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Oscillatory Behavior of First Order Neutral Delay Difference Equations

Research Article

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Abstract: In this paper, we establish some sufficient conditions for the oscillation of all solutions of first order neutral difference equation of the form

$$\Delta[r(n)(x(n)+px(n-\tau))]+q(n)x(n-\sigma)=0, \quad n\geq n_0; \tag{$*$}$$

where $\{r(n)\}$, $\{q(n)\}$ are sequences of positive real numbers, p is a real number, and τ and σ are positive integers. The results proved improve and generalize some of existing results in the literature. Some examples are inserted to illustrate our results.

MSC: 39A10, 39A12.

Keywords: Oscillation, nonoscillation, neutral, delay difference equations.

1. Introduction

A neutral delay difference equation is a difference equation in which the highest order difference of the unknown sequence appears in the equation both with and without delays. Recently, increasing numbers of investigations have been carried out in studying the oscillation of neutral delay difference equation, see for example [1–5]. Consider the first order neutral delay difference equation of the form

$$\Delta[r(n)(x(n)+px(n-\tau))]+q(n)x(n-\sigma)=0, \quad n\geq n_0; \tag{1}$$

where Δ is the forward difference operator defined by $\Delta x(n)=x(n+1)-x(n)$, $\{r(n)\}_{n=n_0}^{\infty}$, $\{q(n)\}_{n=n_0}^{\infty}$ are sequences of positive real numbers, p is a real number, and τ and σ are positive integers. The oscillatory solutions of (1) have been investigated by a number of researchers and some sufficient conditions for the oscillatory and nonoscillatory solutions have been investigated, see [6–10]. Let us choose a positive integer $n^*=\max\{\tau,\sigma\}$. By a solution of (1) on $N(n_0)=\{n_0,n_0+1,...\}$, we mean a real sequence $\{x(n)\}$ which is defined on $n\geq n_0-n^*$ and which satisfies (1) for $n\in N(n_0)$. A solution $\{x(n)\}$ of (1) on $N(n_0)$ is said to be oscillatory if for every positive integers $N_0>n_0$ there exists $n\geq N_0$ such that $x(n)x(n+1)\leq 0$, otherwise $\{x(n)\}$ is said to be nonoscillatory. The main objective of this article is to give some new sufficient conditions for the oscillatory solutions of (1). We present some of the well known Lemmas, which will be needed

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in the proof of our main results. They may also have further applications in the analysis. The Lemma 1.1 and 1.2 are the discrete analogues of the Lemmas 1.5.1 and 1.5.3 respectively in [3].

Lemma 1.1. *Let $\{f(n)\}$ and $\{g(n)\}$ be sequences of real numbers such that $f(n) = g(n) + \mu g(n-c)$; $n \geq n_0 + \max\{0, c\}$, where $\mu \in R$, $\mu \neq 1$ and c is a positive integer Assume that $\lim_{n\to\infty} f(n) = l \in R$ exists and $\lim\inf_{n\to\infty} g(n) = a \in R$. Then $l = (1+\mu)a$.*

Lemma 1.2. *Let $0 \leq \lambda < 1$, c be a positive integer, $n_0 \in N$ and $\{x(n)\}$ be a sequence of positive real numbers and assume that for every $\epsilon > 0$ there exists a $n_\epsilon \geq n_0$ such that $x(n) \leq (\lambda + \epsilon)x(n-c) + \epsilon$ for $n \geq n_\epsilon$. Then $\lim_{n\to\infty} x(n) = 0$.*

Lemma 1.3. *Assume that $p \neq 1$, $r(n) = 1$ and*

$$\sum_{n=n_0}^{\infty} q(n) = \infty. \tag{2}$$

Let $\{x(n)\}$ be an eventually positive solution of the neutral delay difference equation (1). Set

$$z(n) = x(n) + px(n-\tau). \tag{3}$$

Then the following statements hold.

(a) $\{z(n)\}$ is decreasing sequence and either

$$\lim_{n\to\infty} z(n) = -\infty \quad or \tag{4}$$

$$\lim_{n\to\infty} z(n) = 0. \tag{5}$$

(b) The following statements are equivalent:

(i) (4) holds;

(ii) $p < -1$;

(iii)

$$\lim_{n\to\infty} x(n) = \infty. \tag{6}$$

(c) The following statements are equivalent:

(i) (5) holds;

(ii) $p > -1$;

