[\alpha(v_1)-\alpha(u_1)] - A^*(r,x)\Big| \\ &\quad + \Big|\sum_{(P_2)} G^*(\xi_2)[\alpha(v_2)-\alpha(u_2)] - B^*(r,x)\Big| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}\tag{29}$$

□

Theorem 4.4. *Let $\alpha : [a,b] \to \boldsymbol{R}$ be an increasing function. Let $\widetilde{F}, \widetilde{G} : [a,b] \to E^n$ be (FHS) integrable with respect to α on $[a,b]$ and if $\alpha \in C^1[a,b]$. If $\widetilde{F}(t) = \widetilde{G}(t)$ almost everywhere on $[a,b]$, then*

$$(FHS)\int_a^b \widetilde{F}(t)\,d\alpha = (FHS)\int_a^b \widetilde{G}(t)\,d\alpha. \tag{30}$$

Proof. By Definition 4.1, if $\widetilde{F}(t) = \widetilde{G}(t)$ almost everywhere on $[a, b]$, then $F^*(t)(r, x) - G^*(t)(r, x) = 0$ almost everywhere on $[a, b]$ for any $r \in [0, 1]$ and $x \in S^{n-1}$. By Theorem 3.3 and Theorem 3.4 $F^*(t)(r, x) - G^*(t)(r, x)$ is (RHS) integrable with respect to α on $[a, b]$ for any $r \in [0, 1]$ and $x \in S^{n-1}$ such that

$$\int_a^b \big(F^*(t)(r, x) - G^*(t)(r, x)\big) \mathrm{d}\alpha = 0. \tag{31}$$

By Theorem 4.3 the support function

$$G^*(t)(r, x) = F^*(t)(r, x) + \big[G^*(t)(r, x) - F^*(t)(r, x)\big],$$

is (RHS) integrable with respect to α on $[a, b]$ for any $r \in [0, 1]$ and $x \in S^{n-1}$ such that

$$(RHS)\int_a^b G^*(t)(r, x)\mathrm{d}\alpha = (RHS)\int_a^b F^*(t)(r, x)\mathrm{d}\alpha + (RHS)\int_a^b \big[G^*(t)(r, x) - F^*(t)(r, x)\big]\mathrm{d}\alpha = (RHS)\int_a^b F^*(t)(r, x)\mathrm{d}\alpha. \tag{32}$$

□

Theorem 4.5. *(Saks-Henstock Lemma) Let $\alpha : [a, b] \to \boldsymbol{R}$ be an increasing function. Let $\widetilde{F} : [a, b] \to E^n$ be (FHS) integrable with respect to α on $[a, b]$. Then for every $\varepsilon > 0$, there is a function $\delta(t) > 0$, such that*

$$D\Big(\sum_{(P)} \widetilde{F}(\xi)[\alpha(v) - \alpha(u)], \int_a^b \widetilde{F}(t)\, d\alpha\Big) < \varepsilon, \tag{33}$$

for each δ-fine division $P = \{[u_i, v_i]; \xi_i\}_{i=1}^n$ of $[a, b]$. Particulary, if $P' = \{[u_i', v_i']; \xi_i'\}_{i=1}^m$ is an arbitrary δ-fine partial division of $[a, b]$, we have

$$D\Big(\sum_{(P')} \widetilde{F}(\xi')[\alpha(v') - \alpha(u')], \sum_{i=1}^m \int_a^b \widetilde{F}(t)\, d\alpha\Big) \le \varepsilon. \tag{34}$$

Proof. If $\widetilde{F} : [a, b] \to E^n$ is (FHS) integrable with respect to α on $[a, b]$, then $F^*(t)(r, x)$ is (RHS) integrable with respect to α on $[a, b]$ uniformly for any $r \in [0, 1]$ and $x \in S^{n-1}$. Assume $P' = \{[u_i', v_i']; \xi_i'\}_{i=1}^m$ is an arbitrary δ-fine partial division of $[a, b]$, then the complement $[a, b] \setminus \bigcup_{i=1}^m [u_i, v_i]$ can be expressed as a finite collection of closed subintervals and we denote

$$[a, b] \setminus \bigcup_{i=1}^m [u_i, v_i] = \bigcup_{j=1}^k [u_j', v_j']. \tag{35}$$

