International Journal of Mathematics And its Applications
Volume 5, Issue 1–B (2017), 237–245.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Numerical Simulation of Heat Diffusion in a Homogeneous Bar

Research Article

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Abstract: Heat transfer phenomena are caused by the heterogeneity of an intensive physical magnitude between two systems [1, 6]. The temperature parameter T is one of these quantities which will be responsible for a particular transport problem [13]. This will be a transfer of heat energy heat which can be carried out in three processes: conduction, convection and radiation [12, 13]. Our goal is to find the analytical solution $T(x,t)$ of the heat diffusion inside a homogeneous bar of conduction coefficient λ of section S, of length L, maintained at a constant temperature T_0 on each of its ends, and initially heated at a temperature T_i over a length of $2l$ with well-defined boundary conditions and compare it with the numerical solution using the explicit finite difference scheme by studying the stability and consistency of the scheme [2, 3, 5, 11]. The mathematical and numerical modeling results of thermal diffusion in general makes it possible to thoroughly study the application in different fields such as chemistry, metallurgy, and micro-electronics manufacturing [8, 9].

Keywords: Diffusion, heat transfer, conduction, stability of scheme.

1. Introduction

The problem of diffusion of heat in a homogeneous bar is rapidly reduced to solving an equation with partial differentials [10]. For this work, we consider the problem of heat diffusion in a homogeneous bar (Figure 1), conduction coefficient λ, density ρ, heat coefficient C_p, section S and length L, without internal energy production. Assuming that, the two ends are maintained at a constant temperature T_0 and T_i is the temperature distribution at the initial instant, assumed to be symmetrical with respect to $x = 0$

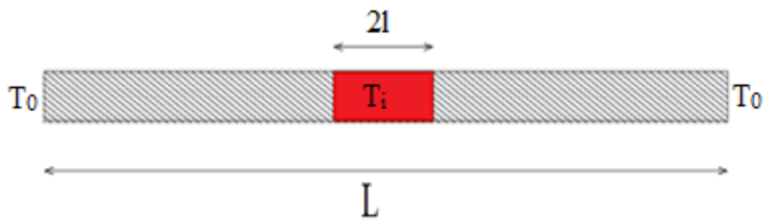

Figure 1: homogeneous bar

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The conservation equation of the energy is written:

$$\begin{cases} \rho C_p S \frac{\partial T}{\partial t} - \lambda S \frac{\partial^2 T}{\partial x^2} = 0 \quad \text{for } \ \mathrm{x} \in [0, \mathrm{L}] \\ \mathrm{T}(0, \mathrm{t}) = \mathrm{T}_0 \\ \mathrm{T}(\mathrm{L}, \mathrm{t}) = \mathrm{T}_0 \end{cases}$$

By performing the change of variable $u = T - T_0$, one obtains the following model problem (1) with homogeneous Dirichlet boundary conditions

$$\begin{cases} \frac{\partial u}{\partial t} - \kappa \frac{\partial^2 T}{\partial x^2} = 0 \quad \text{for } \ \mathrm{x} \in [0, \mathrm{L}] \\ \mathrm{u}(0, \mathrm{t}) = \mathrm{u}(\mathrm{L}, \mathrm{t}) = 0 \quad \text{for } \quad \mathrm{t} > 0 \\ u(x, 0) = C(x) \qquad \text{for } \ \mathrm{x} \in [0, \mathrm{L}] \end{cases} \tag{1}$$

With $\kappa = \frac{\lambda}{\rho C}$ is the thermal diffusivity of the material and $C(x)$ is the relative temperature.

2. Analytical Formulation

In order to obtain a reference solution for the numerical simulations, we will see an analytical solution of the problem (1), using the classical method of variable separation. We consider the problem without any initial condition fixed:

$$\begin{cases} \frac{\partial u}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \\ \mathrm{u}(0, \mathrm{t}) = \mathrm{u}(\mathrm{L}, \mathrm{t}) = 0 \end{cases} \tag{2}$$

