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Solution and Stability of New Type of $(a_{aq}, b_{bq}, c_{aq}, d_{aq})$ Mixed Type Functional Equation in Various Normed Spaces: Using Two Different Methods

Research Article

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Abstract: Using two different methods, the authors are investigate the general solution and generalized Ulam - Hyers stability of new type of $(a_{aq}, b_{aq}, c_{aq}, d_{aq})$ mixed type functional equation of the form $g(ax_1 + bx_2 + cx_3 + dx_4) + g(-ax_1 + bx_2 + cx_3 + dx_4) + g(ax_1 - bx_2 + cx_3 + dx_4) + g(ax_1 + bx_2 - cx_3 + dx_4) + g(ax_1 + bx_2 + cx_3 - dx_4) = g(ax_1 + bx_2) + g(ax_1 + cx_3) + g(ax_1 + dx_4) + g(bx_2 + cx_3) + g(bx_2 + dx_4) + g(cx_3 + dx_4) + g(ax_1) - g(-ax_1) + g(bx_2) - g(-bx_2) + g(cx_3) - g(-cx_3) + g(dx_4) - g(-dx_4) - (ag(x_1) - ag(-x_1) + bg(x_2) - bg(-x_2) + cg(x_3) - cg(-x_3) + dg(x_4) - dg(-x_4)) + a^2(g(x_1) + g(-x_1)) + b^2(g(x_2) + g(-x_2)) + c^2(g(x_3) + g(-x_3)) + d^2(g(x_4) + g(-x_4))$, where a, b, c, d are positive integers with $a \neq b \neq c \neq d$, in Banach space, Fuzzy Normed Space and Instuitionistic Fuzzy normed space.

MSC: 39B52, 32B72, 32B82.

Keywords: Mixed type functional equation, Banach Spaces, Fuzzy Banach Spaces and IFN Space, Generalized Hyers - Ulam stability, Fixed point.

1. Introduction

In [29], Ulam proposed the general stability problem: when is it true that by slightly changing the hypothesis of a theorem one can still assert that the thesis of the theorem remains true or approximately true?. In [30], Hyers gave the first Affirmative answer to the question of Ulam for additive functional equation on Banach Spaces. Hyers result have since then seen many significant generalizations, both interms of the control condition used to define the concept of approximate solution [3, 4, 8, 12–14, 16]. On the other hand, Cadariu and Radu noticed that a fixed point Alternative method is very important for the solutions of the Ulam problem. In otherwords, they employed this fixed point method to the investigation of the Cauchy functional equation [9, 10] and for the quadratic functional equation [19–23]. Nowadays, fixed point and operator theory play an important role in different areas of Mathematics and its applications, particularly in Mathematics, Physics and Differential equation. Since the fuzzy mathematics and fuzzy physics along with the classical one are constantly developing, the fuzzy type of the fixed point and operator theory can also play an important role in new fuzzy area and fuzzy mathematical physics. Many authors [3–6, 16] have also proved some different type of fixed point theorems in fuzzy

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metric spaces and fuzzy normed spaces. The fixed point method was used for the first time by Baker [5] who applied a variant of Banach fixed point theorem to obtain the Ulam - Hyers stability of functional equation in a single variable (for more application of this method see [1, 2, 7, 8, 11–14, 17, 24, 27]). One of the most famous functional equations is the additive functional equation

$$f(x+y) = f(x) + f(y). \tag{1}$$

In 1821, it was first solved by A. L. Cauchy in the class of continuous real valued functions. It is often called an additive cauchy functional equation in honor of cauchy. The theorey of additive functional equations is frequently applied to the developement of theories of the other functional equations. Moreover, the properties of additive functional equations are powerful tools in almost every field of natural and social sciences. Every solutions of the additive functional equation (1) is called the additive function. The second famous functional equation is

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{2}$$

is said to be quadratic functional equation because the quadratic function $f(x) = ax^2$ is a solution of the functional equation (2). In this paper, we consider the following functional equation deriving from additive and quadratic functional equation of the form general solution and generalized Ulam - Hyers - Rassias stability of the new type of cubic functional equation of the form

$$\begin{aligned} &g(ax_1+bx_2+cx_3+dx_4) + g(-ax_1+bx_2+cx_3+dx_4) + g(ax_1-bx_2+cx_3+dx_4) + \\ &g(ax_1+bx_2-cx_3+dx_4) + g(ax_1+bx_2+cx_3-dx_4) = g(ax_1+bx_2) + g(ax_1+cx_3) + \\ &g(ax_1+dx_4) + g(bx_2+cx_3) + g(bx_2+dx_4) + g(cx_3+dx_4) + g(ax_1) - g(-ax_1) + \\ &g(bx_2) - g(-bx_2) + g(cx_3) - g(-cx_3) + g(dx_4) - g(-dx_4) - (ag(x_1) - ag(-x_1) + \\ &bg(x_2) - bg(-x_2) + cg(x_3) - cg(-x_3) + dg(x_4) - dg(-x_4)) + a^2\,(g(x_1) + g(-x_1)) \\ &+ b^2\,(g(x_2) + g(-x_2)) + c^2\,(g(x_3) + g(-x_3)) + d^2\,(g(x_4) + g(-x_4)) \end{aligned} \tag{3}$$

where a, b, c, d are positive integers with $a \neq b \neq c \neq d \neq 0$. We find the general solution of this new type functional equation in Banach space (BS), Fuzzy Normed Space (FNS) and Instuitionistic Fuzzy normed space ($IFNS$) by using Two different methods.

2. General Solution of the Functional Equation (3).

In this section, we obtain the general solution of the functional equation (3).

Lemma 2.1. *Let X and Y be a real vector space. An odd function $g : X \to Y$ satisfies (3) if and only if g is additive.*

Proof. Since g is odd function. Replacing (x, y) by $(0, 0)$ in (1), that $g(0) = 0$ and setting (x, y) by $(x, -x)$ in (1), we get

$$g(-x) = -g(x).$$

Putting (x, y) by (x, x) and $(x, 2x)$ in (1) respectively, we get

$$g(2x) = 2g(x)$$

and

$$g(3x) = 3g(x).$$

In general for any positive integer n, such that

$$g(ax) = ag(x) \tag{4}$$

for all $x \in X$. It is easy to verify that from (4), that

$$g(a^2x) = a^2g(x),$$

$$g(a^3x) = a^3g(x),$$

and

$$g(a^4x) = a^4g(x), \tag{5}$$

for all $x \in X$. Replacing (x, y) by $(ax_1 + bx_2 + cx_3, dx_4)$ in (1), we obtain

$$g(ax_1 + bx_2 + cx_3 + dx_4) = g(ax_1 + bx_2) + g(cx_3) + g(dx_4) \tag{6}$$

for all $x_1, x_2, x_3, x_4 \in X$. Again setting x_4 by $-x_4$ in (6), we get

$$g(ax_1 + bx_2 + cx_3 - dx_4) = g(ax_1 + bx_2) + g(cx_3) - g(dx_4) \tag{7}$$

for all $x_1, x_2, x_3, x_4 \in X$. Replacing (x, y) by $(ax_1 + bx_2 + dx_4, cx_3)$ in (1), we obtain

$$g(ax_1 + bx_2 + cx_3 + dx_4) = g(ax_1 + dx_4) + g(bx_2) + g(cx_3) \tag{8}$$

for all $x_1, x_2, x_3, x_4 \in X$. Changing x_3 by $-x_3$ in (8), we get that

$$g(ax_1 + bx_2 - cx_3 + dx_4) = g(ax_1 + dx_4) + g(bx_2) - g(cx_3) \tag{9}$$

for all $x_1, x_2, x_3, x_4 \in X$. Replacing (x, y) by $(ax_1 + cx_3 + dx_4, bx_2)$ in (1), we obtain

$$g(ax_1 + bx_2 + cx_3 + dx_4) = g(ax_1 + cx_3) + g(dx_4) + g(bx_2) \tag{10}$$

for all $x_1, x_2, x_3, x_4 \in X$. Using (10) and changing x_2 by $-x_2$, we get

$$g(ax_1 - bx_2 + cx_3 + dx_4) = g(ax_1 + cx_3) + g(dx_4) - g(bx_2) \tag{11}$$

for all $x_1, x_2, x_3, x_4 \in X$. Replacing (x, y) by $(bx_2 + cx_3 + dx_4, ax_1)$ in (1), we obtain

$$g(ax_1 + bx_2 + cx_3 + dx_4) = g(bx_2 + dx_4) + g(cx_3) + g(ax_1) \tag{12}$$

for all $x_1, x_2, x_3, x_4 \in X$. Changing x_1 by $-x_1$ in (12), we get

$$g(-ax_1 + bx_2 + cx_3 + dx_4) = g(bx_2 + dx_4) + g(cx_3) - g(ax_1) \tag{13}$$

for all $x_1, x_2, x_3, x_4 \in X$. Adding (6), (7), (8), (9), (10), (11), (12) and (13) and again adding

$g(ax_1) - g(-ax_1) + g(bx2) - g(-bx_2) + g(cx_3) - g(-cx_3) + g(dx_4) - g(-dx_4)$

$-[a(g(x_1) - g(-x_1)) + b(g(x_2) - g(-x_2)) + c(g(x_3) - g(-x_3)) + d(g(x_4) - g(-x_4))]$

on both sides and using oddness, we get desired our result of (3).

