



# On Oscillation of Solutions to Second Order Neutral Difference Equations

Research Article

A.Murugesan<sup>1\*</sup> and K.Ammamuthu<sup>2</sup>

1 Department of Mathematics, Government Arts College (Autonomous), Salem, Tamil Nadu, India.

2 Department of Mathematics, Arignar Anna Government Arts College, Attur, Tamil Nadu, India.

**Abstract:** In this paper, we consider a class of second order neutral difference equation of the form

$$\Delta(r(n)\Delta(x(n) - p(n)x(n + \tau))) + q(n)x(n + \sigma) = 0, \quad n \geq n_0 \quad (*)$$

where  $r(n)$  is a sequence of positive real numbers,  $\{p(n)\}$  and  $\{q(n)\}$  are sequence of nonnegative real numbers, and  $\tau$  and  $\sigma$  are integers. We discuss the oscillatory properties of the equation (\*) relating oscillation of these equations to existence of positive solutions to associated first order neutral inequalities.

**MSC:** 39A10, 39A12.

**Keywords:** Oscillatory properties, positive solutions, second order, delay, advanced, neutral difference equations.

© JS Publication.

## 1. Introduction

In this paper, we discuss the oscillatory behavior of second order neutral difference equation of the form

$$\Delta(r(n)\Delta(x(n) - p(n)x(n + \tau))) + q(n)x(n + \sigma) = 0; \quad n \geq n_0 \quad (1)$$

where  $\Delta$  is the forward difference operator defined by  $\Delta x(n) = x(n + 1) - x(n)$ . Throughout the paper the following conditions are assumed to be hold:

(C<sub>1</sub>)  $\{r(n)\}$  is a sequence of positive real numbers;

(C<sub>2</sub>)  $\{p(n)\}$  is a sequence of nonnegative real numbers and there exist constants  $p_0$  and  $p_1$  such that  $0 \leq p_0 \leq p(n) \leq p_1 < 1$ ;

(C<sub>3</sub>)  $\{q(n)\}$  is a sequence of nonnegative real numbers and  $q(n)$  is not identically zero for all  $n$  sufficiently large;

(C<sub>4</sub>)  $\tau$  and  $\sigma$  are integers.

(C<sub>5</sub>)  $\lim_{n \rightarrow \infty} R(n) < \infty$ , where  $R(n) = \sum_{s=n_0}^{n-1} \frac{1}{r(s)}$ .

\* E-mail: [amurugesan3@gmail.com](mailto:amurugesan3@gmail.com)

If  $\{x(n)\}$  is a solution of (1) then its associated sequence  $\{z(n)\}$  is defined by

$$z(n) = x(n) - p(n)x(n + \tau). \tag{2}$$

By a solution of (1), we mean a real sequence  $\{x(n)\}$  which is defined for  $n^* \geq \min\{n_0, n_0 + \tau, n_0 + \sigma\}$  and satisfies (1) for  $n \geq n^*$ . We consider only such solution which are nontrivial for all large  $n$ . A solution  $\{x(n)\}$  of (1) is said to be nonoscillatory if the terms  $x(n)$  of the sequence are eventually positive or eventually negative. Otherwise it is called oscillatory. In the last few decades there has been an increasing interest in the study of qualitative properties of solutions of neutral difference equations, in particular second order difference equations; see, for example [4, 7–10] and the references cited therein. For the general background of difference equations one can refer to [1, 2]. In [6], we derived sufficient conditions for oscillation of all solutions of the equation (1) where  $p(n) \leq 0$ . Our aim in this paper is to derive sufficient conditions under which every solution of (1) is either oscillatory or tends to zero when  $\tau \leq 0$  and to derive sufficient condition under which every bounded solutions (1) is oscillatory when  $\tau \geq 0$ . Our established results are discrete analogues of some well-known results due to [5]. In the sequel, for our convenience, when we write a fractional inequality without mentioning its domain of validity we assume that it holds for all sufficiently large values of  $n$ . To prove our main results we start with the following lemmas.