Let $\eta > 0$ be arbitrary. From Theorem 3.6 we know $\int_{u_j'}^{v_j'} F^*(t)(r, x)\mathrm{d}\alpha$ exists for each $j = 1, 2, \cdots k$, then there exists δ_j on $[u_j', v_j']$ such that if P_j is a δ_j-fine division of $[u_j', v_j']$, then

$$\Big|\sum_{(P_j)} F^*(\xi_j)(r, x)[\alpha(v_j) - \alpha(u_j)] - \int_{u_j'}^{v_j'} F^*(t)(r, x)\mathrm{d}\alpha\Big| < \frac{\eta}{k}. \tag{36}$$

Assume that $\delta_j \subset \delta$ for all $t \in [a, b]$. Let $P_0 = P' + P_1 + P_2 + \cdots + P_k$, obviously P_0 is δ-fine division of $[a, b]$, we have

$$\begin{aligned}\Big|\sum_{(P_0)} F^*(\xi_0)(r, x)[\alpha(v_0) - \alpha(u_0)] - \int_a^b F^*(t)(r, x)\mathrm{d}\alpha\Big| = \Big|\sum_{(P')} F^*(\xi')(r, x)[\alpha(v') - \alpha(u')] \\ + \sum_{j=1}^k \sum_{(P_j)} F^*(\xi_j)(r, x)[\alpha(v_j) - \alpha(u_j)] - \int_a^b F^*(t)(r, x)\mathrm{d}\alpha\Big| < \varepsilon.\end{aligned} \tag{37}$$

Consequently, we obtain

$$
\begin{aligned}
&|\sum_{(P')} F^*(\xi')(r,x)[\alpha(v')-\alpha(u')]-\sum_{i=1}^{m}\int_{u_i}^{v_i} F^*(t)(r,x)\big)\mathrm{d}\alpha| \\
&=|\sum_{(P_0)} F^*(\xi_0)(r,x)[\alpha(v_0)-\alpha(u_0)]-\sum_{j=1}^{k}\sum_{(P_j)} F^*(\xi_j)(r,x)[\alpha(v_j)-\alpha(u_j)]-\big(\int_a^b F^*(t)(r,x)\mathrm{d}\alpha-\sum_{j=1}^{k}\int_{u_j'}^{v_j'} F^*(t)(r,x)\mathrm{d}\alpha\big)| \\
&\le |\sum_{(P_0)} F^*(\xi_0)(r,x)[\alpha(v_0)-\alpha(u_0)]-\int_a^b F^*(t)(r,x)\mathrm{d}\alpha|+\sum_{j=1}^{k}|\sum_{(P_j)} F^*(\xi_j)(r,x)[\alpha(v_j)-\alpha(u_j)]-\int_{u_j'}^{v_j'} F^*(t)(r,x)\mathrm{d}\alpha| \\
&<\varepsilon+\frac{k\eta}{k} \\
&=\varepsilon+\eta. \qquad (38)
\end{aligned}
$$

□

5. The Characterization of (FHS) Integrability for n-dimensional Fuzzy-Number-Valued Functions

Let $\overline{C}\big(I, C(S^{n-1})\big)=\big\{f|f: I\to C(S^{n-1})\big\}$, f is an left continuously abstract function and f has a right limit for any $t\in[0,1)$, especially f is right continuous at $t=0$. We can proof $\big(\overline{C}(I, C(S^{n-1})), \|\cdot\|\big)$ is a Banach space, where $I=[0,1]$ and $C(S^{n-1})$ is the space of all continuous function on S^{n-1} . $\|f\|=\sup_{r\in[0,1]}\|f(r)\|$. [11, 19].

Lemma 5.1. *[11, 19] For $u\in E^n$, denote $j(u): r\to u^*(r,x)\in C(S^{n-1})$, then $j(u)\in\overline{C}\big(I, C(S^{n-1})\big)$ and:*

(1). $j(su+tv)=sj(u)+tj(v),\ u,v\in E^n,\ s,t\ge 0$;

(2). $\|j(u)-j(v)\|=D(u,v),\ u,v\in E^n$;

(3). $j(E^n)$ is closed set in $\overline{C}\big(I, C(S^{n-1})\big)$.

According to Lemma 5.1, the fuzzy number space E^n can be embedded into a Banach space $\big(\overline{C}(I, C(S^{n-1})), \|\cdot\|\big)$ in the same distance.