Conversely, let $g : X \to Y$ satisfies the functional equation (3), replacing (x, y) by $(0, 0)$ in (3), that

$$g(0) = 0.$$

Setting (x, y) by $\left(\frac{x}{a}, -\frac{x}{b}\right)$ in (3) we get

$$g(-x) = -g(x).$$

for all $x \in X$. Putting (x, y) by $\left(\frac{x}{a}, \frac{x}{b}\right)$ and $\left(\frac{2x}{a}, \frac{x}{b}\right)$ in (3), we obtain

$$g(2x) = 2\ g(x)$$

and

$$g(3x) = 3\ g(x)$$

for all $x \in X$. Again replacing (x, y) by $\left(\frac{x}{a}, \frac{y}{b}\right)$ in (3), and using oddness of (3), we get the desired result of (1). □

Lemma 2.2. *If an even mapping $g : X \to Y$ satisfies the functional equation (2) for all $x, y \in X$ if and only if $g : X \to Y$ satisfies the functional equation (3) for all $x_1, x_2, x_3, x_4 \in X$.*

Proof. Let $f : X \to Y$ satisfies the functional equation (2). Setting (x, y) by $(0, 0)$ in (2), that

$$f(0) = 0.$$

Putting (x, y) by $(0, x)$ in (2), that

$$f(-x) = f(x)$$

for all $x \in X$. Again changing (x, y) by (x, x) and $(2x, x)$ in (2), we obtain

$$f(2x) = 4\ f(x)$$

and

$$f(3x) = 9\ f(x)$$

for all $x \in X$. For any positive integer n, we arrive that

$$f(ax) = a^2 f(x) \tag{14}$$

for all $x \in X$. It is easy that to verify from (14) respectively, we get

$$f(a^2x) = a^4 f(x) \qquad and \qquad f(a^3x) = a^6 f(x) \tag{15}$$

for all $x \in X$. Setting (x, y) by $(ax_1 + dx_4, bx_2 + cx_3)$ in (2), we arrive that

$$g(ax_1 + dx_4 + bx_2 + cx_3) + g(ax_1 + dx_4 - bx_2 - cx_3) = 2\ g(ax_1 + dx_4) + g(bx_2 + cx_3) \tag{16}$$

for all $x_1, x_2, x_3, x_4 \in X$. Again replacing (x, y) by $(ax_1 + cx_3, bx_2 + dx_4)$ in (2), that

$$g(ax_1 + cx_3 + bx_2 + dx_4) + g(ax_1 + cx_3 - bx_2 - dx_4) = 2\ g(ax_1 + cx_3) + g(bx_2 + dx_4) \tag{17}$$

for all $x_1, x_2, x_3, x_4 \in X$. Putting (x, y) by $(ax_1 + bx_2, cx_3 + dx_4)$ in (2), we get

$$g(ax_1 + bx_2 + cx_3 + dx_4) + g(ax_1 + bx_2 - cx_3 - dx_4) = 2\ g(ax_1 + bx_2) + g(cx_3 + dx_4) \tag{18}$$

for all $x_1, x_2, x_3, x_4 \in X$. Putting (x, y) by $(ax_1 - bx_2, cx_3 - dx_4)$ in (2), we obtain that

$$g(ax_1 - bx_2 + cx_3 - dx_4) + g(ax_1 - bx_2 - cx_3 + dx_4) = 2\ g(ax_1 - bx_2) + 2\ g(cx_3 - dx_4) \tag{19}$$

for all $x_1, x_2, x_3, x_4 \in X$. Changing (x, y) by $(cx_3 + dx_4, ax_1 - bx_2)$ in (2), we have

$$g(cx_3 + dx_4 + ax_1 - bx_2) + g(cx_3 + dx_4 - ax_1 + bx_2) = 2\ g(cx_3 + dx_4) + 2\ g(ax_1 - bx_2) \tag{20}$$

for all $x_1, x_2, x_3, x_4 \in X$. Adding (16) and (17), we get

$$g(ax_1 + bx_2 + cx_3 + dx_4) + g(ax_1 - bx_2 - cx_3 + dx_4) + g(ax_1 + bx_2 + cx_3 + dx_4) + g(ax_1 - bx_2 + cx_3 - dx_4)$$

$$= 2\ g(ax_1 + dx_4) + 2\ g(bx_2 + cx_3) + 2\ g(ax_1 + cx_3) + 2\ g(bx_2 + dx_4) \tag{21}$$

for all $x_1, x_2, x_3, x_4 \in X$. Using (19) and (21), we obtain

$$g(ax_1 + bx_2 + cx_3 + dx_4) + 2\ g(ax_1 - bx_2) + 2\ g(cx_3 - dx_4) = 2\ g(ax_1 + dx_4)$$

$$+ 2\ g(bx_2 + cx_3) + 2\ g(ax_1 + cx_3) + 2\ g(bx_2 + dx_4) \tag{22}$$

for all $x_1, x_2, x_3, x_4 \in X$. Using (22) and rearranging, we obtain

$$g(ax_1 + bx_2 + cx_3 + dx_4) + g(ax_1 - bx_2) + g(cx_3 - dx_4) = g(ax_1 + dx_4)$$

$$+ g(bx_2 + cx_3) + g(ax_1 + cx_3) + g(bx_2 + dx_4) \tag{23}$$

for all $x_1, x_2, x_3, x_4 \in X$. Adding $g(ax_1 - bx_2)$ and $g(cx_3 - dx_4)$ on both sides in (23), we get

$$g(ax_1 + bx_2 + cx_3 + dx_4) + 2\ g(ax_1 - bx_2) + 2\ g(cx_3 - dx_4) = g(ax_1 + dx_4) + g(bx_2 + cx_3)$$

$$+ g(ax_1 + cx_3) + g(bx_2 + dx_4) + g(ax_1 - bx_2) + g(cx_3 - dx_4) \tag{24}$$

for all $x_1, x_2, x_3, x_4 \in X$. Adding (18) and (24), we arrive

$$g(ax_1 + bx_2 + cx_3 + dx_4) + 2\ g(ax_1 - bx_2) + 2\ g(cx_3 - dx_4) + g(ax_1 + bx_2 + cx_3 + dx_4)$$

$$+g(ax_1 + bx_2 - cx_3 - dx_4) = 2\ g(ax_1 + bx_2) + 2\ g(cx_3 + dx_4) + g(ax_1 + dx_4) + g(bx_2 + cx_3)$$

$$+ g(ax_1 + cx_3) + g(bx_2 + dx_4) + g(ax_1 - bx_2) + g(cx_3 - dx_4) \tag{25}$$

for all $x_1, x_2, x_3, x_4 \in X$. Using (25) in (3), that

$$g(ax_1 + bx_2 + cx_3 + dx_4) + 2\ g(ax_1 - bx_2) + 2\ g(cx_3 - dx_4) + 2\ g(ax_1 + bx_2) + 2\ g(cx_3 + dx_4)$$

$$= g(ax_1 + bx_2) + g(cx_3 + dx_4) + g(ax_1 + bx_2) + g(cx_3 + dx_4) + g(ax_1 + dx_4) + g(bx_2 + cx_3)$$

$$+ g(ax_1 + cx_3) + g(bx_2 + dx_4) + g(ax_1 - bx_2) + g(cx_3 - dx_4) \tag{26}$$

for all $x_1, x_2, x_3, x_4 \in X$. Using (20) in (26), we get

$$g(ax_1 + bx_2 + cx_3 + dx_4) + g(ax_1 - bx_2 + cx_3 + dx_4) + g(-ax_1 + bx_2 + cx_3 + dx_4) + 2\ g(ax_1 + bx_2)$$

$$+2\ g(cx_3 - dx_4) = g(ax_1 + bx_2) + g(ax_1 + cx_3) + g(ax_1 + dx_4) + g(bx_2 + cx_3) + g(bx_2 + dx_4) +$$

$$g(cx_3 + dx_4) + g(ax_1 + bx_2) + g(ax_1 - bx_2) + g(cx_3 + dx_4) + g(cx_3 - dx_4) \tag{27}$$

for all $x_1, x_2, x_3, x_4 \in X$. Replacing (x, y) by (ax_1, bx_2) in (2), we obtain

$$g(ax_1 + bx_2) + g(ax_1 - bx_2) = 2\ g(ax_1) + 2\ g(bx_2) \tag{28}$$

for all $x_1, x_2 \in X$. Putting (x, y) by (cx_3, dx_4) in (2), we get

$$g(cx_3 + dx_4) + g(cx_3 - dx_4) = 2\ g(cx_3) + 2\ g(dx_4) \tag{29}$$

for all $x_3, x_4 \in X$. Setting (x, y) by $(ax_1 + bx_2, cx_3 - dx_4)$ in (2), we obtain

$$g(ax_1 + bx_2 + cx_3 - dx_4) + g(ax_1 + bx_2 - cx_3 + dx_4) = 2\ g(ax_1 + bx_2) + 2\ g(cx_3 - dx_4) \tag{30}$$

for all $x_1, x_2, x_3, x_4 \in X$. Substitute (28), (29) and (30) in (27) respectively, we get

$$g(ax_1 + bx_2 + cx_3 + dx_4)\ +\ g(-ax_1 + bx_2 + cx_3 + dx_4)\ +\ g(ax_1 - bx_2 + cx_3 + dx_4)\ +$$

$$g(ax_1 + bx_2 - cx_3 + dx_4)\ +\ g(ax_1 + bx_2 + cx_3 - dx_4)\ =\ g(ax_1 + bx_2) + g(ax_1 + cx_3) +$$

$$g(ax_1 + dx_4) + g(bx_2 + cx_3) + g(bx_2 + dx_4) + g(cx_3 + dx_4) + 2\ a^2\ g(x_1) + 2\ b^2\ g(x_2)$$

$$+ 2\ c^2\ g(x_3) + 2\ d^2\ g(x_4) \tag{31}$$

for all $x_1, x_2, x_3, x_4 \in X$. Adding

$$g(ax_1) - g(-ax_1) + g(bx_2) - g(-bx_2) + g(cx_3) - g(-cx_3) + g(dx_4) - g(-dx_4) -$$

$$[a\,(g(x_1) - g(-x_1)) + b\,(g(x_2) - g(-x_2)) + c\,(g(x_3) - g(-x_3)) + d\,(g(x_4) - g(-x_4))]$$

and using eveness, we get desired result for (3). Conversely, Let $g : X \to Y$ satisfies the functional equation (2). Replacing (x_1, x_2, x_3, x_4) by $(x_1, x_2, 0, 0)$ in (3), we get

$$g(ax_1 + bx_2) + g(-ax_1 + bx_2) + g(ax_1 - bx_2) + g(ax_1 + bx_2) + g(ax_1 + bx_2) = g(ax_1 + bx_2) +$$

$$g(ax_1) + g(ax_1) + g(bx_2) + g(bx_2) + 2\ a^2\ g(x_1) + 2\ b^2\ g(x_2) \tag{32}$$

for all $x_1, x_2 \in X$. Using eveness and it is easy to verify from (14), we arrive (2). Hence this completes the proof of the Theorem. □

3. Stability Results for (3): Odd Case-Direct Method

In this section, we present the generalized Ulam - Hyers stability of the functional equation (3) for odd case.

