International Journal of Mathematics And its Applications
Volume 5, Issue 1–C (2017), 291–300.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Degradation and Subsequent Regeneration of Forestry Biomass using Genetic Resource

Research Article

Anuj Kumar[1]* and Manju Agarwal[1]

1 Department of Mathematics and Astronomy, University of Lucknow, Lucknow, India.

Abstract: In present paper, a nonlinear mathematical model is proposed and analyzed to study the depletion of forestry resources by the human population and industrial pressure. It shows that density of forestry resource biomass decreases as the population and industrial pressure increases. We also observed the effect of genetic resources on the depletion of forestry biomass. The model is analyzed by using stability theory of differential equations. The numerical simulation analysis of the model confirms the analytical results.

MSC: 00A71, 34L30.

Keywords: Mathematical Model, Stability Analysis, Genetic Resource.

1. Introduction

Forests play an important role in the human life. It is one of the major natural resources used by human population for their day to day livelihood with the alarming rate of increase of population abundantly, more space, food, shelter one required for their survival. This requirement can be made by cutting trees for space, wood and setup industries for live hood. Conservation and management of forest resources would be essential for sustainable development and adaptation to the changing environment. Genetic resources are one of the most important components for sustainable forestry and food and fodder security. In general, genetic resources include plant species that do not only provide food, fodder and medicine but also as a source of shelter, energy, environmental protection and other supports to the livelihood. The effect of the forest biomass has been studied by many investigators. Shukla and Dubey [8] had studied the combine effect of population and pollution on the forestry resources by proposing a nonlinear mathematical model. Dubey et al. [4] proposed the mathematical model and analyzed to study the depletion of forestry biomass caused by population and population pressure augmented industrialization. Mishra and Lata [6] developed a mathematical model and studied the depletion of forestry resources, caused simultaneously by population and population pressure augmented industrialization. In their model for controlling the population pressure, they also considered economic efforts. Some other authors also investigated about, effect of population and industrial on forestry biomass, [1, 2, 5]. In the recently Agarwal and Pathak [3] studied the effect of the alternative resource and time delay on conservation of forestry biomass, interaction between forestry biomass, industrialization pressure, toxicant pressure and technological effort proposed and analyzed. In this paper, therefore we

* *E-mail: guptaanujkm89@mail.com*

proposed and analyzed a mathematical model to study the effect of genetic resources on the forestry biomass, which is depleted due to population and industrialization. One of our aims is to show the detrimental effect of genetic of forestry biomass. In the view of above, a mathematical model for the depletion of forestry biomass is proposed and analyzed using the stability theory to see the effect of population and industrial pressure on the depletion of forestry biomass and its conservation by genetic resources.

2. Mathematical Model

An important fact about mathematical model is given as;

H_1. Carrying capacity of forestry biomass is $K(I,P)$, which depends on population biomass and industrial effect. $K(I,P) = K_0 - K_1 + K_1 e^{-c_2 P - c_1 I}$, $K(0,0) = K_0$ and $K_\infty = K_0 - K_1$, where $K_1 \in (0, K_0)$.

H_2. k_1 and k_2 are describe the growth rate of genetic resource due to forestry; population and forestry; industrial effect.

Considering all above point the mathematical model of the following variables is given as follow,

$$\frac{dB}{dt} = B\left(1 - \frac{B}{K(I,P)}\right) - S_2 B^2 I - S_1 B^2 P + kF, \tag{1}$$

$$\frac{dP}{dt} = P\left(1 - \frac{P}{L}\right) + S_{10} B^2 P, \tag{2}$$

$$\frac{dI}{dt} = \mu\theta P + S_{20} B^2 I - QI, \tag{3}$$

$$\frac{dF}{dt} = k_1 S_1 B^2 P + k_2 S_2 B^2 I - k_0 F, \tag{4}$$

Where $B(0) \geq 0$, $P(0) \geq 0$, $I(0) \geq 0$, $F(0) \geq 0$. Here B is forestry biomass, P is population biomass, I is the industrial effect and F is genetic resource. S_2 is the depletion rate of forestry biomass due to industrial effect, S_1 is the depletion rate of forestry biomass due to population biomass, k is the growth rate of forestry biomass due to effort of genetic resource, L is the carrying capacity of population biomass. S_{10} is the growth rate of population biomass due to forestry biomass, μ is the growth coefficient, θ is the constant parameter, S_{20} is the growth rate of industrial effect due to forestry biomass, Q is the loss rate of industrial effect and k_0 is the depletion rate of genetic resource.