Theorem 5.2. *Let $\alpha:[a,b]\to\boldsymbol{R}$ be an increasing function. If $\widetilde{F}:[a,b]\to E^n$ is fuzzy-number-valued function on $[a,b]$, then the following statements are equivalent:*

(1). The fuzzy-number-valued function $\widetilde{F}(t)$ is (FHS) integrable on $[a,b]$;

(2). The set-valued function $F_r(t)$ is (IHS) integrable on $[a,b]$ uniformly for any $r\in[0,1]$;

(3). The vector-valued function $j\circ\widetilde{F}(t)$ is (VHS) integrable on $[a,b]$, and $(VHS)\int_a^b j\circ\widetilde{F}(t)d\alpha\in j(E^n)$;

(4). The vector-valued function $F^*(t)(\cdot,x)$ is (VHS) integrable on $[a,b]$ for any $x\in S^{n-1}$;

(5). The real-valued function $F^*(t)(r,x)$ is (RHS) integrable on $[a,b]$ uniformly for any $r\in[0,1]$ and $x\in S^{n-1}$.

Proof. (1) implies (2): Since $\widetilde{F}(t)$ is (FHS) integrable with respect to α on $[a,b]$, then there exists a fuzzy-number $\widetilde{A}\in E^n$, for every $\varepsilon>0$, there is a function $\delta(t)>0$, such that for any δ-fine division $P=\{[u,v],\xi\}$ of $[a,b]$, we have

$$D\big(\sum_{(P)}\widetilde{F}(\xi)[\alpha(v)-\alpha(u)], \widetilde{A}\ \big)<\varepsilon. \qquad (39)$$

And because, $[\widetilde{A}]^r \in P_k(R^n)$ for any $r \in [0,1]$, then according to Lemma 2.7 and Lemma 2.9,

$$\sup_{r\in[0,1]} d\big(\sum_{(P)} F_r(\xi)[\alpha(v)-\alpha(u)], [\widetilde{A}]^r\big) = \sup_{r\in[0,1]} d\big(\big[\sum_{(P)} \widetilde{F}(\xi)[\alpha(v)-\alpha(u)]\big]^r, [\widetilde{A}]^r\big)$$
$$= D\big(\sum_{(P)} \widetilde{F}(\xi)[\alpha(v)-\alpha(u)], \widetilde{A}\big) < \varepsilon.$$

Therefore, the set-valued function $F_r(t)$ is (IHS) integrable on $[a,b]$ uniformly for any $r \in [0,1]$.

(2) implies (5): Since the set-valued function $F_r(t)$ is (IHS) integrable with respect to α on $[a,b]$ uniformly for any $r \in [0,1]$, then there exists $I_r \in P_k(R^n)$, for every $\varepsilon > 0$, there is a function $\delta(t) > 0$, such that for any δ-fine division $P = \{[u,v],\xi\}$ of $[a,b]$, we have

$$d\big(\sum_{(P)} F_r(\xi)[\alpha(v)-\alpha(u)], I_r\big) < \varepsilon. \tag{40}$$

Uniformly for any $r \in [0,1]$. And because, $\sigma(x,I_r) \in R$, for any $r \in [0,1]$ and $x \in S^{n-1}$, then by Lemma 2.7, Lemma 2.10 (6),(7), Definition 2.5 and Lemma 2.12, we have

$$\big|\sum_{(P)} F^*(\xi)(r,x)[\alpha(v)-\alpha(u)] - \sigma(x,I_r)\big| = \big|(\sum_{(P)} \widetilde{F}(\xi)[\alpha(v)-\alpha(u)])^*(r,x) - \sigma(x,I_r)\big|$$
$$= \big|\sup_{a\in\big[\sum_{(P)} \widetilde{F}(\xi)[\alpha(v)-\alpha(u)]\big]^r} \langle a,x\rangle - \sigma(x,I_r)\big|$$
$$= \big|\sigma(x,\big[\sum_{(P)} \widetilde{F}(\xi)[\alpha(v)-\alpha(u)]\big]^r) - \sigma(x,I_r)\big|$$
$$\leq \sup_{x\in S^{n-1}} \big|\sigma(x,\big[\sum_{(P)} \widetilde{F}(\xi)[\alpha(v)-\alpha(u)]\big]^r) - \sigma(x,I_r)\big|$$
$$= d\big(\big[\sum_{(P)} \widetilde{F}(\xi)[\alpha(v)-\alpha(u)]\big]^r, I_r\big)$$
$$= d\big(\sum_{(P)} F_r(\xi)[\alpha(v)-\alpha(u)], I_r\big) < \varepsilon. \tag{41}$$

Uniformly for any $r \in [0,1]$. That is the real-valued function $F^*(t)(r,x)$ is (RHS) integrable on $[a,b]$ uniformly for any $r \in [0,1]$ and $x \in S^{n-1}$.