Theorem 3.1. *Let $l \in \{-1, 1\}$ and $\alpha : X^4 \to [0, \infty)$ be a function such that*

$$\sum_{k=0}^{\infty} \frac{\alpha\left(a^{kj}x_1, a^{kj}x_2, a^{kj}x_3, a^{kj}x_4\right)}{a^{kj}}$$

converges in R and

$$\lim_{k\to\infty} \frac{\alpha\left(a^{kj}x_1, a^{kj}x_2, a^{kj}x_3, a^{kj}x_4\right)}{a^{kj}} \tag{33}$$

for all $x_1, x_2, x_3, x_4 \in X$. Let $g_a : X \to Y$ be an odd function satisfying the inequality

$$\|Dg_a(x_1, x_2, x_3, x_4)\| \leq \alpha(x_1, x_2, x_3, x_4) \tag{34}$$

for all $x_1, x_2, x_3, x_4 \in X$. There exists a unique additive mapping $A : X \to Y$ which satisfies the functional equation (3) and

$$\|g_a(x) - A(x)\| \leq \frac{1}{2a} \sum_{k=\frac{i-j}{2}}^{\infty} \frac{\alpha\left(a^{kj}x, 0, 0, 0\right)}{a^{kj}} \tag{35}$$

for all $x \in X$. The mapping $A(x)$ is defined by

$$A(x) = \lim_{k\to\infty} \frac{g_a\left(a^{kj}x\right)}{a^{kj}} \tag{36}$$

for all $x \in X$.

Proof. Assume that $l = 1$. Replacing (x_1, x_2, x_3, x_4) by $(x, 0, 0, 0)$ in (34) and using oddness of g_a, we get

$$\|2g_a(ax) - 2a\ g_a(x)\| \leq \alpha(x, 0, 0, 0) \tag{37}$$

for all $x \in X$. It is follows from (37) that

$$\left\|\frac{g_a(ax)}{a} - g_a(x)\right\| \leq \frac{1}{2a}\ \alpha(x, 0, 0, 0) \tag{38}$$

for all $x \in X$. Replacing x by ax in (38) and dividing by a, we obtain

$$\left\|\frac{g_a(a^2x)}{a^2} - \frac{g_a(ax)}{a}\right\| \leq \frac{1}{2a^2}\ \alpha(ax, 0, 0, 0) \tag{39}$$

for all $x \in X$. It is follows from (38) and (39) that

$$\left\|\frac{g_a(a^2x)}{a^2} - g_a(ax)\right\| \leq \frac{1}{2a}\ \left\{\alpha(x, 0, 0, 0) + \frac{\alpha(ax, 0, 0, 0)}{a}\right\} \tag{40}$$

for all $x \in X$. Generalizing the above, we arrive that

$$\left\|g_a(ax) - \frac{g_a(a^kx)}{a^k}\right\| \leq \frac{1}{2a}\ \left\{\sum_{k=0}^{n-1} \frac{\alpha(a^kx, 0, 0, 0)}{a^k}\right\}$$

$$\leq \frac{1}{2a} \left\{ \sum_{k=0}^{\infty} \frac{\alpha(a^k x, 0, 0, 0)}{a^k} \right\} \tag{41}$$

for all $x \in X$. In order to prove that the convergence of the sequence

$$\left\{ \frac{g_a(a^k x)}{a^k} \right\}$$

replace x by $a^l x$ and dividing a^l (41), for any $k, l > 0$, to deduce that, we will arrive

$$\left\| \frac{g_a(a^l x)}{a^l} - \frac{g_a(a^{k+l} x)}{a^{k+l}} \right\| = \frac{1}{2l} \left\| g_a(a^l x) - \frac{g_a(a^k a^l x)}{a^k} \right\|$$

$$\leq \frac{1}{2a} \left\{ \sum_{k=0}^{n-1} \frac{\alpha(a^{k+l} x, 0, 0, 0)}{a^{k+l}} \right\}$$

$$\leq \frac{1}{2a} \left\{ \sum_{k=0}^{\infty} \frac{\alpha(a^{k+l} x, 0, 0, 0)}{a^{k+l}} \right\} \rightarrow 0 \qquad as \quad l \rightarrow \infty \tag{42}$$

for all $x \in X$. Hence the sequence

$$\left\{ \frac{g_a(a^k x)}{a^k} \right\}$$

is a Cauchy sequence. Since Y is complete, there exists a mapping $A : X \rightarrow Y$ such that

$$A(x) = \lim_{k \rightarrow 0} \frac{g_a(a^k x)}{a^k}$$

for all $x \in X$. Letting $k \rightarrow \infty$ in (33), we see that (35) holds for all $x \in X$. To prove that A satisfies (3), replacing (x_1, x_2, x_3, x_4) by $(a^k x_1, a^k x_2, a^k x_3, a^k x_4)$ and dividing a^k in (34) we obtain

$$\frac{1}{a^k} \left\| Dg_a(a^k x_1, a^k x_2, a^k x_3, a^k x_4) \right\| \leq \frac{1}{a^k} \alpha(a^k x_1, a^k x_2, a^k x_3, a^k x_4)$$

for all $x_1, x_2, x_3, x_4 \in X$. Letting $k \rightarrow \infty$ in the above inequality and using the definition of $A(x)$, we see that

$$D_A(x_1, x_2, x_3, x_4) = 0.$$

Hence, A satisfies the equation (3) for all $x_1, x_2, x_3, x_4 \in X$. To show that A is unique, let B be another additive mapping satisfying (3) and (34), then

$$\|A(x) - B(x)\| \leq \frac{1}{a^l} \left\{ \left\| A(a^l x) - g_a(a^l x) \right\| + \left\| g_a(a^l x) - B(a^l x) \right\| \right\}$$

$$\leq \frac{1}{2a} \sum_{k=0}^{\infty} \frac{\alpha(a^{k+l} x, 0, 0, 0)}{a^{k+l}} \rightarrow 0 \quad as \qquad l \rightarrow \infty,$$

for all $x \in X$. Hence A is Unique. Now, replacing x by $\left(\frac{x}{a}\right)$ in (37), we get

$$\left\| 2\ g_a(x) - 2\ g_a\left(\frac{x}{a}\right) \right\| \leq \alpha\left(\frac{x}{a}, 0, 0, 0\right) \tag{43}$$

for all $x \in X$. It is follows from (43) that

$$\left\| g_a(x) - g_a\left(\frac{x}{a}\right) \right\| \leq \frac{1}{2} \alpha\left(\frac{x}{a}, 0, 0, 0\right) \tag{44}$$

for all $x \in X$. The rest of proof is similar to that $l = 1$. Hence for $l = -1$ also the theorem is true. This completes the proof of the Theorem. □

The following corollary is an immediate consequence of of Theorem 3.1 concerning the stability of (3).

Corollary 3.2. *Let μ and p be a non negative real number. Let an odd function $g_a : X \to Y$ satisfying the inequality*

$$\|Dg_a(x_1,x_2,x_3,x_4)\| \leq \begin{cases} \mu; \\ \mu\left\{\|x_1\|^p + \|x_2\|^p + \|x_3\|^p + \|x_4\|^p\right\}; \\ \mu\left\{\{\|x_1\|^p \|x_2\|^p \|x_3\|^p \|x_4\|^p\} + \{\|x_1\|^{4p} + \|x_2\|^{4p} + \|x_3\|^{4p} + \|x_4\|^{4p}\}\right\}; \end{cases} \tag{45}$$

for all $x_1,x_2,x_3,x_4 \in X$. Then there exists a unique additive function $A : X \to Y$ such that

$$\|g_a(x) - A(x)\| \leq \begin{cases} \frac{\mu}{2\ |a-1|}; \\ \frac{\mu\ \|x\|^p}{2\ |a-a^p|}; \\ \frac{\mu\ \|x\|^{4p}}{2\ |a-a^{4p}|}; \end{cases} \tag{46}$$

for all $x \in X$.

Proof. If we putting

$$\alpha(x_1,x_2,x_3,x_4) = \begin{cases} \mu; \\ \mu\left\{\|x_1\|^p + \|x_2\|^p + \|x_3\|^p + \|x_4\|^p\right\}; \\ \mu\left\{\{\|x_1\|^p \|x_2\|^p \|x_3\|^p \|x_4\|^p\} + \{\|x_1\|^{4p} + \|x_2\|^{4p} + \|x_3\|^{4p} + \|x_4\|^{4p}\}\right\}; \end{cases} \tag{47}$$

in theorem 3.1, we get (46). □

4. Stability Results for (3): Even Case-Direct Method

In this section, we present the generalized Ulam - Hyers stability of the functional equation (3) for even case.

Theorem 4.1. *Let $l \in \{-1,1\}$ and $\alpha : X^4 \to [0,\infty)$ be a function such that*

$$\sum_{k=0}^{\infty} \frac{\alpha\left(a^{kj}x_1, a^{kj}x_2, a^{kj}x_3, a^{kj}x_4\right)}{a^{2kj}}$$

converges to R and

$$\lim_{k\to\infty} \frac{\alpha\left(a^{kj}x_1, a^{kj}x_2, a^{kj}x_3, a^{kj}x_4\right)}{a^{2kj}} = 0 \tag{48}$$

for all $x_1,x_2,x_3,x_4 \in X$. Let $g_q : X \to Y$ be an even function satisfying the inequality

$$\|Dg_q(x_1,x_2,x_3,x_4)\| \leq \alpha(x_1,x_2,x_3,x_4) \tag{49}$$

for all $x_1,x_2,x_3,x_4 \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ which satisfies the functional inequality (3) and

$$\|g_q(x) - Q(x)\| \leq \frac{1}{4a^2} \sum_{k=\frac{i-j}{2}}^{\infty} \frac{\alpha\left(a^{kj}x, 0, 0, 0\right)}{a^{2kj}} \tag{50}$$

for all $x \in X$. The mapping $Q(x)$ is defined by

$$Q(x) = \lim_{k\to\infty} \frac{g_q\left(a^{kj}x\right)}{a^{2kj}} \tag{51}$$

for all $x \in X$.