An elementary solution of this problem 2 is sought in the form of separate variables:

$$u(x, t) = f(x)g(t)$$

Substituting this relation in equation (2) we obtain: $f(x)g'(x) = \kappa f''(x)g(t)$. By dividing by $f(x)g(x)$, it comes:

$$\frac{1}{\kappa}\frac{g'(t)}{g(t)} = \frac{f''(x)}{f(x)} \tag{3}$$

Examination of equation (3) shows that the left-hand member is a function of t only, while the right-hand member is a function of x. Consequently, if x varies and t remains constant, $\frac{1}{\kappa}\frac{g'(t)}{g(t)}$ remains constant. Similarly, if t varies and x remains constant $\frac{f''(x)}{f(x)}$ remains constant. Each of these members must therefore be constant, say equal to α. Hence the two equations:

$$\begin{cases} \frac{g'(t)}{g(t)} = \alpha\kappa \\ \frac{f''(x)}{f(x)} = \alpha \end{cases}$$

Which is rewritten as:

$$\begin{cases} g'(t) - \alpha\kappa g(t) = 0 & \text{(4a)} \\ f''(x) - \alpha f(x) = 0 & \text{(4b)} \end{cases}$$

By integrating the equation (4a) for g (t), it comes: $g(t) = \beta_1 \exp(\alpha\kappa t)$. Now the sign of the constant α gives the evolution of the temperature over time (since $\kappa > 0$). If the constant α is positive, the temperature increases exponentially, which is not physically acceptable. On the other hand, if the constant α is negative, the temperature decreases exponentially, which is a priori the solution to be retained. We will, however, consider the two cases, and show that only the second case leads to a solution that is not identically zero. Assuming $\alpha > 0$, we put $\alpha = \omega^2$. By rewriting the equation (4b), we obtain:

$$f''(x) - \omega^2 f(x) = 0 \tag{5}$$

This equation (5) is a homogeneous second order equation with constant coefficients. Let us look for the solution in the form: $f(x) = e^{kx}$. We then obtain the characteristic equation: $k^2 - \omega^2 = 0$, which fixes the possible values of $k = \omega$, whence the solution $f(x)$:

$$f(x) = c_1 e^{\omega x} + c_2 e^{-\omega x}$$

Or

$$f(x) = c' ch(\omega x) + c'' sh(\omega x)$$

The solution $u(x,t)$ of (2) is then written:

$$u(x,t) = e^{-(\kappa\omega^2 t)}[Ach(\omega x) + Bsh(\omega x)]$$

With ω, A and B are constants. It can be shown without difficulty that the application of the boundary conditions of the problem (1) leads to $A = B = 0$ and therefore to an identically zero solution. Assuming now $\alpha < 0$, we put $\alpha = -\omega^2 < 0$, and in this case following the same approach as before, we show that the solution is written:

$$u(x,t) = e^{-(\kappa\omega^2 t)}[A\cos(\omega x) + B\sin(\omega x)]$$

The application of the first boundary condition $u(0,t) = 0$ for $x = 0$, given

$$e^{-(\kappa\omega^2 t)}[A] = 0 \Rightarrow A = 0$$

$u(x,t)$ is then written: $u(x,t) = e^{-(\kappa\omega^2 t)}[B\sin(\omega x)]$. The application of the second condition to $x = L$ implies:

$$e^{-(\kappa\omega^2 t)}[B\sin(\omega x)] = 0 \Rightarrow \omega = \tfrac{n\pi}{L} n \in \mathrm{Z}$$

If $B = 0$ is chosen, an identically zero solution is found. It is therefore assumed that $B \neq 0$ and we get $e^{-(\kappa\omega^2 t)} \neq 0$. The solution of (2) is written:

$$u(x,t) = Be^{-(\kappa(\frac{n\pi}{L})^2 t)}\sin(\frac{n\pi}{L}x) \tag{6}$$

For each value of n we thus have an elementary solution dependent on n and where B is an arbitrary constant that will be denoted C_n:

$$u(x,t) = C_n e^{-\kappa(\frac{n\pi}{L})^2 t}\sin(\frac{n\pi}{L}x) \tag{7}$$

And according to the principle of superposition, any linear combination of solutions of the form (7) is also solution of (1). The general solutions of (7) are written:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$
$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\kappa(\frac{n\pi}{L})^2 t}\sin(\frac{n\pi}{L}x) \tag{8}$$

Where C_n are arbitrary constant. It is found that since no initial condition has been fixed, problem (2) admits infinity of solutions.