**Lemma 1.1** ([3]). *Assume that  $\tau \leq 0$ . If  $\{x(n)\}$  is eventually positive solution of (1) such that  $\limsup_{n \rightarrow \infty} x(n) > 0$ , then  $z(n) > 0$ , eventually, where  $z(n)$  is defined by (2).*

**Lemma 1.2.** *Assuem that  $\tau \geq 0$ . Let  $\{x(n)\}$  be a bounded and eventually positive solution of (1) and  $\{z(n)\}$  be its associated sequence defined by (2). Then  $z(n) > 0$  eventually.*

*Proof.* From (1) and (2), we have

$$\Delta(r(n)\Delta z(n)) \leq 0, \tag{3}$$

which implies that  $\Delta z(n) > 0$  or  $\Delta z(n) < 0$ . In either case we have  $z(n) > 0$  or  $z(n) < 0$  eventually. If  $z(n) < 0$  then

$$x(n) < p(n)x(n + \tau) \leq p_1x(n + \tau) \text{ and have } x(n + k\tau) > \left(\frac{1}{p_1}\right)^k x(n).$$

This implies that  $x(n) \rightarrow +\infty$  as  $n \rightarrow \infty$ , which contradicts our assumption that  $\{x(n)\}$  is bounded. We use the following notations for our convenience.

$$y(n) = -v(n) = r(n)\Delta z(n) \text{ and } \delta(n) = \sum_{s=n}^{\infty} \frac{1}{r(s)}.$$

□

## 2. Main Results

In this section we establish sufficient conditions for oscillation of solution to (1).

**Theorem 2.1.** *Let  $n_1$  be large enough. Suppose that there exist two integers  $\alpha$  and  $\beta$  such that  $\alpha \leq \sigma \leq \beta$ ,  $\tau \leq 0$  and  $q(n) \geq q(n + \tau)$ , eventually. If the first order neutral difference inequalities*

$$\Delta(y(n) - p_0y(n + \tau)) + q(n)(R(n + \alpha) - R(n_1))y(n + \alpha) \leq 0 \text{ and} \tag{4}$$

$$\Delta(w(n) - p_0w(n + \tau)) - q(n)\delta(n + \beta)w(n + \beta) \geq 0 \tag{5}$$

*has no positive solutions, then every solutions of (1) is either oscillatory or tends to zero.*

*Proof.* Assume the contrary. Without loss of generality we may suppose that  $\{x(n)\}$  is an eventually positive solution of (1) such that  $\limsup_{n \rightarrow \infty} x(n) > 0$ . Then by Lemma 1.1,  $z(n) > 0$ , eventually where  $\{z(n)\}$  is defined by (2). From (1) and (2), we have

$$\begin{aligned} \Delta(r(n)\Delta z(n)) - p_0\Delta(r(n+\tau)\Delta z(n+\tau)) + q(n)x(n+\sigma) - p_0q(n+\tau)x(n+\tau+\sigma) &= 0 \\ \Delta(r(n)\Delta z(n)) - p_0\Delta(r(n+\tau)\Delta z(n+\tau)) + q(n)(x(n+\tau) - p_0x(n+\tau+\sigma)) &\leq 0 \text{ or} \\ \Delta(r(n)\Delta z(n)) - p_0\Delta(r(n+\tau)\Delta z(n+\tau)) + q(n)z(n+\sigma) &\leq 0 \end{aligned} \tag{6}$$

Equation (1) yields that, for some  $n_1$  large enough and for all  $n \geq n_1$ , either

$$\Delta z(n) > 0, \quad \Delta(r(n)\Delta z(n)) < 0 \text{ or} \tag{7}$$

$$\Delta z(n) < 0, \quad \Delta(r(n)\Delta z(n)) < 0. \tag{8}$$

Assume that (7) holds. Inequality (6) and the fact that  $\alpha \leq \sigma$  yields that

$$\Delta(r(n)\Delta z(n)) - p_0\Delta(r(n+\tau)\Delta z(n+\tau)) + q(n)z(n+\alpha) \leq 0. \tag{9}$$

It follows from (7) that

$$z(n) \geq \sum_{s=n_1}^{n-1} \frac{r(s)\Delta z(s)}{r(s)} \geq r(n)\Delta z(n) \sum_{s=n_1}^{n-1} \frac{1}{r(s)} = y(n)(R(n) - R(n_1)). \tag{10}$$

Using (10) in (9), we see that  $\{y(n)\}$  is an eventually positive solution of the inequality (4), which contradiction to our assumption that (4) has no positive solutions. Consider now the second case. It follows from (8) that

$$\Delta z(s) \leq \frac{r(n)\Delta z(n)}{r(s)} \text{ for all } s \geq n. \tag{11}$$