Proof. Assume that $l = 1$. Setting (x_1, x_2, x_3, x_4) by $(x, 0, 0, 0)$ in (49) and using the evenness of g_q, we get

$$\left\|4g_q(ax) - 4a^2\ g_q(x)\right\| \leq \alpha(x, 0, 0, 0) \tag{52}$$

for all $x \in X$. The rest of the proof is similar to that Theorem 3.1. □

Corollary 4.2. *Let μ and p be a non negative real number. Let an even function $g_q : X \to Y$ satisfying the inequality*

$$\|Dg_q(x_1, x_2, x_3, x_4)\| \leq \begin{cases} \mu; \\ \mu \left\{\|x_1\|^p + \|x_2\|^p + \|x_3\|^p + \|x_4\|^p\right\}; & p \neq 2 \\ \mu \left\{\left\{\|x_1\|^p \|x_2\|^p \|x_3\|^p \|x_4\|^p\right\} + \left\{\|x_1\|^{4p} + \|x_2\|^{4p} + \|x_3\|^{4p} + \|x_4\|^{4p}\right\}\right\}; \\ \qquad p \neq \frac{2}{4} \end{cases} \tag{53}$$

for all $x_1, x_2, x_3, x_4 \in X$. Then there exists a unique quadratic function $Q : X \to Y$ such that

$$\|g_q(x) - Q(x)\| \leq \begin{cases} \frac{\mu}{4\ \left|a^2-1\right|}; \\ \frac{\mu\ \|x\|^p}{4\ \left|a^2-a^p\right|}; & p \neq 2 \\ \frac{\mu\ \|x\|^{4p}}{4\ \left|a^2-a^{4p}\right|}; & p \neq \frac{1}{2} \end{cases} \tag{54}$$

for all $x \in X$.

5. Stability Results for (3): Mixed Case-Direct Method

In this section, we investigate the generalized Ulam - Hyers stability lf the functional equations (3) for the mixed case.

Theorem 5.1. *Let $l \in \{-1, 1\}$ and $\alpha : X^4 \to [0, \infty)$ be a function satisfying (33) and (48) for all $x_1, x_2, x_3, x_4 \in X$. Let $g : X \to Y$ be a function satisfying the inequality such that*

$$\|Dg(x_1, x_2, x_3, x_4)\| \leq \alpha(x_1, x_2, x_3, x_4) \tag{55}$$

for all $x_1, x_2, x_3, x_4 \in X$. Then there exists a unique additive mapping and unique quadratic mapping $Q : X \to Y$ which satisfies the functional inequality (3) and

$$\|g(x) - A(x) - Q(x)\| \leq \frac{1}{2}\left[\frac{1}{2a}\sum_{k=\frac{i-j}{2}}^{\infty}\left\{\frac{\alpha\left(a^{kj}x, 0, 0, 0\right)}{a^{kj}} + \frac{\alpha\left(-a^{kj}x, 0, 0, 0\right)}{a^{kj}}\right\}\right.$$
$$\left. + \frac{1}{4a^2}\sum_{k=\frac{i-j}{2}}^{\infty}\left\{\frac{\alpha\left(a^{kj}x, 0, 0, 0\right)}{a^{2kj}} + \frac{\alpha\left(-a^{kj}x, 0, 0, 0\right)}{a^{2kj}}\right\}\right] \tag{56}$$

for all $x \in X$. The mapping $A(x)$ and $Q(x)$ is defined in (36) and (51) respectively for all $x \in X$.

Proof. Let us take

$$g_o(x) = \frac{g_a(x) - g_a(x)}{2}$$

for all $x \in X$. Then $g_o(0) = 0$ and $g_o(-x) = -g_o(x)$ for all $x \in X$. Hence

$$\|Dg_o(x_1, x_2, x_3, x_4)\| \leq \frac{1}{2}\left\{\|Dg_o(x_1, x_2, x_3, x_4)\| + \|Dg_o(-x_1, -x_2, -x_3, -x_4)\|\right\}$$

$$\leq \left\{\frac{\alpha(x_1, x_2, x_3, x_4)}{2} + \frac{\alpha(-x_1, -x_2, -x_3, -x_4)}{2}\right\} \tag{57}$$

for all $x_1, x_2, x_3, x_4 \in X$. By Theorem 3.1, we obtain

$$\|g_o(x) - A(x)\| \leq \frac{1}{4a}\sum_{k=\frac{i-j}{2}}^{\infty}\left\{\frac{\alpha\left(a^{kj}x, 0, 0, 0\right)}{a^{kj}} + \frac{\alpha\left(-a^{kj}x, 0, 0, 0\right)}{a^{kj}}\right\} \tag{58}$$

for all $x \in X$. Also let,

$$g_e(x) = \frac{g_q(x) + g_q(-x)}{2}$$

for all $x \in X$. Then $g_e(0) = 0$ and $g_e(-x) = g_e(x)$ for all $x \in X$. Hence

$$\|Dg_e(x_1, x_2, x_3, x_4)\| \leq \frac{1}{2}\left\{\|Dg_q(x_1, x_2, x_3, x_4)\| + \|Dg_q(-x_1, -x_2, -x_3, -x_4)\|\right\}$$

$$\leq \left\{\frac{\alpha(x_1, x_2, x_3, x_4)}{2} + \frac{\alpha(-x_1, -x_2, -x_3, -x_4)}{2}\right\} \tag{59}$$

for all $x_1, x_2, x_3, x_4 \in X$. By Theorem 4.1, we obtain

$$\|g_e(x) - Q(x)\| \leq \frac{1}{8a^2}\sum_{k=\frac{i-j}{2}}^{\infty}\left\{\frac{\alpha\left(a^{kj}x, 0, 0, 0\right)}{a^{2kj}} + \frac{\alpha\left(-a^{kj}x, 0, 0, 0\right)}{a^{2kj}}\right\} \tag{60}$$

for all $x \in X$. Define

$$g(x) = g_e(x) + g_e(-x) \tag{61}$$

for all $x \in X$. It is follows from (58), (60) and (61), we get

$$\|g(x) - A(x) - Q(x)\| = \|g_e(x) + g_o(-x) - A(x) - Q(x)\|$$

$$\leq \|g_o(-x) - A(x)\| + \|g_e(x) - Q(x)\|$$

$$\leq \frac{1}{4a}\sum_{k=\frac{i-j}{2}}^{\infty}\left\{\frac{\alpha\left(a^{kj}x, 0, 0, 0\right)}{a^{kj}} + \frac{\alpha\left(-a^{kj}x, 0, 0, 0\right)}{a^{kj}}\right\}$$

$$+ \frac{1}{8a^2}\sum_{k=\frac{i-j}{2}}^{\infty}\left\{\frac{\alpha\left(a^{kj}x, 0, 0, 0\right)}{a^{2kj}} + \frac{\alpha\left(-a^{kj}x, 0, 0, 0\right)}{a^{2kj}}\right\} \tag{62}$$

for all $x \in X$. Hence the Theorem is proved. □

Using the corollaries 3.2 and 4.2, we have the following corolary concerning the stability of (3).

Corollary 5.2. *Let μ and p be a non negative real number. Let an even function $g : X \to Y$ satisfying the inequality*

$$\|Dg(x_1, x_2, x_3, x_4)\| \leq \begin{cases} \mu; & \\ \mu\left\{\|x_1\|^p + \|x_2\|^p + \|x_3\|^p + \|x_4\|^p\right\}; & p \neq 1, 2 \\ \mu\left\{\left\{\|x_1\|^p\,\|x_2\|^p\,\|x_3\|^p\,\|x_4\|^p\right\} + \left\{\|x_1\|^{4p} + \|x_2\|^{4p} + \|x_3\|^{4p} + \|x_4\|^{4p}\right\}\right\}; & \\ & p \neq \frac{1}{4}, \frac{2}{4} \end{cases} \tag{63}$$

for all $x_1, x_2, x_3, x_4 \in X$. Then there exists a unique additive mapping $A : X \to Y$ function $Q : X \to Y$ such that

$$\|g(x) - A(x) - Q(x)\| \leq \begin{cases} \frac{\mu}{2}\left\{\frac{1}{|a-1|} + \frac{1}{2\,|a^2-1|}\right\}; \\ \frac{\mu\ \|x\|^p}{2}\left\{\frac{1}{|a-a^p|} + \frac{1}{2|a^2-a^p|}\right\}; \\ \frac{\mu\ \|x\|^{4p}}{2}\left\{\frac{1}{|a-a^{4p}|} + \frac{1}{|a^2-a^{4p}|}\right\}; \end{cases} \tag{64}$$

for all $x \in X$.

For next sections 6, 7 and 8, let us consider X and Y to be a normed space and a Banach space, respectively. Define a mapping $Dg : X \to Y$ by

$$Dg(x_1, x_2, x_3, x_4) = 0.$$

for all $x_1, x_2, x_3, x_4 \in X$.

Fixed Point Stability Results for the Functional Equation (3).

The following theorem are useful to prove our fixed point stability result.

Theorem A: (Banach Contradiction Principle)[4].

Let (X, d) be a complete metric spaces and consider a mapping $P : X \to X$ which is strictly contractive mapping, that is

$(A1)$. $d(Px, Py) \leq d(x, y)$ for some (**Lipschitz Constant**) $L < 1$, then

1. The mapping P has one and only fixed point $x^* = P(x^*)$.

2. The fixed point for each given element x^* is globally contractive that is

$(A2)$. $\lim_{n\to\infty} P^n x = x^*$

For any starting point $x \in X$. One has the following estimation inequalities

$(A3)$. $d(P^n x, x^*) \leq \frac{1}{1-L}\ d(P^n x, P^{n+1}x)$, for all $n \geq 0$ and for all $x \in X$.

$(A4)$. $d(x, x^*) \leq \frac{1}{1-L}\ d(x, x^*)$, for all $x \in X$.

Theorem B: (The Alternative Fixed Point Theorem) [4].

Suppose that for a complete generalized metric space (X, d) and strictly contractive mapping $T : X \to Y$ with Lipschitz constant L, then for each given element $x \in X$, either,

$(B1)$. $d(P^n x, P^{n+1}x) = \infty$; for all $n \geq 0$

$(B2)$. There exists a natural number n_0 such that

1. $d(P^n x, P^{n+1}x) = \infty$; for all $n \geq 0$.

2. The sequence $\{T^n x\}$ is convergent to a fixed point y^* of T,

3. y^* is the unique fixed point of T in the set

$$Y \ = \ \{y \in Y \ : \ d(P^{n_0}x, y) \ < \ \infty\}$$

4. $d(y^*, y) \ \leq \frac{1}{1-L}\ d(y, Py)$ for all $y \in Y$.

6. Fixed Point Stability of (3) : odd case - Fixed Point Method

In this section, we present the generalized Ulam - Hyers stability of the functional equation (3) for odd case using fixed point method.