(iii)

$$\lim_{n\to\infty} x(n) = 0. \tag{7}$$

Proof. From (1) and (3) we obtain

$$\Delta z(n) = -q(n)x(n-\sigma) \tag{8}$$

and so eventually $\Delta z(n) \leq 0$. Hence either (4) holds or

$$\lim_{n\to\infty} z(n) \equiv l \in R. \tag{9}$$

If (9) holds, then by summing (8) from n_1 to ∞, with n_1, sufficiently large, we find

$$l - z(n_1) = -\sum_{s=n_1}^{\infty} q(s)x(s-\sigma). \tag{10}$$

In view of (2) this implies that $\liminf_{n\to\infty} x(n) = 0$ and so by Lemma 1.1, $l = (1+p)0 = 0$. The proof of (a) is complete. Now we turn to the proofs of (b) and (c). First assume that (4) holds. Then p must be negative and $\{x(n)\}$ is unbounded. Therefore there exists a $n^* \geq n_0$ such that $z(n^*) < 0$ and

$$x(n^*) \geq \max_{s\leq n^*} x(s) > 0.$$

Then

$$0 > z(n^*) = x(n^*) + px(n-\tau) \geq x(n^*)(1+p)$$

which implies that $p < -1$. Also $z(n) = x(n) + px(n-\tau) > px(n-\tau)$ and (4) implies that $\lim_{n\to\infty} x(n) = \infty$. Now assume that (5) holds. If $p \geq 0$, then from (3), it follows that $\lim_{n\to\infty} x(n) = 0$. Next assume that $p \in (-1, 0)$. Then by Lemma 1.2, $\lim_{n\to\infty} x(n) = 0$.

Finally if $p < -1$, then $x(n) > -px(n-\tau) \geq x(n-\tau)$ which shows that $\{x(n)\}$ is bounded from below by a positive constant, say m. Then (10) yields.

$$l - z(n_1) + m\sum_{s=n_1}^{\infty} q(s) \leq 0,$$

which is a contradiction. Therefore, if (5) hold, $p > -1$. On the basis of the above discussion, the proof of (b) and (c) follow immediately. □

Lemma 1.4. *Assume that* $-1 < p < 0$ *and* $\lim_{n\to\infty} r(n) = r_0$ *exists. Let* $\{x(n)\}$ *be an eventually positive solution of* (1) *and* $\{z(n)\}$ *be its associated sequence defined by* (3). *Then* $z(n) > 0$ *eventually.*

Proof. From (1) and (3) we have

$$\Delta(r(n)z(n)) < 0, \quad \text{eventually}. \tag{11}$$

This shows that $\{r(n)z(n)\}$ is decreasing sequence. Assume the contrary. That is $z(n) < 0$ then $x(n) < -px(n-\tau)$. This implies that $x(n+k\tau) < (-p)^k x(n)$, and hence $x(n) \to 0$ as $n \to \infty$. Together with this we have $\lim_{n\to\infty} z(n) = 0$. Since $\lim_{n\to\infty} r(n) = r_0$ exists, we obtain

$$\lim_{n\to\infty} (r(n)z(n)) = 0.$$

This is a contradiction to the fact that $\{r(n)z(n)\}$ decreasing and eventually negative sequence. This completes the proof. □

Lemma 1.5 ([3]). *Assume that* k *is a positive integer. Let* $\{h(n)\}$ *be a sequence of nonnegative real numbers and suppose that*

$$\liminf_{n\to\infty} \sum_{s=n-k}^{n-1} h(s) > \left(\frac{k}{k+1}\right)^{k+1}. \tag{12}$$

Then

(i). the delay difference inequality

$$\Delta x(n) + h(n)x(n-k) \leq 0, \quad n \geq n_0$$

has no eventually positive solution.

(ii). the delay difference inequality

$$\Delta x(n) + h(n)x(n-k) \geq 0, \quad n \geq n_0$$

has no eventually negative solution.

Lemma 1.6 ([3]). *Assume that k is a positive integer with $k > 1$. Let $\{h(n)\}$ be a sequence of nonnegative real numbers and suppose that*

$$\liminf_{n\to\infty} \sum_{s=n+1}^{n+k-1} h(s) > \left(\frac{k-1}{k}\right)^k. \tag{13}$$

Then

(i). the advanced difference inequality

$$\Delta x(n) - h(n)x(n+k) \leq 0, \quad n \geq n_0$$

has no eventually negative solution.

(ii). the advanced difference inequality

$$\Delta x(n) - h(n)x(n+k) \geq 0, \quad n \geq n_0$$

has no eventually positive solution.

2. Main Results

In this section, we give some new sufficient conditions for oscillations of all solutions of (1).