(5) implies (1): See Theorem 3.3

Summary (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (5). In the following section, we shall prove that (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (1).

(5) implies (4): Since the real valued function $F^*(t)(r,x)$ is (RHS) integrable with respect to α on $[a,b]$ uniformly for any $r \in [0,1]$ and $x \in S^{n-1}$, then there is $a(r,x) \in R$, for every $0 < \varepsilon < 1$, there is a function $\delta(t) > 0$ (independently to r), such that for any δ-fine division $P = \{[u,v],\xi\}$ of $[a,b]$, we have

$$\big|\sum_{(P)} F^*(\xi)(r,x)[\alpha(v)-\alpha(u)] - a(r,x)\big| < \varepsilon. \tag{42}$$

Uniformly for any $r \in [0,1]$. In the following, we shall prove $(RHS)\int_a^b F^*(t)(\cdot,x)\mathrm{d}\alpha = a(\cdot,x) \in \overline{C}\big(I, C(S^{n-1})\big)$. Since $\widetilde{F}(t) \in E^n$, then by using Lemma 2.10 (2), for any $r \in [0,1]$ and $x \in S^{n-1}$, we have

$$\big|a(r,x)\big| = \big|\int_a^b F^*(t)(r,x)\mathrm{d}\alpha\big| \leq \big|\int_a^b F^*(t)(0,x)\mathrm{d}\alpha\big|, \tag{43}$$

i.e. $a(\cdot,x)$ is bounded on $[a,b]$. According to the Lemma 2.10 (3), if a positive sequence $\{r_m\}$ is nondecreasing convergent to r, for any $r\in(0,1]$ and $x\in S^{n-1}$, we have

$$\lim_{m\to\infty} F^*(t)(r_m,x)=F^*(t)(r,x). \tag{44}$$

By the (RH) integral-dominated convergence theorem [9, 10], for any $x\in S^{n-1}$, we have

$$\lim_{m\to\infty}(RHS)\int_a^b F^*(t)(r_m,x)\mathrm{d}\alpha=(RHS)\int_a^b F^*(t)(r,x)\mathrm{d}\alpha, \tag{45}$$

i.e. $\lim\limits_{m\to\infty} a(r_m,x)=a(r,x)$. Therefore $a(\cdot,x)$ is left continuous. By using Lemma 2.10 (2),(3), it is clear the right limit value of $a(\cdot,x)$ exists. Then according to Lemma 2.10 (3), similarly to the proof of the left continuity of $a(\cdot,x)$, we have $a(\cdot,x)$ is right continuous at $r=0$. Summary, $a(\cdot,x)\in\overline{C}\big(I,C(S^{n-1})\big)$. Since $\big(\overline{C}(I,C(S^{n-1})),\|\cdot\|\big)$ is Banach space in the sense of the norm taking the supremum, so $\sum\limits_{(P)} F^*(\xi)(\cdot,x)[\alpha(v)-\alpha(u)]-a(\cdot,x)\in\overline{C}\big(I,C(S^{n-1})\big)$. Then for any $x\in S^{n-1}$, we have

$$\begin{aligned}\Big\|\sum_{(P)} F^*(\xi)(\cdot,x)[\alpha(v)-\alpha(u)]-a(\cdot,x)\Big\| &= \sup_{r\in[0,1]}\Big|\sum_{(P)} F^*(\xi)(r,x)[\alpha(v)-\alpha(u)]-a(r,x)\Big|\\ &\le\varepsilon.\end{aligned} \tag{46}$$

It is clear (4) implies (3) by using Lemma 5.1

(3) implies (1): Since the vector-valued function $j\circ\widetilde{F}(t)$ is (VHS) integrable with respect to α on $[a,b]$, and $(VHS)\int_a^b j\circ\widetilde{F}(t)\mathrm{d}\alpha\in j(E^n)$, then there exists a fuzzy-number $\widetilde{A}\in E^n$, for every $\varepsilon>0$, there is a function $\delta(t)>0$, such that for any δ-fine division $P=\{[u,v],\xi\}$ of $[a,b]$, we have

$$\Big\|\sum_{(P)} j\circ\widetilde{F}(\xi)[\alpha(v)-\alpha(u)]-j\circ\widetilde{A}\Big\|<\varepsilon. \tag{47}$$