Theorem 6.1. *Let $g_a : X \to Y$ be an odd mapping for which there exists a function $\alpha : X^4 \to [0,\infty)$ with the condition*

$$\lim_{k\to\infty}\left\{\frac{\alpha\left(\nu_i^k x_1, \nu_i^k x_2, \nu_i^k x_3, \nu_i^k x_4\right)}{\nu_i^k}\right\} = 0 \tag{65}$$

where

$$\nu_i = \begin{cases} a & ; \ i = 0 \\ \frac{1}{a} & ; \ i = 1 \end{cases}$$

such that the functional inequality

$$\|Dg_a(x_1, x_2, x_3, x_4)\| \ \leq \ \alpha(x_1, x_2, x_3, x_4) \tag{66}$$

for all $x_1, x_2, x_3, x_4 \in X$. If there exists $L = L(i)$ such that the function

$$x \to \delta(x) = \frac{1}{2}\alpha\left(\frac{x}{a}, 0, 0, 0\right)$$

has the property

$$\frac{1}{\nu_i} \ \delta(\nu_i x) \ = \ L \ \delta(x) \tag{67}$$

for all $x \in X$. Then there exists a unique additive function $A : X \to Y$ satisfying the functional equaiton (3) and

$$\|g_a(x) - A(x)\| \leq \frac{L^{1-i}}{1-L} \ \delta(x) \tag{68}$$

holds for all $x \in X$.

Proof. Consider the set

$$X = \{q \ / \ q : X \to Y \ ; \ q(0) = 0\}$$

and introduced the generalized metric on X.

$$d(r, q) = inf\left\{k \in (0,\infty) \ : \ \|q(x) - r(x)\| \leq k \ \delta(x), \ \forall \ x \in X\right\}.$$

It is easy to see that (X, d) is complete. Define $T : X \to X$ by

$$T_q(x) = \frac{1}{\nu_i} \ q(\nu_i x)$$

for all $x \in X$. Now $r, q \in X$, then we have $d(q, r) \ \leq k$

$$\Rightarrow \qquad \|q(x) - r(x)\| \ \leq \ k \ \delta(x), \qquad x \in X$$

$$\Rightarrow \qquad \left\|\frac{1}{\nu_i} \ q(\nu_i x) \ - \ \frac{1}{\nu_i} \ r(\nu_i x)\right\| \ \leq \ \frac{1}{\nu_i} \ k \ \delta(\nu_i x) \qquad x \in X$$

$$\Rightarrow \qquad \|T_q(x) - T_r(x)\| \ \leq \ L \ k \ \delta(x), \qquad x \in X$$

$$\Rightarrow \quad d(T_q, T_r) \leq L\ K.$$

$$\Rightarrow \quad d(T_q, T_r) \leq L\ d(q, r)$$

for all $q, r \in X$. That is, T is strictly contractive mapping on X with Lipschitz constant L. It is follows from (37) that

$$\|2\ g_a(ax) - 2\ a\ g_a(x)\| \leq \alpha(x, 0, 0, 0) \tag{69}$$

for all $x \in X$. It is follows from (69) that

$$\left\| g_a(x) - \frac{g_a(x)}{a} \right\| \leq \frac{1}{2a}\ \alpha(x, 0, 0, 0) \tag{70}$$

for all $x \in X$. using (67), for the case $i = 0$, it it reduces to

$$\left\| g_a(x) - \frac{1}{a}\ g_a(x) \right\| \leq \frac{1}{a}\ \delta(x) \tag{71}$$

for all $x \in X$. That is, $d(g_a, Tg_a) \leq \frac{1}{a}$, therefore,

$$d(g_a, Tg_a) \leq \frac{1}{a} = L = L^1 < \infty.$$

Again Replacing x by $\frac{x}{a}$ in (69), we arrive that

$$\left\| g_a(x) - a\ g_a(\frac{x}{a}) \right\| \leq \frac{1}{2}\ \alpha(\frac{x}{a}, 0, 0, 0) \tag{72}$$

for all $x \in X$. using (67), for the case $i = 1$, it it reduces to

$$\left\| g_a(x) - a\ g_a(\frac{x}{a}) \right\| \leq \delta(x) \tag{73}$$

for all $x \in X$. Therefore we arrive that $d(g_a, Tg_a) \leq 1$,

$$\Rightarrow \quad d(g_a, Tg_a) \leq 1 = L^0 < \infty.$$

In the above case, we arrive that

$$d(g_a, Tg_a) \leq L^{1-i}.$$

Therefore $(B2(1))$ holds. By $(B2(2))$, it follows that there exists a fixed point A of T in X, such that

$$A(x) = \lim_{k\to\infty} \frac{g_a(\nu_i^k x)}{\nu_i^k} \tag{74}$$

for all $x \in X$. In order to prove $A : X \to Y$ is additive.

Replacing (x_1, x_2, x_3, x_4) by $(\nu_i^k x_1, \nu_i^k x_2, \nu_i^k x_3, \nu_i^k x_4)$ in (66) and dividing by ν_i^k, it follows from (65) and (74), we see that A satisfies (3) for all $x_1, x_2, x_3, x_4 \in X$. Hence A satisfies the functional equation (3).

By $(B2(3))$, A is the unique fixed point of T in the set,

$$Y = \{g_a \in X\ :\ d(Tg_a, A)\ \infty\}.$$

Using the fixed point Alternataive result, A is the unique function such that,

$$\|g_a(x) - A(x)\| \leq \ k \ \delta(x),$$

for all $x \in X$, and $k > 0$. Finally by $(B2(4))$, we obtain

$$d(g_a, A) \ \leq \ \frac{1}{1-L} \ d(g_a, Tg_a).$$

Therefore,

$$d(g_a, A) \ \leq \ \frac{L^{1-i}}{1-L}.$$

Hence, we conclude that

$$\|g_a(x) - A(x)\| \leq \ \frac{L^{1-i}}{1-L} \ \delta(x).$$

for all $x \in X$. □

Corollary 6.2. *Let $g_a : X \to Y$ be an odd mapping and there exists a real numbers p and μ such that*

$$\|Dg(x_1,x_2,x_3,x_4)\| \leq \begin{cases} \mu; \\ \mu\left\{\|x_1\|^p + \|x_2\|^p + \|x_3\|^p + \|x_4\|^p\right\}; \\ \mu\left\{\{\|x_1\|^p \|x_2\|^p \|x_3\|^p \|x_4\|^p\} + \left\{\|x_1\|^{4p} + \|x_2\|^{4p} + \|x_3\|^{4p} + \|x_4\|^{4p}\right\}\right\}; \end{cases} \tag{75}$$

for all $x_1, x_2, x_3, x_4 \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|g_a(x) - A(x)\| \leq \begin{cases} \frac{\mu}{2\ |a-1|}; & \\ \frac{\mu\ \|x\|^p}{2\ |a-a^p|}; & p \neq 1 \\ \frac{\mu\ \|x\|^{4p}}{2\ \left|a-a^{4p}\right|}; & p \neq \frac{1}{4} \end{cases} \tag{76}$$

for all $x \in X$.

Proof. Replacing

$$\alpha(x_1,x_2,x_3,x_4) = \begin{cases} \mu; \\ \mu\left\{\|x_1\|^p + \|x_2\|^p + \|x_3\|^p + \|x_4\|^p\right\}; \\ \mu\left\{\{\|x_1\|^p \|x_2\|^p \|x_3\|^p \|x_4\|^p\} + \left\{\|x_1\|^{4p} + \|x_2\|^{4p} + \|x_3\|^{4p} + \|x_4\|^{4p}\right\}\right\}; \end{cases}$$

for all $x_1, x_2, x_3, x_4 \in X$. Now $\alpha(\nu_i^k x_1, \nu_i^k x_2, \nu_i^k x_3, \nu_i^k x_4)$

$$= \begin{cases} \frac{\mu}{\nu_i^k}; \\ \frac{\mu}{\nu_i^k}\left\{\left\|\nu_i^k x_1\right\|^p + \left\|\nu_i^k x_2\right\|^p + \left\|\nu_i^k x_3\right\|^p + \left\|\nu_i^k x_4\right\|^p\right\}; \\ \frac{\mu}{\nu_i^k}\left\{\left\{\left\|\nu_i^k x_1\right\|^p \left\|\nu_i^k x_2\right\|^p \left\|\nu_i^k x_3\right\|^p \left\|\nu_i^k x_4\right\|^p\right\} + \left\{\left\|\nu_i^k x_1\right\|^{4p} + \left\|\nu_i^k x_2\right\|^{4p} + \left\|\nu_i^k x_3\right\|^{4p} + \left\|\nu_i^k x_4\right\|^{4p}\right\}\right\}; \end{cases}$$

$$= \begin{cases} \longrightarrow 0 \quad as \quad k \to \infty \\ \longrightarrow 0 \quad as \quad k \to \infty \\ \longrightarrow 0 \quad as \quad k \to \infty \end{cases} \tag{77}$$

Therefore (65) holds. But we have

$$\delta(x) = \frac{1}{2}\ \alpha\left(\frac{x}{a}, 0, 0, 0\right).$$

Hence

$$\delta(x) = \frac{1}{2}\ \alpha\left(\frac{x}{a}, 0, 0, 0\right) = \begin{cases} \frac{\mu}{2}; \\ \frac{\mu}{2a^p}\ \ \|x\|^p; \\ \frac{\mu}{2a^{4p}}\ \ \|x\|^{4p}. \end{cases}$$

Also,

$$\frac{1}{\nu_i x}\delta(\nu_i x) = \begin{cases} \frac{\mu}{2\nu_i}; \\ \frac{\mu}{2\nu_i}\ \ \|\nu_i x\|^p; \\ \frac{\mu}{2\nu_i}\ \ \|\nu_i x\|^{4p}. \end{cases} = \begin{cases} \nu_i^{-1}\delta(x) \\ \nu_i^{p-1}\delta(x) \\ \nu_i^{4p-1}\delta(x) \end{cases} \tag{78}$$

Hence the inequality (67) holds. For

$L = a^{-1}$ for $p = 0$ if $i = 0$ and $L = \frac{1}{a^{-1}}$ for $p = 0$; if $i = 1$ and for

$L = a^{p-1}$ for $p < 1$ if $i = 0$ and $L = \frac{1}{a^{p-1}}$ for $p > 1$; if $i = 1$ and for

$L = a^{4p-1}$ for $p < 1$ if $i = 0$ and $L = \frac{1}{a^{4p-1}}$ for $p > 1$; if $i = 1$.