Summing from  $n$  to  $l-1$  we have  $z(l) \leq z(n) + r(n)\Delta z(n) \sum_{s=n}^{l-1} \frac{1}{r(s)}$ . Taking limit  $l \rightarrow \infty$ , we get  $z(n) + r(n)\Delta z(n)\delta(n) \geq 0$  or

$$z(n) \geq -r(n)\Delta z(n)\delta(n). \tag{12}$$

Using (12) in (6) and the fact that  $\sigma \leq \beta$ , we have

$$\Delta(r(n)\Delta z(n)) - p_0\Delta(r(n+\tau)\Delta z(n+\tau)) + q(n)z(n+\beta) \leq 0. \tag{13}$$

Then,  $y(n) < 0$  and by virtue of (12) and (13), we have

$$\begin{aligned} \Delta(r(n)y(n)) - p_0y(n+\tau) - q(n)y(n+\beta) &\leq 0 \text{ or} \\ \Delta(u(n)) - p_0u(n+\tau) - q(n)u(n+\beta) &\geq 0, \end{aligned} \tag{14}$$

which shows that  $\{u(n)\}$  is a positive solution of the inequality (5) which, according to our assumption, has no positive solutions. This is a contradiction and this completes the proof. □

**Theorem 2.2.** Assume that  $\tau \geq 0$  and  $\sum_{n=n_0}^{\infty} q(n) < \infty$ ,  $q(n) \geq q(n + \tau)$ , eventually and let  $n_1$  be large enough. Suppose that there exist integers  $\alpha$  and  $\beta$  such that  $\alpha \leq \sigma \leq \beta$ . If the first order difference inequalities

$$\Delta g(n) + q(n)(R(n + \alpha) - R(n_1))g(n + \alpha) \leq 0 \quad \text{and} \tag{15}$$

$$\Delta h(n) - \frac{1}{1 - p_0} q(n)\delta(n + \beta)h(n + \beta - \tau) \geq 0 \tag{16}$$

have no positive solutions, then every bounded solution of (1) is oscillatory.

*Proof.* Assume the contrary. Without loss of generality we may suppose that  $\{x(n)\}$  is a bounded and eventually positive solution of (1). Then by Lemma 1.2 the sequence  $\{z(n)\}$  defined by (2) is eventually positive. As in the proof Theorem 2.1, one arrive at the inequality (6). Equation (1) yields that for some  $n_1$  sufficiently large enough and for all  $n \geq n_1$  either

$$\Delta z(n) > 0, \quad \Delta(r(n)\Delta z(n)) < 0, \quad \text{or} \tag{17}$$

$$\Delta z(n) < 0, \quad \Delta(r(n)\Delta z(n)) < 0. \tag{18}$$

Assume first that (17) holds. By repeating the procedure as we followed in Theorem 2.1, we arrive at the inequality (4). Set

$$v(n) = y(n) - p_0 y(n + \tau). \tag{19}$$

We can assert that  $\{v(n)\}$  is an eventually positive sequence. Otherwise,  $v(n) < 0$  implies  $y(n) \rightarrow +\infty$  as  $n \rightarrow \infty$  which is a contradiction to the fact that  $\{y(n)\}$  is eventually positive and decreasing. Also,

$$v(n) \leq y(n). \tag{20}$$

Using (20) in (4), we see that  $\{v(n)\}$  is an eventually positive solution of (15), which contradicts our assumption. Consider the second case. It follows from (18) as we have shown in Theorem 2.1,  $\{u(n)\}$  is positive increasing and satisfies (5). That is  $\Delta(u(n) - p_0 u(n + \tau)) - q(n)\delta(n + \beta)u(n + \beta) \geq 0$ . Set

$$w(n) = u(n) - p_0 u(n + \tau). \tag{21}$$

We claim that  $w(n) > 0$ , eventually. Otherwise  $w(n) < 0$ , which implies that  $u(n) \rightarrow +\infty$  as  $n \rightarrow \infty$ . On the other hand, from equation (1), we have

$$\Delta u(n) = q(n)x(n + \sigma). \tag{22}$$

Since  $\{x(n)\}$  is bounded, then there exist  $M > 0$  such that  $x(n) \leq M$  for all sufficiently large values of  $n$ . Using this and the assumption that  $\sum_{n=n_0}^{\infty} q(n) < \infty$ , we have from the inequality (22) that  $\lim_{n \rightarrow \infty} u(n) < \infty$ , which leads to a contradiction. Using the fact that  $\tau \geq 0$ , we obtain

$$w(n) \leq (1 - p_0)u(n + \tau). \tag{23}$$

Substituting (23) in (21), we see that  $\{w(n)\}$  is a positive solution of (16), which leads to a contradiction. This completes the proof. □

Combining Theorem 2.2 with the oscillation results presented in Gyori et al. [2], we obtain the following result.