- General solution checking initial condition

Solutions (8) satisfy the equation and boundary conditions of Problem A. To satisfy the initial condition $(u(x,0) = C(x))$, we must have:

$$\sum_{n=1}^{\infty} C_n \sin(\frac{n\pi}{L}x) = C(x) \quad \text{for } 0 \le x \le L \tag{9}$$

The problem now comes back to the determination of C_n that the series (9) converges to $C(x)$ for all $x \in [0, L]$. To calculate the constants C_n, multiply the equation (9) by $\sin(\frac{n\pi}{L}x)$, and integrate on the interval $[0, L]$. Using the orthogonality of the sinus functions:

$$\int_0^L \sin(\frac{n\pi}{L}x)\sin(\frac{k\pi}{L}x)dx = \frac{L}{2}\delta_{k,n}$$

We obtain the value of C_k

$$C_k = \frac{2}{L}\int_0^L C(x)\sin(\frac{k\pi}{L}x)dx \tag{10}$$

with C_k are the Fourier coefficients of the periodic function of period $2L$ coinciding with $C(x)$ on the interval $[0, L]$. The general solution of the problem (1) is written:

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\kappa(\frac{n\pi}{L})^2 t}\sin\left(\frac{n\pi}{L}x\right) \tag{11}$$

$$C_n = \frac{2}{L}\int_0^L C(y)\sin\left(\frac{k\pi}{L}y\right)dy \tag{12}$$

In the particular case $C(x) = \sin\left(n\frac{\pi x}{L}\right)$, we find the eigenmode of order n of the problem (1)

$$u(x,t) = \sin\left(\frac{n\pi}{L}x\right)e^{-\kappa(\frac{n\pi}{L})^2 t} \tag{13}$$

This eigenmode decreases self-similar and exponentially over time in proportion to the square of its length $\left(\frac{L}{n\pi}\right)$. This shows that a fluctuation in temperature is all the more damped over time as its wavelength $\left(\frac{L}{n\pi}\right)$ is short. It is a phenomenon characteristic of diffusion processes.

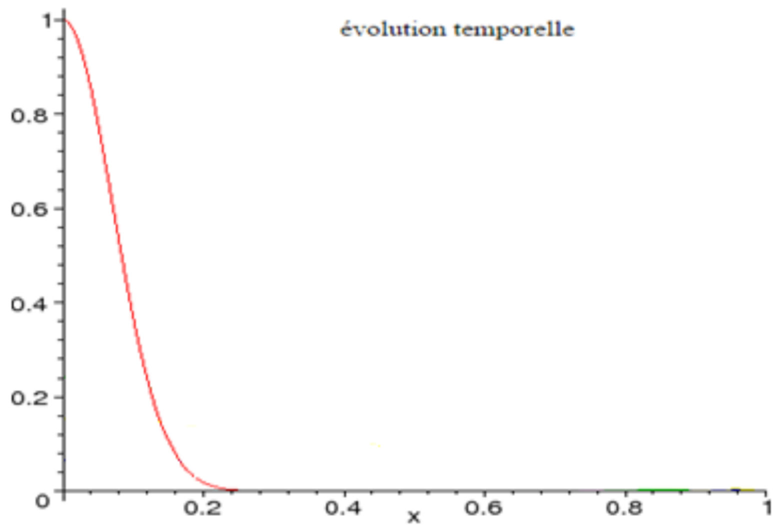


Figure 2: Analytical solution of temporal evolution

3. Study of Consistency and Stability

- Consistency

We notice $u(x,t)$ is the solution of the partial differential equation (PDE) and u_i^n is the solution of the finite difference equation (FDE). We will calculate the difference E_{rr} between the EDP and the EDF at the nodes of the mesh and for the exact solution u.

$$E_{rr} = \frac{U_i^{n+1} - U_i^n}{\Delta t} - \kappa \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{\Delta x^2} \tag{14}$$