Now from (78), we prove the following cases:

Case : I Let $L = a^{-1}$; $i = 0$

$$\|g_a(x) - A(x)\| \le \frac{L^{1-i}}{1-L}\ \delta(x) = \frac{(a^{-1})^{1-0}}{1-a^{-1}} \cdot \frac{\mu}{2}$$

$$= \frac{a^{-1}}{1-\frac{1}{a}} \cdot \frac{\mu}{2} = \frac{a^{-1}}{\frac{a-1}{a}} \cdot \frac{\mu}{2} = \frac{\mu}{2(a-1)}.$$

Case : II Let $L = \left(\frac{1}{a}\right)^{-1}$; $i = 1$

$$\|g_a(x) - A(x)\| \le \frac{L^{1-i}}{1-L}\ \delta(x) = \frac{((a)^{1-1})}{(1-a)} \cdot \frac{\mu}{2} = \frac{\mu}{2(1-a)}.$$

Case : III

Let $L = a^{p-1}$; $p < 1$; $i = 0$

$$\|g_a(x) - A(x)\| \le \frac{L^{1-i}}{1-L}\ \delta(x) = \frac{((a)^{p-1})^{1-0}}{(1-a^{p-1})} \cdot \frac{\mu}{2a^p}\ \ \|x\|^p$$

$$= \frac{(a)^p\ a^{-1}}{1-\frac{a^p}{a^{-1}}} \cdot \frac{\mu}{2a^p}\ \ \|x\|^p = \frac{\mu}{2\left(a-a^p\right)}\ \ \|x\|^p$$

Case : IV

Let $L = \frac{1}{a^{p-1}}$; $p > 1$; $i = 1$

$$\|g_a(x) - A(x)\| \le \frac{L^{1-i}}{1-L}\ \delta(x) = \frac{\left(\frac{1}{a^{p-1}}\right)^{1-1}}{1-\frac{1}{a^{p-1}}} \cdot \frac{\mu}{2a^p}\ \ \|x\|^p$$

$$= \frac{1}{1-\frac{1}{a^p}} \cdot \frac{\mu}{2a^p}\ \ \|x\|^p = \frac{\mu}{2\left(a^p-a\right)}\ \ \|x\|^p$$

Case : V

Let $L = a^{4p-1}$; $p < \frac{1}{4}$; $i = 0$

$$\|g_a(x) - A(x)\| \le \frac{L^{1-i}}{1-L}\ \delta(x) = \frac{((a)^{4p-1})^{1-0}}{(1-a^{4p-1})} \cdot \frac{\mu}{2a^{4p}}\ \ \|x\|^{4p}$$

$$= \frac{(a)^{4p}\ a^{-1}}{1 - \frac{a^{4p}}{a^{-1}}} \cdot \frac{\mu}{2a^{4p}}\ \|x\|^{4p} = \frac{\mu}{2\left(a - a^{4p}\right)}\ \|x\|^{4p}$$

Case : VI

Let $L = \frac{1}{a^{4p-1}}$; $p > \frac{1}{4}$; $i = 1$

$$\|g_a(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\ \delta(x) = \frac{\left(\frac{1}{a^{4p-1}}\right)^{1-1}}{1 - \frac{1}{a^{4p-1}}} \cdot \frac{\mu}{2a^{4p}}\ \|x\|^{4p}$$

$$= \frac{1}{1 - \frac{1}{a^{4p}}} \cdot \frac{\mu}{2a^{4p}}\ \|x\|^{4p} = \frac{\mu}{2\left(a^{4p} - a\right)}\ \|x\|^{4p}$$

□

7. Fixed Point Stability of (3) : Even Case-Fixed point method

In this section, we give the generalized Ulam - Hyers stability of the functional equation (3), for even case.

Theorem 7.1. *Let $g_q : X \to Y$ be an even mapping for which there exists a function $\alpha : X^4 \to [0, \infty)$ with the condition*

$$\lim_{k\to\infty} \left\{ \frac{\alpha\left(\nu_i^k x_1, \nu_i^k x_2, \nu_i^k x_3, \nu_i^k x_4\right)}{\nu_i^{2k}} \right\} = 0 \tag{79}$$

where

$$\nu_i = \begin{cases} a & ;\ i = 0 \\ \frac{1}{a} & ;\ i = 1 \end{cases}$$

such that the functional inequality

$$\|Dg_q(x_1, x_2, x_3, x_4)\| \ \leq\ \alpha(x_1, x_2, x_3, x_4) \tag{80}$$

for all $x_1, x_2, x_3, x_4 \in X$. If there exists $L = L(i) < 1$ such that the function

$$x \to \delta(x) = \frac{1}{2}\alpha\left(\frac{x}{a}, 0, 0, 0\right)$$

has the property

$$\frac{1}{\nu_i^2}\ \ \delta(\nu_i x) \ = \ L\ \delta(x) \tag{81}$$

for all $x \in X$. Then there exists a unique quadratic function $Q : X \to Y$ satisfying the functional equaiton (3) and

$$\|g_q(x) - Q(x)\| \leq \frac{L^{1-i}}{1-L}\ \delta(x) \tag{82}$$

holds for all $x \in X$.

Proof. The rest of the proof is similar to that of Theorem 6.1. □

Corollary 7.2. *Let $g_q : X \to Y$ be an even mapping and there exists a real numbers p and μ such that*

$$\|Dg_q(x_1, x_2, x_3, x_4)\| \leq \begin{cases} \mu; \\ \mu\left\{\|x_1\|^p + \|x_2\|^p + \|x_3\|^p + \|x_4\|^p\right\}; \\ \mu\left\{\left\{\|x_1\|^p\,\|x_2\|^p\,\|x_3\|^p\,\|x_4\|^p\right\} + \left\{\|x_1\|^{4p} + \|x_2\|^{4p} + \|x_3\|^{4p} + \|x_4\|^{4p}\right\}\right\}; \end{cases}$$

for all $x_1, x_2, x_3, x_4 \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|g_q(x) - Q(x)\| \leq \begin{cases} \frac{\mu}{4\ \left|a^2-1\right|}; & \\ \frac{\mu\ \|x\|^p}{4\ \left|a^2-a^p\right|}; & p \neq 1,2 \\ \frac{\mu\ \|x\|^{4p}}{4\ \left|a^2-a^{4p}\right|}; & p \neq \frac{1}{2} \end{cases}$$

for all $x \in X$.

8. Fixed Point Stability of (3) : Mixed Case-Fixed Point Method

In this section, we discuss the generalized Ulam - Hyers stability of the functional equation (3), for the mixed case.

Theorem 8.1. *Let $g : X \to Y$ be mapping for which there exists a function $\alpha : X^4 \to [0, \infty)$ with the condition (65) and (79), where*

$$\nu_i = \begin{cases} a & ;\ i = 0 \\ \frac{1}{a} & ;\ i = 1 \end{cases}$$

such that the functional inequality

$$\|Dg(x_1, x_2, x_3, x_4)\| \ \leq \ \alpha(x_1, x_2, x_3, x_4) \tag{83}$$

for all $x_1, x_2, x_3, x_4 \in X$. If there exists $L = L(i)$ such that the function for all $x \in X$, such that the condition

$$x \to \delta(x) = \frac{1}{2}\alpha\left(\frac{x}{a}, 0, 0, 0\right)$$

has the property (67) and (sbb3) for all $x \in X$. Then there exists a unique additive function $A : X \to Y$ and a unique quadratic function $Q : X \to Y$ satisfying the functional equaiton (3) and

$$\|g(x) - A(x) - Q(x)\| \leq \frac{L^{1-i}}{1-L}\ [\delta(x) + \delta(-x)] \tag{84}$$

holds for all $x \in X$.

Proof. The rest of the proof is similar to that of Theorem 6.1 and 7.1. □

Corollary 8.2. *Let $g : X \to Y$ be a mapping and there exists a real numbers p and μ such that*

$$\|Dg(x_1, x_2, x_3, x_4)\| \leq \begin{cases} \mu; \\ \mu\left\{\|x_1\|^p + \|x_2\|^p + \|x_3\|^p + \|x_4\|^p\right\}; \\ \mu\left\{\left\{\|x_1\|^p\|x_2\|^p\|x_3\|^p\|x_4\|^p\right\} + \left\{\|x_1\|^{4p} + \|x_2\|^{4p} + \|x_3\|^{4p} + \|x_4\|^{4p}\right\}\right\}; \end{cases} \tag{85}$$

for all $x_1, x_2, x_3, x_4 \in X$. Then there exists a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that

$$\|g(x) - A(x) - Q(x)\| \leq \begin{cases} \frac{\mu}{1\ |a-1|} + \frac{\mu}{4\ \left|a^2-1\right|}; & \\ \frac{\mu\ \|x\|^p}{2\ |a-a^p|} + \frac{\mu\ \|x\|^p}{4\ \left|a^2-a^p\right|}; & p \neq 1,2 \\ \frac{\mu\ \|x\|^{4p}}{2\ \left|a-a^{4p}\right|} + \frac{\mu\ \|x\|^{4p}}{4\ \left|a^2-a^{4p}\right|}; & p \neq \frac{1}{2}, \frac{1}{4} \end{cases}$$

for all $x \in X$.

9. Fuzzy Stability Results: Odd Case-Direct Method

In this section, the authors present the basic definitions in Fuzzy normed space and investigate the fuzzy stability of the additive functional equation, using direct method.

We use the definition of fuzzy normed spaces given in [15] and [7].

Definition 9.1. *Let X be a real linear space. A function $N : X \times R \to [0,1]$ is said to be fuzzy norm on X, if for all $x, y \in X$ and $s, t \in R$,*

$(FNS1)$ $N(x,c) = 0$ *for* $c \leq 0$;

$(FNS2)$ $x = 0$ *if and only if* $N(x,c) = 1$ *for all* $c > 0$;

$(FNS3)$ $N(cx,t) = N\left(x, \frac{t}{|c|}\right)$ *if* $c \neq 0$

$(FNS4)$ $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$;

$(FNS5)$ $N(x,.)$ *is non - decreasing function on R and* $\lim_{n\to\infty} N(x,t) = 1$;

$(FNS6)$ *for* $x \neq 0$, *and $N(x,.)$ is (α - upper semi) continuous on R.*

The pair (X,N) is called fuzzy normed space.