**Corollary 2.3.** Assume that  $\tau \geq 0$ ,  $\sum_{n=n_0}^{\infty} q(n) < \infty$ . Suppose that there exist two integers  $\alpha$  and  $\beta$  such that  $q(n) \geq q(n + \tau)$ ,  $\alpha \leq 0$ ,  $\beta > \tau + 1$  and  $\alpha \leq \sigma \leq \beta$ . If for all sufficiently large  $n \geq n_0$ ,

$$\liminf_{n \rightarrow \infty} \sum_{s=n+\alpha}^{n-1} q(s) (R(s + \alpha) - R(n_1)) > \left(\frac{\alpha - 1}{\alpha}\right)^{\alpha-1} \quad \text{and} \tag{24}$$

$$\liminf_{n \rightarrow \infty} \sum_{s=n+1}^{n+\beta-\tau-1} q(s)\delta(s + \beta) > (1 - p_0) \left(\frac{\beta - \tau - 1}{\beta - \tau}\right)^{\beta-\tau}, \tag{25}$$

then every bounded solution of (1) is oscillatory.

*Proof.* By [2], assumption (24) ensures that the delay difference inequality (15) has no positive solution. On the other hand, by [2] condition (25) guarantees that the advanced difference inequality (16) has no positive solution. Application of Theorem 2.2 yields the result.  $\square$

**Theorem 2.4.** Assume that  $\tau \leq 0$ ,  $q(n) \geq q(n + \tau)$  and let  $n_1$  be an integer large enough. Suppose that there exist two integers  $\alpha$  and  $\beta$  such that  $\alpha \leq \sigma \leq \beta$ . If the first order difference inequalities

$$\Delta g(n) + \frac{1}{1 - p_0} q(n) (R(n + \alpha) - R(n_1)) g(n + \alpha - \tau) \leq 0 \quad \text{and} \tag{26}$$

$$\Delta h(n) - q(n)\delta(n + \beta)h(n + \beta) \geq 0 \tag{27}$$

have no positive solutions, then every solution of (1) is either oscillatory or tends to zero.

*Proof.* Assume the contrary. Without loss of generality we may suppose that  $\{x(n)\}$  is an eventually positive solution of (1) such that  $\limsup_{n \rightarrow \infty} x(n) > 0$ . Then by Lemma 1.1, the sequence  $\{z(n)\}$  defined by (2) is eventually positive. As in the proof of Theorem 2.1, we arrive at the inequality (6). Equation (1) yields that, for some  $n_1$  sufficiently large enough and for all  $n \geq n_1$  either

$$\Delta z(n) > 0, \quad \Delta (r(n)\Delta z(n)) < 0, \quad \text{or} \tag{28}$$

$$\Delta z(n) < 0, \quad \Delta (r(n)\Delta z(n)) < 0. \tag{29}$$

Assume that (28) holds. As we proved in the Theorem 2.1, we deduce the inequality (4). Set

$$v(n) = y(n) - p_0 y(n + \tau). \tag{30}$$

We claim that  $v(n) > 0$ , eventually. For,  $\{y(n)\}$  is eventually positive and decreasing sequence. Then  $\lim_{n \rightarrow \infty} y(n) = L \geq 0$ . From this and (30), we have  $\lim_{n \rightarrow \infty} v(n) = (1 - p_0)L \geq 0$ , which implies that  $\{v(n)\}$  decreases to a limit and hence  $\{v(n)\}$  is an eventually positive sequence. Also, from (30), we have

$$v(n) \leq (1 - p_0)y(n + \tau). \tag{31}$$

Using (31) in (4), we see that the inequality (26) has a positive solution  $\{v(n)\}$ , which contradicts our assumption. Consider now the second case. It follows from (29) as we have shown in Theorem 2.1,  $\{u(n)\}$  is a positive increasing and satisfies (5). That is,

$$\Delta (u(n) - p_0 u(n + \tau)) - q(n)\delta(n + \beta)u(n + \beta) \geq 0. \tag{32}$$