3. Region of Attraction

Theorem 3.1. *The region of attraction of the system of differential equation as,*

$$R = \left[(B,P,I,F) : 0 < B + F \leq \frac{K_0}{\delta_0}, 0 < P \leq L(1 + \delta_1^2 S_{10}), 0 < I \leq \frac{\mu\theta(1 + S_{10}\delta_1^2)L}{Q}\right], \tag{5}$$

Where $\delta_0 = \min(1, k_0 - k)$ *and* $\delta_1 = B_{max}$.

Proof. From the equations (1), (4) of model

$$\frac{d}{dt}(B+F) = \left\{B\left(1 - \frac{B}{K(I,P)}\right) - S_2 B^2 I - S_1 B^2 P + kF\right\} + \left\{\left(k_1 S_1 B^2 P + k_2 S_2 B^2 I\right) - k_0 F\right\}$$

$$\frac{d}{dt}(B+F) = B\left(1 - \frac{B}{K(I,P)}\right) - S_1(1-k_1)B^2 P - S_2(1-k_2)B^2 I - (k_0 - k)F$$

$$\frac{d}{dt}(B+F) \le B\left(1-\frac{B}{K(I,P)}\right)-(k_0-k)F$$
$$\frac{d}{dt}(B+F) \le B-\left(\frac{B}{K_0}\right)B-(k_0-k)F$$
$$\frac{d}{dt}(B+F)+\delta_0(B+F) \le 2B-\frac{B^2}{K_0}$$

where $\delta_0 = min\,(1, k_0-k)$,

$$\frac{d}{dt}(B+F)+\delta_0(B+F) \le \phi(B) \tag{6}$$

where $\phi(B) = 2B - \left(\frac{B}{K_0}\right)B$, then maximum value of $\phi(B) = K_0$, then

$$\frac{d}{dt}(B+F)+\delta(B+F) \le K_0$$

then $\limsup B+F \le \dfrac{K_0}{\delta_0}$. Now form equation (3)

$$\frac{dP}{dt} \le \left(1+S_{10}B_{max}^2-\frac{P}{L}\right)P,$$

We obtained $\limsup P \le (1+S_{10}\delta_1^2)L$, where δ_1 is the maximum value of B. From (3),

$$\frac{dI}{dt} \le \mu\theta P_{max} - QI$$

we get $\limsup I \le \dfrac{\mu\theta(1+S_{10}\delta_1^2)L}{Q}$. □

4. Equilibrium and Their Existence

Above system of equations has the five possible points $E_0(0,0,0,0)$, $E_1(K_0,0,0,0)$, $E_2(0,L,\mu\theta L/Q,0)$, $E_3(B_3,0,I_3,F_3)$, $E^*(B^*,P^*,I^*,F^*)$.

1. Equilibrium points E_0, E_1 and E_2 are obvious.

2. The existence of $E_3(B_3,0,I_3,F_3)$ can be seen by the system of equations as,

$$B\left(1-\frac{B}{K(I,0)}\right)-S_2B^2I=0,$$
$$S_{20}B^2I-QI=0,$$
$$k_2S_2B^2I-k_0F=0,$$

From second and third equations of the above system of equations

(i). $B^2 = \frac{Q}{S_{20}}$

(ii). $F = \frac{k_2S_2QI}{k_0S_{20}}$

using (i)and (ii) conditions in first equation of the above system, we get,

$$\gamma(I) = \left[\frac{Q}{S_{20}}\left(\frac{1}{K(I,0)}+S_2I\right)-\frac{kk_2S_2QI}{k_0S_{20}}\right]^2-\frac{Q}{S_{20}} \tag{7}$$

then, $\gamma(0)<0$, if $S_{20}K_0>Q$, $\gamma(\eta)>0$ where η is the maximum value of I, then there exist a positive $I_3\in(0,\eta)$ such that $\gamma(I_3)=0$. Now, the sufficient condition for the uniqueness of $E_3(B_3,0,I_3,F_3)$ is $\gamma'(I)>0$ the above condition is hold if $kk_2<k_0$.