To calculate this truncation error, all quantities are expressed as a function of U_i^n by carrying out Taylor series development in the vicinity of the point $(i\Delta x, n\Delta t)$

$$\begin{aligned}
U_i^{n+1} &= U_i^n + \frac{\Delta t}{1!}\left.\frac{\partial U}{\partial t}\right|_{(i,n)} + \frac{\Delta t^2}{2!}\left.\frac{\partial^2 U}{\partial t^2}\right|_{(i,n)} + O(\Delta t^3)\\
U_{i+1}^n &= U_i^n + \frac{\Delta x}{1!}\left.\frac{\partial U}{\partial x}\right|_{(i,n)} + \frac{\Delta x^2}{2!}\left.\frac{\partial^2 U}{\partial x^2}\right|_{(i,n)} + \frac{\Delta x^3}{3!}\left.\frac{\partial^3 U}{\partial x^3}\right|_{(i,n)} + \frac{\Delta x^4}{4!}\left.\frac{\partial^4 U}{\partial x^4}\right|_{(i,n)} + O(\Delta x^5)\\
U_{i-1}^n &= U_i^n - \frac{\Delta x}{1!}\left.\frac{\partial U}{\partial x}\right|_{(i,n)} + \frac{\Delta x^2}{2!}\left.\frac{\partial^2 U}{\partial x^2}\right|_{(i,n)} - \frac{\Delta x^3}{3!}\left.\frac{\partial^3 U}{\partial x^3}\right|_{(i,n)} + \frac{\Delta x^4}{4!}\left.\frac{\partial^4 U}{\partial x^4}\right|_{(i,n)} + O(\Delta x^5)
\end{aligned}$$

By deferring these developments in the truncation error, it comes

$$E_{rr} = \left.\frac{\partial U}{\partial t}\right|_{(i,n)} + \frac{\Delta t}{2!}\left.\frac{\partial^2 U}{\partial t^2}\right|_{(i,n)} - \kappa\left(\left.\frac{\partial^2 U}{\partial x^2}\right|_{(i,n)} + \frac{\Delta x^2}{12}\left.\frac{\partial^4 U}{\partial x^4}\right|_{(i,n)}\right) + O\left(\Delta t^2, \Delta x^4\right) \tag{15}$$

We then use the fact that U_i^n satisfies the exact equation at point (i,n)

$$\left.\frac{\partial U}{\partial t}\right|_{(i,n)} - \kappa\left.\frac{\partial^2 U}{\partial x^2}\right|_{(i,n)} = 0$$

Which gives the expression of the truncation error of the explicit scheme :

$$E_{rr} = \frac{\Delta t}{2}\left.\frac{\partial^2 U}{\partial t^2}\right|_{(i,n)} - \kappa\frac{\Delta x^2}{12}\left.\frac{\partial^4 U}{\partial x^4}\right|_{(i,n)} + O\left(\Delta t^2, \Delta x^4\right) \tag{16}$$

$$E_{rr} = \kappa\left(\kappa\frac{\Delta t}{2} - \frac{\Delta x^2}{12}\right)\left.\frac{\partial^4 U}{\partial x^4}\right| + O\left(\Delta t^2, \Delta x^4\right) \tag{17}$$

It is found that this truncation error tends towards zero, when x and t tend towards zero independently. We note that for the particular value of r such that:

$$\boxed{\frac{\Delta t}{\Delta x^2} = \frac{\kappa}{6}} \tag{18}$$

The first term of the truncation error vanishes.

- Stability of scheme

Let U_i^n be the numerical solution of our problem

$$U_i^{n+1} = U_i^n + r\left(U_{i+1}^n - 2U_i^n + U_{i-1}^n\right) \tag{19}$$

In our study, we consider ε_i^n a perturbation of the solution in step n with the boundary condition on the perturbation is a homogeneous condition of Dirichlet: $\varepsilon_0^n = \varepsilon_N^n = 0$. The perturbed solution $U_i^{n+1} + \varepsilon_i^{n+1}$ in step $n+1$ is a solution of the finite difference equation:

$$U_i^{n+1} + \varepsilon_i^{n+1} = U_i^n + \varepsilon_i^n + r\left(U_{i+1}^n + \varepsilon_{i+1}^n - 2(U_i^n + \varepsilon_i^n) + U_{i-1}^n + \varepsilon_{i-1}^n\right) \tag{20}$$

The equation on the perturbation is obtained by making the difference (19) - (20):

$$\varepsilon_i^{n+1} = \varepsilon_i^n + r(\varepsilon_{i+1}^n - 2\varepsilon_i^n + \varepsilon_{i-1}^n) \tag{21}$$

This perturbation is decomposed at any point $x_i = i\Delta x$ of the mesh and at any instant t_n in the form of a series of Fourier modes with $J = \sqrt{-1}$:

$$\varepsilon_i^n = \sum_{j=1}^{\infty} \lambda_j^n \exp(J\omega_j i\Delta x) \tag{22}$$