According to Lemma 5.1, we have

$$\begin{aligned}D\Big(\sum_{(P)}\widetilde{F}(\xi)[\alpha(v)-\alpha(u)],\widetilde{A}\Big) &= \Big\|j\circ\Big(\sum_{(P)}\widetilde{F}(\xi)[\alpha(v)-\alpha(u)]\Big)-j\circ\widetilde{A}\Big\|\\ &=\Big\|\sum_{(P)} j\circ\widetilde{F}(\xi)[\alpha(v)-\alpha(u)]-j\circ\widetilde{A}\Big\|\\ &<\varepsilon.\end{aligned} \tag{48}$$

□

Corollary 5.3. *Let $\alpha:[a,b]\to\boldsymbol{R}$ be an increasing function. Let $\widetilde{F},\widetilde{G}:[a,b]\to E^n$ be a fuzzy-number-valued function. If $\widetilde{F},\ \widetilde{G}\in FHS_\alpha[a,b]$ and $D\big(\widetilde{F}(t),\widetilde{G}(t)\big)\in LS_\alpha[a,b]$, then*

$$D\Big((FHS)\int_a^b\widetilde{F}(t)d\alpha,(FHS)\int_a^b\widetilde{G}(t)d\alpha\Big)\le(LS)\int_a^b D\big(\widetilde{F}(t),\widetilde{G}(t)\big)d\alpha. \tag{49}$$

Proof. Since $D\big(\widetilde{F}(t),\widetilde{G}(t)\big)=\sup\limits_{r\in[0,1]}\sup\limits_{x\in S^{n-1}}\big|F^*(t)(r,x)-G^*(t)(r,x)\big|$ is Lebesgue-Stieltjes integrable with respect to α on $[a,b]$ and $F^*(t)(r,x),\ G^*(t)(r,x)$ are (RHS) integrable for any $r\in[0,1]$ and $x\in S^{n-1}$. Hence

$$\begin{aligned}
D\Big((FHS)\int_a^b\widetilde{F}(t)\mathrm{d}\alpha,(FHS)\int_a^b\widetilde{G}(t)\mathrm{d}\alpha\Big) &= \sup_{r\in[0,1]} d\Big\{\Big[(FHS)\int_a^b\widetilde{F}(t)\mathrm{d}\alpha\Big]^r,\Big[(FHS)\int_a^b\widetilde{G}(t)\mathrm{d}\alpha\Big]^r\Big\}\\
&= \sup_{r\in[0,1]}\sup_{x\in S^{n-1}}\Big|\Big[(FHS)\int_a^b F(t)\Big]^*(r,x)\mathrm{d}\alpha-\Big[(FHS)\int_a^b G(t)\Big]^*(r,x)\mathrm{d}\alpha\Big|\\
&= \sup_{r\in[0,1]}\sup_{x\in S^{n-1}}\Big|(RHS)\int_a^b F(t)^*(r,x)\mathrm{d}\alpha-(RHS)\int_a^b G(t)^*(r,x)\mathrm{d}\alpha\Big|\\
&\le \sup_{r\in[0,1]}\sup_{x\in S^{n-1}}(RHS)\int_a^b\big|F(t)^*(r,x)-G(t)^*(r,x)\big|\mathrm{d}\alpha\\
&= \sup_{r\in[0,1]}\sup_{x\in S^{n-1}}(LS)\int_a^b\big|F(t)^*(r,x)-G(t)^*(r,x)\big|\mathrm{d}\alpha\\
&\le \sup_{r\in[0,1]}(LS)\int_a^b\sup_{x\in S^{n-1}}\big|F(t)^*(r,x)-G(t)^*(r,x)\big|\mathrm{d}\alpha\\
&\le (LS)\int_a^b\sup_{r\in[0,1]} d\big(\big[\widetilde{F}(t)\big]^r,\big[\widetilde{G}(t)\big]^r\big)\mathrm{d}\alpha=(LS)\int_a^b D\big(\widetilde{F}(t),\widetilde{G}(t)\big)\mathrm{d}\alpha. \qquad (50)
\end{aligned}$$

□

6. Conclusions

In this paper, the Henstock-Stieltjes (HS) integral for n-dimensional fuzzy-number-valued functions is defined and some basic properties of this integral are discussed. Finally, by using the embedding theorem the fuzzy number space E^n can be embedded into a concrete Banach space, and some characterized theorems for this integral are obtained.

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