Theorem 2.1. *Assume that $r(n) \equiv r > 0$ and $p = -1$ and (3) holds. Then every solution of (1) is oscillatory.*

Proof. Assume the contrary. Without loss of generality that we may assume that $\{x(n)\}$ is an eventually positive solution of (1). Then there exists an integer $n_1 \geq n_0$ such that $x(n-\sigma) > 0$ for all $n \geq n_1 \geq n_0$. Set

$$z(n) = x(n) - x(n-\tau). \tag{14}$$

Then from (1) we have

$$\Delta z(n) = \frac{-q(n)x(n-\sigma)}{r} < 0. \tag{15}$$

Hence for all $n \geq n_1$, we have $z(n) > 0$ or $\Delta z(n) < 0$. Let $z(n) > 0$. This implies that

$$\sum_{s=n_1}^{\infty} q(s)x(s-\sigma) < rz(n_1) < \infty. \tag{16}$$

On the other hand, $z(n) > 0$ gives $x(n) > x(n-\tau)$ and hence $\liminf_{n\to\infty} x(n) > 0$. Thus, there exists a positive constant k such that $x(n) > k > 0$. Then

$$\sum_{s=n+\sigma}^{\infty} q(s)x(s-\sigma) > k \sum_{s=n+\sigma}^{\infty} q(s),$$

which leads to

$$\sum_{s=n+\sigma}^{\infty} q(s)x(s-\sigma) = \infty.$$

This is a contradiction with (16). Therefore $z(n) < 0$, which implies that $x(n) < x(n-\tau)$. Then $\{x(n)\}$ is bounded and hence $\liminf_{n\to\infty} x(n)$ and $\liminf_{n\to\infty} z(n)$ exists. From Lemma 1.1, we get $\lim_{n\to\infty} z(n) = 0$. This contradicts the fact that $\{z(n)\}$ is a negative and monotonic decreasing sequence. □

Theorem 2.2. *Assume that* $p \neq \pm 1$, $r(n) \equiv r > 0$, *and* $\{q(n)\}$ *is a* τ*- periodic sequence of positive real numbers. Suppose that one of the following conditions holds.*

$$\liminf_{n\to\infty} \sum_{s=n-\sigma+\tau}^{n-1} \frac{q(s)}{r(1+p)} > \left(\frac{\sigma-\tau}{\sigma-\tau+1}\right)^{\sigma-\tau+1}, \quad \sigma-\tau \geq 1 \quad and \quad 1+p>0 \tag{17}$$

or

$$\liminf_{n\to\infty} \sum_{s=n+1}^{n+\tau-\sigma-1} \left(\frac{-q(s)}{r(1+p)}\right) > \left(\frac{\tau-\sigma-1}{\tau-\sigma}\right)^{\tau-\sigma}, \quad \tau-\sigma>1 \quad and \quad 1+p<0. \tag{18}$$

Then every solution of (1) *is oscillatory.*

Proof. Assume the contrary. Without loss of generality, we may suppose that $\{x(n)\}$ is an eventually positive solution of (1). Set

$$z(n) = x(n) + px(n-\tau) \;\text{ and}$$
$$w(n) = z(n) + pz(n-\tau).$$

Then by direct substitution, we can show that $\{z(n)\}$ and $\{w(n)\}$ are solutions of (1). Then

$$r\Delta z(n) + prz(n-\tau) + q(n)z(n-\sigma) = 0, \quad n \geq n_0 \;\text{ and} \tag{19}$$
$$r\Delta w(n) + prw(n-\tau) + q(n)w(n-\sigma) = 0. \tag{20}$$

By Lemma 1.3, $\{z(n)\}$ is decreasing and either (4) or (5) holds. In either case we claim that

$$\Delta w(n-\tau) \leq \Delta w(n) \tag{21}$$

Indeed,

$$\begin{aligned} \Delta w(n) &= -q(n)z(n-\sigma) \\ &\geq -q(n)z(n-\sigma-\tau) \\ &= -q(n-\tau)z(n-\sigma-\tau) \\ &= -\Delta w(n-\tau). \end{aligned}$$

Furthermore, it follows from Lemma 1.3 that as long as $p \neq \pm 1$, $w(n) > 0$. By using (21) in (20) we obtain

$$r(1+p)\Delta w(n-\tau) + q(n)w(n-\sigma) \leq 0.$$

In view of the τ -periodicity of $\{q(n)\}$ we find

$$\Delta w(n) + \frac{q(n)}{r(1+p)} w(n-(\sigma-\tau)) \leq 0 \quad if \quad 1+p>0 \tag{22}$$

or

$$\Delta w(n) - \left(\frac{-q(n)}{r(1+p)}\right) w(n+\tau-\sigma) \geq 0 \quad if \quad 1+p<0. \tag{23}$$

In view of Lemmas 1.5 and 1.6 and the conditions (17) and (18), it is impossible for (22) and (23) to have eventually positive solution. This contradicts the fact that $w(n) > 0$ eventually. □