Set

$$w(n) = u(n) - p_0 u(n + \tau). \quad (33)$$

We can assert that  $\{w(n)\}$  is an eventually positive sequence. Otherwise,  $w(n) < 0$  implies that  $u(n) < p_0 u(n + \tau)$  and hence  $u(n) \rightarrow 0$  as  $n \rightarrow \infty$ , which is a contradiction to the fact that  $\{u(n)\}$  is positive and increasing sequence. From (33), we have  $w(n) \leq u(n)$ . Using the above inequality in (32), we have  $\Delta w(n) - q(n)\delta(n + \beta)w(n + \beta) \geq 0$ . This shows that the equation (27) has a positive solution  $\{w(n)\}$  which contradicts our assumption. The proof is complete.  $\square$

Combining Theorem 2.4 with results in Gyori et al. [2], we obtain the following oscillation results.

**Corollary 2.5.** *Assume that  $\tau \leq 0$ ,  $q(n) \geq q(n + \tau)$  and let  $n_1$  be an integer large enough. Suppose that there exists integers  $\alpha$  and  $\beta$  such that  $\alpha \leq \sigma \leq \beta$ ,  $\alpha \leq \tau$  and  $\beta > 1$ . If, for all sufficiently large  $n_1 \geq n_0$*

$$\liminf_{n \rightarrow \infty} \sum_{s=n-\tau+\alpha}^{n-1} q(s) (R(s + \alpha) - R(n_1)) > (1 - p_0) \left( \frac{\tau - \alpha}{\tau - \alpha + 1} \right)^{\tau - \alpha + 1}, \quad \text{and} \quad (34)$$

$$\liminf_{n \rightarrow \infty} \sum_{s=n+1}^{n+\beta-1} q(s)\delta(s + \beta) > \left( \frac{\beta - 1}{\beta} \right)^\beta, \quad (35)$$

then every solution of (1) is either oscillatory or tends to zero.

*Proof.* By [2] condition (34) ensures that the difference inequality (26) has no positive solution. On the otherhand, it follows from [2] that condition (35) guarantees that the difference inequality (27) has no positive solutions. Application of Theorem 2.4 completes the proof.  $\square$

## References

- [1] R.P.Agarwal, *Difference Equations and Inequalities: Theory, Methods and Applications*, Marcel Dekker, New York, (1999).
- [2] I.Gyori and G.Ladas, *Oscillation Theory of Delay Differential Equations with Applications*, Clarendon Press, Oxford, (1991).
- [3] W.T.Li and S.S.Cheng, *Classifications and existence or positive solutions of second order nonlinear neutral difference equations*, Funkcial. Ekvac., 40(1997), 371-393.
- [4] E.Tunc and S.R.Grace, *On oscillatory and asymptotic behavior of a second order nonlinear damped neutral differential equation*, International Journal of Differential Equations, 2016(2016), Article ID 3746368.
- [5] T.Li, Y.V.Rogovchenko and C.Zhang, *Oscillation of second-order neutral differential equations*, Funkcial. Ekvac., 56(2013), 111-120.
- [6] A.Murugesan and K.Ammamuthu, *New oscillation criteria for second order neutral difference equations*, International Journal of Mathematics and Statistics Invention, 5(2)(2017), 53-57.
- [7] P.J.Y.Wong and R.P.Agarwal, *Oscillation and monotone solutions of second-order quasilinear difference equations*, Funkcial. Ekvac., 39(1996), 491517.
- [8] W.T.Li and X.L.Fan, *Oscillation criteria for second-order nonlinear difference equations with damped term*, Comp. Math. Appl., 37(1999), 1730.
- [9] R.N.Rath, Seshadev Padhi and B.L.S.Barik, *Oscillatory and asymptotic behaviour of a homogeneous neutral delay difference equation of second order*, Bulletin of the Institute of Mathematics Academia Sinica (New Series), 3(3)(2008), 453-467.

- [10] E.Thandapani, I.Gyori and B.S.Lalli, *An application of discrete inequality to second-order nonlinear oscillation*, J. Math. Anal. Appl., 186(1994), 200208.