Since the problem is linear, each of the modes satisfies equation (20), which is written for a mode j.

$$\lambda_j^{n+1} e^{(I\omega_j i\Delta x)} = \lambda_j^n e^{(I\omega_j i\Delta x)} + r\lambda_j^n \left(e^{(I\omega_j (i+1)\Delta x)} - 2e^{(I\omega_j i\Delta x)} + e^{(I\omega_j (i-1)\Delta x)} \right) \tag{23}$$

By simplifying by $e^{(I\omega_j i\Delta x)}$, the equation (**??**) is written:

$$\lambda_j^{n+1} = \lambda_j^n \left[1 + r\left(e^{I\omega_j \Delta x} - 2 + e^{-I\omega_j \Delta x)} \right) \right] \tag{24}$$

By positing $\alpha = \omega_j \Delta x$ and by using the trigonometric relations $\cos\alpha = \frac{e^{J\alpha}+e^{-J\alpha}}{2}$, equation (23) becomes:

$$\lambda_j^{n+1} = \lambda_j^n \left[1 - 4r\sin^2(\frac{\alpha}{2}) \right] \tag{25}$$

It is clear that if the amplitude λ_j^n of each mode of the perturbation decreases from iteration to the next, the error decreases and the disturbance eventually disappears. This results in the condition:

$$\left| \frac{\lambda_j^{n+1}}{\lambda_j^n} \right| = \left| 1 - 4r\sin^2(\frac{\alpha}{2}) \right| \leq 1 \tag{26}$$

This must be verified for each mode j. Which give

$$0 \leq r \leq \frac{1}{2\sin^2(\alpha)}, \quad \forall \alpha$$

This inequality imposes a maximum value at r

$$r \leq \frac{1}{2} \tag{27}$$

Hence the explicit scheme is conditionally stable, with a stability criterion given by the condition:

$$\kappa \frac{\Delta t}{\Delta x^2} \leq \frac{1}{2} \tag{28}$$

This condition is used in the following way. The mesh is fixed in space Δx and the step Δt in integration time must then be chosen such that:

$$\boxed{\Delta t \leq \frac{\Delta x^2}{2\kappa}} \tag{29}$$

4. Numerical Solution

The numerical solution is calculated for two values of the mode based on the finite difference scheme, which makes it possible to see the convergence order of the solution.