Theorem 2.3. *Assume that* $-1 < p < 0$ *and* $\lim_{n\to\infty} r(n) = r_0$ *exists. Suppose that*

$$\liminf_{n\to\infty} \sum_{s=n-\sigma}^{n-1} \frac{q(s)}{r(s-\sigma)} > \left(\frac{\sigma}{\sigma+1}\right)^{\sigma+1}. \tag{24}$$

Then every solution of (1) *oscillates.*

Proof. Assume the contrary. Without loss of generality, we may assume that $\{x(n)\}$ is an eventually positive solution of (1). Then there exists an integer $n_1 \geq n_0$ such that $x(n-\tau) > 0$ and $x(n-\sigma) > 0$ for all $n \geq n_1$. Set $z(n) = x(n)+px(n-\tau)$. Then by Lemma 1.4, $z(n) > 0$ eventually. As $x(n) > z(n)$, it follows from (1) that

$$\Delta(r(n)z(n)) + q(n)z(n-\sigma) \leq 0, \quad n \geq n_1. \tag{25}$$

Let $y(n) = r(n)z(n)$. Then (25) becomes

$$\Delta y(n) + \frac{q(n)}{r(n-\sigma)} y(n-\sigma) \leq 0, \quad n \geq n_1. \tag{26}$$

In view of Lemma 1.5 and (24), it is impossible for (26) have an eventually positive solution. This contradicts the fact that $y(n) > 0$ and the proof of Theorem 2.3 is completed. □

3. Example

Example 3.1. *Consider the following neutral delay difference equation*

$$\Delta[x(n) - x(n-2)] + \frac{4}{n-4} x(n-4) = 0; \quad n \geq 5. \tag{27}$$

Clearly $r(n) = 1$, $q(n) = \frac{4}{n-4}$, $\tau = 2$ *and* $\sigma = 4$. *Clearly*

$$\sum_{s=5}^{\infty} q(s) = \infty.$$

Then by Theorem 2.1, every solution of (27) *is oscillatory. One such solution of* (27) *is* $x(n) = n(-1)^n$.

Example 3.2. *Consider the first order neutral difference equation*

$$\Delta\left[r(x(n) - \frac{1}{2}x(n-2))\right] + q(n)x(n-4) = 0; \quad n = 5, 6, 7, \ldots \tag{28}$$

where r *is positive real number,* $p = \frac{-1}{2}$, $\tau = 2$, $\sigma = 4$ *and*

$$q(n) = \begin{cases} \frac{5r}{4} & if \quad n \quad is \quad even; \\ \frac{5r}{6} & if \quad n \quad is \quad odd. \end{cases}$$

Clearly $\sigma \geq \tau + 1$ *and*

$$\begin{aligned} \liminf_{n\to\infty} \sum_{s=n-\sigma+\tau}^{n-1} \frac{q(s)}{r(1+p)} &= \liminf_{n\to\infty} \sum_{s=n-2}^{n-1} \frac{q(s)}{r(1+p)} \\ &= \liminf_{n\to\infty} \frac{2}{r}[q(n-2) + q(n-1)] \\ &= \frac{2}{r}\left[\frac{5r}{4} + \frac{5r}{6}\right] \\ &= \frac{25}{6} \\ &> \left(\frac{\sigma-\tau}{\sigma-\tau+1}\right)^{\sigma-\tau+1} = \left(\frac{2}{3}\right)^3. \end{aligned}$$

Hence, by Theorem 2.2, every solution of (28) *is oscillatory. One such solution is*

$$x(n) = \begin{cases} 2, & n \quad is \quad even; \\ -3, & n \quad is \quad odd. \end{cases}$$

Example 3.3. *Consider the neutral delay difference equation*

$$\Delta[\frac{1}{n}(x(n) - \frac{1}{2}x(n-4))] + \frac{1}{2}\left(\frac{1}{n} + \frac{1}{n+1}\right)x(n-2) = 0; \quad n \geq 4. \tag{29}$$

Clearly, $r(n) = \frac{1}{n}$, $q(n) = \frac{1}{2}\left(\frac{1}{n} + \frac{1}{n+1}\right)$, $\tau = 4$ *and* $\sigma = 2$. *We can see that* $\lim_{n\to\infty} r(n) = 0$. *Also,*

$$\liminf_{n\to\infty} \sum_{s=n-2}^{n-1} \frac{q(s)}{r(s-2)} = 2 > \left(\frac{\sigma}{\sigma+1}\right)^{\sigma+1} = \left(\frac{2}{3}\right)^3.$$

Hence by Theorem 2.3, every solution of (29) *is oscillatory. One such solution of* (29) *is* $x(n) = (-1)^n$.

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