Definition 9.2. *Let (X,N) be a fuzzy normed linear space. Let $\{x_n\}$ be sequence in X. Then x_n is said to be convergent if there exists $x \in X$ such that the*

$$\lim_{n\to\infty} N(x_n - x, t) = 1$$

for all $t > 0$. In that case, x is called the limit of the sequaence x_n and we denote it by

$$N - \lim_{n\to\infty} x_n = x.$$

Definition 9.3. *A sequence $\{x_n\}$ in X is called Cauchy if for each $\epsilon > 0$ and each $t > 0$ there exists n_0 such that for $n \geq n_0$ and all $p > 0$, we have*

$$N(x_{n+p} - x_n, t) > 1 - \epsilon.$$

Definition 9.4. *Every convergent sequaence in a fuzzy normed space is Cauchy. If each Cauchy sequence is converegent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.*

Definition 9.5. *A mapping $f : X \to Y$ between fuzzy normed space X and Y is continuous at a point x_0 if for each sequence $\{x_n\}$ converging to x_0 in X, the sequence f is said to continuous on X.*

The stability of various functional equaiton in fuzzy normed spaces was investigated in [18–20, 24–26, 28].

Thoroughout this paper, assume that X, (X, N') and (Y, N') are linear space, fuzzy normed space and fuzzy Banach space respectively. Now the authors investigate the generalized Ulam - Hyers stability of the functional equation (3).

Theorem 9.6. *Let $\beta \in \{-1, 1\}$ and let $\alpha : X^4 \to Z$ be a mapping with $0 < \left(\frac{d}{a}\right)^\beta < 1$*

$$N'\left(\alpha(a^\beta x, 0, 0, 0), r\right) \geq N'\left(d^\beta\ \alpha(x, 0, 0, 0), r\right) \tag{86}$$

for all $x \in X$, and all $d > 0$, and

$$\lim_{n\to\infty} N'\left(\alpha\left(a^{\beta n}x_1, a^{\beta n}x_2, a^{\beta n}x_3, a^{\beta n}x_4\right), a^{\beta n}r\right) = 1 \tag{87}$$

for all $x_1, x_2, x_3, x_4 \in X$, and all $r > 0$. Suppose that a function $g : X \to Y$ satisfies the inequality

$$N\left(Dg_a(x_1, x_2, x_3, x_4), r\right) \geq N^{'}\left(\alpha(x_1, x_2, x_3, x_4), r\right) \tag{88}$$

for all $x_1, x_2, x_3, x_4 \in X$, and all $r > 0$. Then the limit

$$A(x) = N - \lim_{n\to\infty} \frac{g(a^{\beta n} r)}{a^{\beta n}} \tag{89}$$

exists for all $x \in X$ and the mapping $A : X \to Y$ is a unique additive mapping satisfying (3) and

$$N(g(x) - A(x), r) \geq N^{'}\left(\alpha(x, 0, 0, 0), r\,|a - d|\right) \tag{90}$$

for all $x \in X$ and all $r > 0$.

Proof. First assume that $\beta = 1$. Replacing (x_1, x_2, x_3, x_4) by $(x, 0, 0, 0)$ in (88), we get

$$N\left(2g(ax) - 2a^2 g(x),\ r\right)\ \geq\ N^{'}\left(\alpha(x, 0, 0, 0), r\right) \tag{91}$$

for all $x \in X$ and all $r > 0$. The rest of the proof to that of Theorem 3.1. Hence this completes the proof of the Theorem. □

From the Theorem 9.6, we obtain the following corollary concerning the Ulam - Hyers stabilities of the functional equation (3).

Corollary 9.7. *Suppose that a function $g : X \to Y$ satisfies the inequality*

$$N\left(Dg_a(x_1, x_2, x_3, x_4), r\right) \geq \begin{cases} N^{'}\left(\mu\left\{\|x_1\|^p + \|x_2\|^p + \|x_3\|^p + \|x_4\|^p\right\}, r\right) & p < 1 \quad or \quad p > 1 \\ N^{'}\left(\mu\left\{\{\|x_1\|^p \|x_2\|^p \|x_3\|^p \|x_4\|^p\} + \left(\{\|x_1\|^{4p} + \|x_2\|^{4p} + \|x_3\|^{4p} + \|x_4\|^{4p}\right\}, r\right)\right\}\right) & \\ & p < \frac{1}{4} \quad or \quad p > \frac{1}{4} \end{cases} \tag{92}$$

for all $r > 0$ and all $x_1, x_2, x_3, x_4 \in X$, where p and μ are constant. Then there exists a unique additive mapping $A : X \to Y$ such that

$$N(g(x) - A(x), r) \geq \begin{cases} N^{'}\left(\mu\ \|x\|^p,\ r\,|a - a^p|\right); \\ N^{'}\left(\mu\ \|x\|^{4p},\ r\left|a - a^{4p}\right|\right); \end{cases} \tag{93}$$

for all $x \in X$ and all $r > 0$.

Proof. If we define that

$$\alpha(x_1, x_2, x_3, x_4) = \begin{cases} \mu\left\{\|x_1\|^p + \|x_2\|^p + \|x_3\|^p + \|x_4\|^p\right\}; \\ \mu\left\{\{\|x_1\|^p \|x_2\|^p \|x_3\|^p \|x_4\|^p\} + \left\{\|x_1\|^{4p} + \|x_2\|^{4p} + \|x_3\|^{4p} + \|x_4\|^{4p}\right\}\right\}; \end{cases}$$

for all $x_1, x_2, x_3, x_4 \in X$. Then the corollary is followed from Theorem 9.6 by if we define

$$d = \begin{cases} a^p; \\ a^{4p}; \end{cases}$$

□

10. Fuzzy Stability Results: Even-Case: Direct Method

In this section, the authors present the generalized Ulam - Hyers stability of the functional equation (3) in Fuzzy Normed space even case using direct method.

Theorem 10.1. *Let $\beta \in \{-1,1\}$ and let $\alpha : X^4 \to Z$ be a mapping with $0 < \left(\frac{d}{a^2}\right)^\beta < 1$*

$$N^{'}\left(\alpha(a^\beta x,0,0,0),r\right) \geq N^{'}\left(d^\beta\ \alpha(x,0,0,0),r\right) \tag{94}$$

for all $x \in X$, and all $r > 0$, and

$$\lim_{n\to\infty} N^{'}\left(\alpha\left(a^{\beta n}x_1, a^{\beta n}x_2, a^{\beta n}x_3, a^{\beta n}x_4\right), a^{\beta n}r\right) = 1 \tag{95}$$

for all $x_1,x_2,x_3,x_4 \in X$, and all $r > 0$. Suppose that a function $g : X \to Y$ satisfies the inequality

$$N\left(Dg_q(x_1,x_2,x_3,x_4),r\right) \geq N^{'}\left(\alpha(x,0,0,0),r\right) \tag{96}$$

for all $x_1,x_2,x_3,x_4 \in X$, and all $x \in X$, $r > 0$. Then the limit

$$Q(x) = N - \lim_{n\to\infty} \frac{g(a^{\beta_n} r)}{a^{2\beta_n}} \tag{97}$$

exists for all $x \in X$ and the mapping $Q : X \to Y$ is a unique quadratic mapping satisfying (3) and

$$N(g(x) - Q(x),r) \geq N^{'}\left(\alpha(x,0,0,0), r\left|a^2 - d\right|\right) \tag{98}$$

for all $x \in X$ and all $r > 0$.

Proof. First assume that $\beta = 1$. Replacing (x_1,x_2,x_3,x_4) by $(x,0,0,0)$ in (96), we obtain

$$N\left(4g(ax) - 4a^2 g(x),\ r\right)\ \geq\ N^{'}\left(\alpha(x,0,0,0),r\right) \tag{99}$$

for all $x \in X$ and all $r > 0$. The rest of the proof to that of Theorem 9.6. Hence this completes the proof of the Theorem. □

From the Theorem 10.1, we obtain the following corollary concerning the Ulam - Hyers stabilities for the functional equation (3).

Corollary 10.2. *Suppose that a function $g : X \to Y$ satisfies the inequality*

$$N\left(Dg_q(x_1,x_2,x_3,x_4),r\right)$$
$$\geq \begin{cases} N^{'}\left(\mu\left\{\|x_1\|^p + \|x_2\|^p + \|x_3\|^p + \|x_4\|^p\right\},r\right) & p < 2 \quad or \quad p > 2 \\ N^{'}\left(\mu\left\{\left\{\|x_1\|^p\|x_2\|^p\|x_3\|^p\|x_4\|^p\right\} + \left(\left\{\|x_1\|^{4p} + \|x_2\|^{4p} + \|x_3\|^{4p} + \|x_4\|^{4p}\right\},r\right)\right\}\right) & \\ & p < \frac{1}{2} \quad or \quad p > \frac{1}{2} \end{cases} \tag{100}$$

for all $r > 0$ and all $x_1,x_2,x_3,x_4 \in X$, where p and μ are constant. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$N(g(x) - Q(x),r) \geq \begin{cases} N^{'}\left(\mu\ \|x\|^p,\ r\left|a^2 - a^p\right|\right); \\ N^{'}\left(\mu\ \|x\|^{4p},\ r\left|a^2 - a^{4p}\right|\right); \end{cases} \tag{101}$$

for all $x \in X$ and all $r > 0$.

11. Stability of the Functional Equation (3): Odd Case-Fixed Point Method

In this section, the authors investigate the generalized Ulam - Hyers stability of the functional equation (3) in fuzzy normed space using the fixed point method.

Theorem 11.1. *Let $g : X \to Y$ be a mapping for which there exists a function $\alpha : X^4 \to Z$ with the condition*

$$\lim_{n\to\infty} N^{'}\left(\alpha\left(\mu_i^n x_1, \mu_i^n x_2, \mu_i^n x_3, \mu_i^n x_4\right), \mu_i^n r\right) = 1 \tag{102}$$

for all $x_1, x_2, x_3, x_4 \in X$ and all $r > 0$ and satisfying the functional inequality

$$N\left(\delta(x_1, x_2, x_3, x_4), r\right) \geq N^{'}\left(\alpha(x_1, x_2, x_3, x_4), r\right) \tag{103}$$

for all $x_1, x_2, x_3, x_4 \in X$ and all $r > 0$. If there exists $L = L(i)$ such that the function

$$x \longrightarrow \delta(x) = \alpha\left(\frac{x}{a}, 0, 0, 0\right)$$

has the property

$$N^{'}\left(L\ \frac{\delta \mu_i x}{\mu_i}\ r\right) = N^{'}\left(\delta(x),\ r\right) \tag{104}$$

for all $x \in X$; and all $r > 0$. Then there exists a unique additive function $A : X \to Y$ satisfying the functional equation (3) and

$$N\left(\delta(x) - A(x),\ r\right) \geq\ N^{'}\left(\delta(x),\ \frac{L^{1-i}}{1-L}\right); \tag{105}$$

for all $x \in X$; and all $r > 0$.