3. Now the existence of equilibrium point $E^*(B^*, P^*, I^*, F^*)$ can be obtained by the equations,

$$B\left(1 - \frac{B}{K(I,P)}\right) - S_2B^2I - S_1B^2P + kF = 0, \tag{8}$$

$$P\left(1 - \frac{P}{L}\right) + S_{10}B^2P = 0, \tag{9}$$

$$\mu\theta P + S_{20}B^2I - QI = 0, \tag{10}$$

$$\left(k_1S_1B^2P + k_2S_2B^2I\right) - k_0F = 0 \tag{11}$$

above equations have the positive equilibrium points, it can be seen by following algebraic simulation. Form equations (9) - (11), we obtained

$$P = (1 + S_{10}B^2)L = w_1(B)$$
$$I = \frac{\mu\theta L(1 + S_{10}B^2)}{Q - S_{20}B} = w_2(B)$$
$$F = \frac{(k_1S_1P + k_2S_2I)B^2}{k_0} = w_3(B)B^2$$

using values of P, I and F in equation (8), we obtained

$$\phi(B) = 1 - B\left[h(B) + S_1w_1(B) + S_2w_2(B) - kw_3(B)\right] \tag{12}$$

where $h(B) = \frac{1}{K(w_1(B), w_2(B))}$, $\phi(0) = 1 > 0$ and $\phi(\delta_1) < 0$ from equation (12) there will be a positive root $B^* \in (0, \delta_1)$ such that $\phi(B^*) = 0$ and this root must be unique for $\phi'(B) < 0$. After evaluating the value of B^*, we can obtained the positive values of P^*, I^* and F^*.

5. Stability Analysis of the System

Stability of the system can be studied by the method of Jacobian matrices, variation matrix of the equilibrium point. Then the Jacobian matrix at interior equilibrium point $E^*(B^*, P^*, I^*, F^*)$ is given as

$$V(E^*) = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & 0 & 0 \\ A_{31} & A_{32} & A_{33} & 0 \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \tag{13}$$

where

$$A_{11} = \left(1 - \frac{2B^*}{K(I^*, P^*)}\right) - 2S_2B^*I^* - 2S_1B^*P^*;\ A_{12} = -\frac{B^{*2}K_1c_2e^{-c_1I^*-c_2P^*}}{K^2(I^*, P^*)} - S_1{B^*}^2$$
$$A_{13} = -\frac{B^{*2}K_1c_1e^{-c_1I^*-c_2P^*}}{K^2(I^*, P^*)} - S_2{B^*}^2;\ A_{14} = k;\ A_{21} = 2S_{10}B^*P^*$$
$$A_{22} = 1 - \frac{2P^*}{L} + S_{10}{B^*}^2;\ A_{31} = 2S_{20}B^*I^*;\ A_{32} = \mu\theta;\ A_{33} = S_{20}{B^*}^2 - Q$$
$$A_{41} = 2B^*(k_1S_1P^* + k_2S_2I^*);\ A_{42} = k_1S_1{B^*}^2;\ A_{43} = k_2S_2{B^*}^2;\ A_{44} = -k_0$$

then the characteristic equation of the above matrix is given as,

$$\lambda^4 + \lambda^3a_{11} + \lambda^2a_{22} + \lambda a_{33} + a_{44} = 0 \tag{14}$$

where

$$a_{11} = -A_{44} - A_{11}A_{22} - A_{11}A_{33} - A_{22}A_{33} = k_0 - A_{11}A_{22} - A_{11}A_{33} - A_{22}A_{33},$$

$$a_{22} = A_{11} + A_{22} + A_{33} - A_{21}A_{14} - A_{13}A_{31} + A_{41}A_{14} + A_{44}(A_{11}A_{22} + A_{11}A_{33} + A_{22}A_{33}),$$