- Numerical solution for $N = 100$

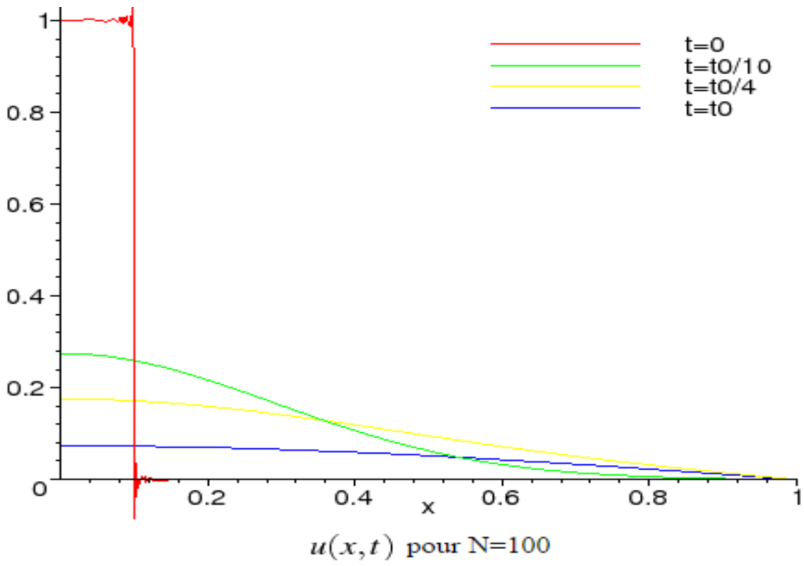


Figure 3: Numerical solution for $N = 100$

- Numerical solution for $N = 20$

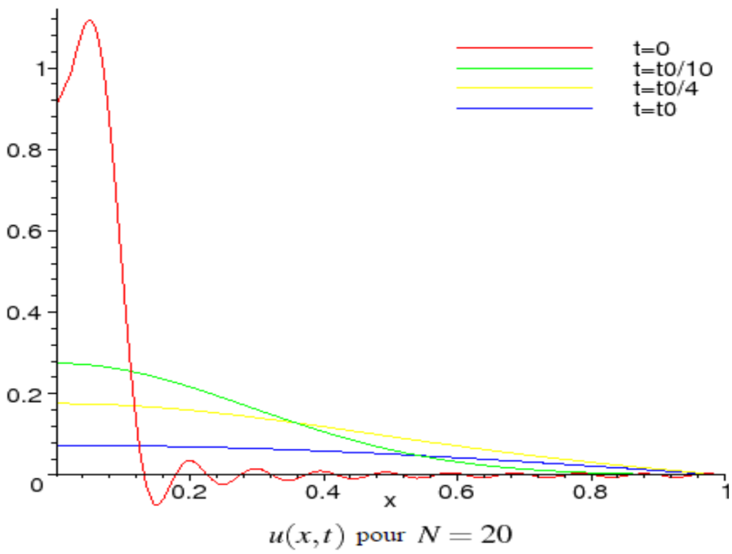


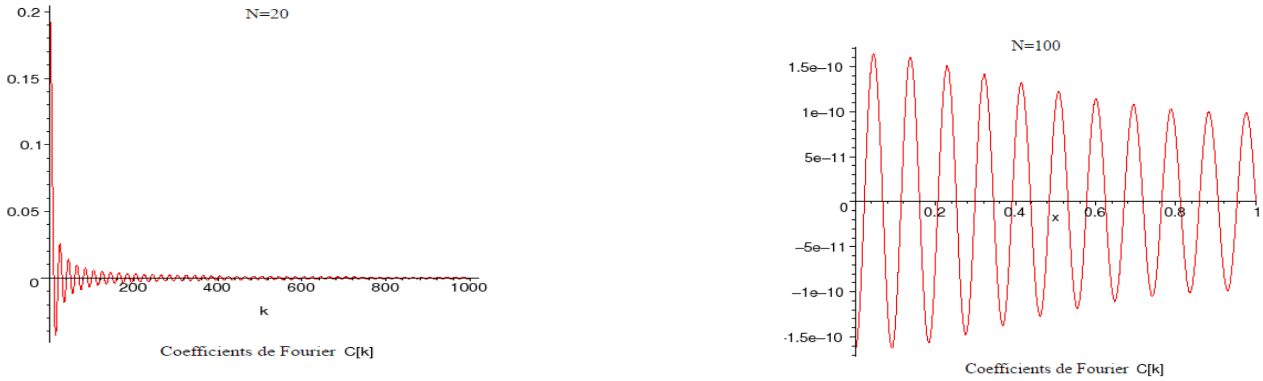
Figure 4: Numerical solution for $N = 20$

We observe that the convergence of the Fourier series is very slow, since the amplitude of the coefficients C_k decreases very slowly towards zero. This is also what is seen in Figure (1.2b), where the initial solution calculated with $N = 100$ modes present oscillations on the edges.

We note also that the solutions for $t > 0$ no longer exhibit its parasitic oscillations. Indeed, the modes associated with the large values of k have an amplitude which decreases very rapidly, and therefore the Fourier series converges very rapidly to the solution $u(x, t)$ for $t > 0$.

It is found that the solutions resemble solutions calculated with $N = 20$ modes (Figure 5b). For the initial solution, it has more oscillations than the solution with $N = 100$ modes, which confirms the low convergence of the Fourier series in this case.

In (Figure 5a and 5b), we have plotted the difference between the solution calculated with $N = 20$ modes and that with $N = 100$ modes for values of t. We find that the error is very small $(2 < 10 \wedge (-10))$ and decreases with time, which shows that the solution computed with $N = 20$ modes is a very good approximation of the exact solution for $t > 0$.



(a) Fourier Coefficient for N=20

(b) Fourier Coefficient for N=100

Figure 5:

5. Conclusion

In this manuscript, the attempt has been made to find the analytical and numerical solution of heat diffusion in a homogenous bar independent of thermal conductivity in heat with well-defined conditions. To find numerical solution of problem, the finite difference scheme has been developed for governing linear partial differential equations. The convergence and stability analysis of finite difference solution has been done by fundamental theorems of numerical analysis.

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