Proof. The rest of the proof to that pf the Theorem 6.1. □

From the Theorem 11.1, we get the following corollary concerning the stability for the functional equation (3).

Corollary 11.2. *Suppose that a odd function $g : X \to Y$ satisfies the inequality*

$$N\left(\delta(x_1, x_2, x_3, x_4), r\right)$$
$$\geq \begin{cases} N^{'}\left(\mu\left\{\|x_1\|^p + \|x_2\|^p + \|x_3\|^p + \|x_4\|^p\right\}, r\right) \\ N^{'}\left(\mu\left\{\{\|x_1\|^p \|x_2\|^p \|x_3\|^p \|x_4\|^p\} + \left(\left\{\|x_1\|^{4p} + \|x_2\|^{4p} + \|x_3\|^{4p} + \|x_4\|^{4p}\right\}, r\right)\right\}\right) \end{cases} \tag{106}$$

for all $r > 0$ and all $x_1, x_2, x_3, x_4 \in X$, where p and μ are constant with $\mu > 0$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$N(g(x) - A(x), r) \geq \begin{cases} N^{'}\left(\mu\ \|x\|^p,\ \frac{a}{|a-a^p|}\right); & p < 1 \quad or \quad p > 1 \\ N^{'}\left(\mu\ \|x\|^{4p},\ \frac{a}{|a-a^{4p}|}\right); & p < \frac{1}{4} \quad or \quad p > \frac{1}{4} \end{cases} \tag{107}$$

for all $x \in X$ and all $r > 0$.

12. Stability of the Functional Equation (3): Even Case-Fixed Point Method

In this section, the authors present the generalized Ulam - Hyers stability of the functional equation (3) in Fuzzy Normed space using fixed point method.

Theorem 12.1. *Let $g : X \to Y$ be a mapping for which there exists a function $\alpha : X^4 \to Z$ with the condition*

$$\lim_{n\to\infty} N^{'}\left(\alpha\left(\mu_i^n x_1, \mu_i^n x_2, \mu_i^n x_3, \mu_i^n x_4\right), \mu_i^n r\right) = 1 \tag{108}$$

for all $x_1, x_2, x_3, x_4 \in X$ and all $r > 0$ and satisfying the functional inequality

$$N\left(Dg_q(x_1, x_2, x_3, x_4), r\right) \geq N^{'}\left(\alpha(x_1, x_2, x_3, x_4), r\right) \tag{109}$$

for all $x_1, x_2, x_3, x_4 \in X$ and all $r > 0$. If there exists $L = L(i)$ such that the function

$$x \longrightarrow \delta(x) = \alpha\left(\frac{x}{a}, 0, 0, 0\right)$$

has the property

$$N^{'}\left(L\,\frac{\delta\mu_i x}{\mu_i^2}\ r\right) = N^{'}\left(\delta(x),\ r\right) \tag{110}$$

for all $x \in X$; and all $r > 0$. Then there exists a unique quadratic function $Q : X \to Y$ satisfying the functional equation (3) and

$$N\left(\delta(x) - Q(x),\ r\right) \geq\ N^{'}\left(\delta(x),\ \frac{L^{1-i}}{1-L}\right); \tag{111}$$

for all $x \in X$; and all $r > 0$.

Corollary 12.2. *Suppose that a odd function $g : X \to Y$ satisfies the inequality*

$$N\left(\delta(x_1, x_2, x_3, x_4), r\right)$$
$$\geq \begin{cases} N^{'}\left(\mu\left\{\|x_1\|^p + \|x_2\|^p + \|x_3\|^p + \|x_4\|^p\right\}, r\right) \\ N^{'}\left(\mu\left\{\left\{\|x_1\|^p\,\|x_2\|^p\,\|x_3\|^p\,\|x_4\|^p\right\} + \left(\left\{\|x_1\|^{4p} + \|x_2\|^{4p} + \|x_3\|^{4p} + \|x_4\|^{4p}\right\}, r\right)\right\}\right) \end{cases} \tag{112}$$

for all $r > 0$ and all $x_1, x_2, x_3, x_4 \in X$, where p and μ are constant with $\mu > 0$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$N(g(x) - Q(x), r) \geq \begin{cases} N^{'}\left(\frac{\mu}{a^p}\ \|x\|^p,\ \frac{a^2}{|a^2 - a^{4p}|}\right); & p < 2 \quad or \quad p > 2 \\ N^{'}\left(\frac{\mu}{a^{4p}}\ \|x\|^{4p},\ \frac{a^2}{|a^2 - a^{4p}|}\right); & p < \frac{1}{2} \quad or \quad p > \frac{1}{2} \end{cases} \tag{113}$$

for all $x \in X$ and all $r > 0$.

References

[1] J.Aczel and J.Dhomberes, *Functional Equations in several variables*, Cambridge university press, (1989).

[2] T.Aoki, *On the stability of linear transformation in Banach spaces*, J. Math. Soc. Japan, 2(1950), 64-66.

[3] M.Arunkumar, S.Murthy and G.Ganapathy, *Solution and Stability of n - dimensional Quadratic functional equation*, ICCMSC, 2012 Computer and Information science (CIS), springer verlag - Germany, 238(2012), 384-394.

[4] M.Arunkumar, S.Murthy and V.Govindan, *General solution and generalized Ulam - Hyers stability of a generalized n - type additive quadratic functional equation in Banach spaces and Banach Algebra: Using direct and fixed point method*, International journal of Advanced Mathematical sciences, 3(1) (2015), 25-64.

[5] T.Bag and S.K.Samanta, *Finite dimensional fuzzy normed linear spaces*, Journal of Fuzzy Mathematics, 11(3)(2003), 687705.

[6] L.Cadariu and V.Radu, *On the stability of the Cauchy functional equation: a fixed point approach*, in Iteration Theory, vol. 346 of Grazer Math. Ber., pp. 4352, Karl-Franzens-Universitaet, Graz, Austria, (2004).

[7] M.Eshaghi Gordji and H.Khodaei, *The Fixed Point Method for Fuzzy Approximation of a Functional Equation Associated with Inner Product Spaces*, Hindawi Publishing Corporation Discrete Dynamics in Nature and Society, 2010(2010), Article ID 140767.

[8] P.Guvruta, *A Generalization of the Hyers - Ulam - Rassias stability of approximately additive mappings*, J. Math. Anal. Appl., 184(1994), 431-436.

[9] D.H.Hyers, *On the stability of linear functional equations*, Proc. Natl. Acad. Sci. USA, 27(1941), 222-224.

[10] D.H.Hyers, G.Issac and Th.M.Rassias, *Stability of functional equations in several variables*, Birkhauser, Basel, (1998).

[11] S.M.Jung, *Hyers - Ulam - Rassias stability of functional equations*, Mathematical Analysis, Hadronic press, palm harbor, (2001).

[12] S.S.Jin and Y.H.Lee, *Fuzzy stability of a general quadratic functional equation deriving from quadratic and additive mappings*, Abstract Appl. Anal., 2011(2011), Article ID 534120.

[13] S.S.Jin and Y.H.Lee, *Fuzzy stability of a general quadratic functional equation*, Adv. Fuzzy syst., 2011(2011), Article ID 791695.

[14] S.S.Jin and Y.H.Lee, *Fuzzy stability of the Cauchy additive and quadratic type functional equation*, Common korean math. Soc., 27(2012), 523-535.

[15] Y.H.Lee and S.M.Jung, *Fuzzy Stability of an n-Dimensional Quadratic and Additive Functional Equation*, Hindawi Publishing Corporation Advances in Fuzzy Systems, 2012(2012), Article ID 150815.

[16] S.Murthy, M.Arunkumar and G.Ganapathy, *Solution and stability of n - dimensional cubic functional equation in F - spaces: Direct and Fixed point methods*, Proceeding of the international conferences of Mathematical Sciences (Elsevier India), 81-88.

[17] A.Najati, *On the stability of quartic functional equation*, J. Math. Anal. App., 340(1)(2008), 569-574.

[18] A.Najati, *The generalized Hyers - Ulam stability of a cubic functional equation*, Turk. J. math., 31(2007), 1-14.

[19] J.M.Rassias, *On approximately of approximately linear mappings of linear mappings*, J. Functional Anal., USA, 46(1982), 126-130.

[20] J.M.Rassias and H.M.Kim, *Generalized Hyers - Ulam stability for general additive functional equations in quasi- -normed spaces*, J. Math. Anal. Appl., 356(1)(2009), 302-309.

[21] Th.M.Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., 72(1978), 297-300.

[22] Th.M.Rassias, *On the stability of the functional equations in Banach spaces*, J. Math. Anal. Appl. , 251(2000), 264-284.

[23] Th.M.Rassias, *Functional Equations, Inequalities and Applications*, Kluwer Acedamic Publishers, Dordrecht, Bostan London, (2003).

[24] K.Ravi and M.Arunkumar, *Hyers - Ulam - Rassias stability of a Quartic functional Equation*, International Journal of Pure and Applied mathematics, 34(2)(2007), 247-260.

[25] K.Ravi and M.Arunkumar, *On a General solution of a Quartic functional equation*, J. of Combinatorics Information and system sciences, 33(1)(2008), 373-386.

[26] K.Ravi, M.Arunkumar and J.M.Rassias, *On the Ulam stability for the orthogonally general Euler-Lagrange type functional equation*, International Journal of Mathematical Sciences, 3(8)(2008), 36-47.

[27] P.K.Sahoo and J.K.Chung, *On the general solution of a quartic functional equation*, Bull. Korean. Math. Soc., 40(4)(2003), 565-576.

[28] E.Thandapani, K.Ravi, and M.Arunkumar, *On the solution of the generalized Quartic functional equation*, Far East Journal of Applied Mathematics, 24(3)(2006), 297-312.

[29] S.M.Ulam, *A Collection of Mathematical Problems*, Interscience Publ., New York, (1960).

[30] S.M.Ulam, *Problem in Modern mathematics*, Sciences editions, Joha Wiley and sons, Inc., New York, (1969).