$$a_{33} = -A_{14}A_{21}A_{42} - A_{14}A_{43}A_{31} + A_{14}A_{41}(A_{22} + A_{33}) - A_{13}A_{21}A_{32} + A_{13}A_{31}(A_{22} + A_{44}) - A_{44}(A_{11} + A_{22} + A_{33}) + A_{11}A_{22}A_{33} + A_{21}A_{14}(A_{33} + A_{44}), \text{ and}$$

$$a_{44} = A_{14}A_{21}A_{43}A_{33} - A_{14}A_{21}A_{32}A_{33} + A_{14}A_{22}A_{43}A_{31} - A_{14}A_{41}A_{33}A_{22} + A_{44}A_{13}A_{21}A_{33} - A_{13}A_{31}A_{22}A_{44} - A_{11}A_{22}A_{44}A_{33} - A_{14}A_{21}A_{44}A_{33}$$

then by Routh Hurwitz criterion equilibrium point $E^\star$ is locally asymptotically stable if, $a_{11} > 0$, $a_{11}a_{22} - a_{33} > 0$ and $(a_{11}a_{22} - a_{33})a_{33} - a_{11}^2 a_{44} > 0$, unstable if either of these conditions is not satisfied. Now characteristic equation for equilibrium point $E_0(0,0,0,0)$ is given as

$$(\lambda - 1)^2(\lambda - \mu\theta)(\lambda + k_0) = 0 \tag{15}$$

then characteristic roots are given as 1, 1, $\mu\theta$ and $-k_0$, it is unstable in $B-P-I$ plane but stable in direction F so this point is saddle point. For the point $E_1(B_1,0,0,0)$ characteristics values are given as, $1 - \frac{2B_1}{K_0} < 0$, $1 + S_{10}B_1^2 > 0$, $\mu\theta > 0$, $-k_0 < 0$ then this is asymptotically stable in plane $B-F$ but unstable in plane $P-I$, so it is saddle point. For equilibrium point $E_2(0, L, \mu\theta\text{L}/Q, 0)$, eigen values are given as $1,-1,\mu\theta$ and $-k_0$ then this point is asymptotically stable in plane $P-F$ but unstable in plane $B-I$, so it is saddle point. At equilibrium point $E_3(B_3, 0, I_3, F_3)$ jacobian matrix is given as

$$V(E_3) = \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ 0 & B_{22} & 0 & 0 \\ B_{31} & B_{32} & B_{33} & 0 \\ B_{41} & B_{42} & B_{43} & B_{44} \end{bmatrix} \tag{16}$$

where

$$B_{11} = \left(1 - \frac{2B_3}{K(I_3,0)}\right) - 2S_2B_3I_3;\ B_{12} = -\frac{B_3^2K_1c_2e^{-c_1I_3}}{K^2(I_3,0)} - S_1{B_3}^2;\ B_{13} = -\frac{B_3^2K_1c_1e^{-c_1I_3}}{K^2(I_3,0)} - S_2{B_3}^2$$

$$B_{14} = k;\ B_{22} = 1 + S_{10}{B_3}^2;\ B_{31} = 2S_{20}B_3I_3;\ B_{32} = \mu\theta;\ B_{33} = S_{20}{B_3}^2 - Q;\ B_{41} = 2B_3k_2S_2I_3$$

$$B_{42} = k_1S_1{B_3}^2;\ B_{43} = k_2S_2{B_3}^2;\ B_{44} = -k_0$$

then the characteristic equation is given as

$$(\lambda - B_{22})(\lambda^3 + \lambda^2 b_1 + \lambda b_2 + b_3) = 0 \tag{17}$$

where

$$b_1 = -B_{11}B_{44} - B_{11}B_{33} - B_{33}B_{44}$$

$$b_2 = B_{11} + B_{44} + B_{33} - B_{41}B_{14} - B_{33}B_{13}$$

$$b_3 = B_{31}B_{13} + B_{43}B_{14} + B_{33}B_{41}B_{14} - B_{11}B_{33}B_{44}$$

then eigen values are given as $\lambda_2 = B_{22} > 0$ and other eigen values is given by this equation

$$(\lambda^3 + \lambda^2 b_1 + \lambda b_2 + b_3) = 0 \tag{18}$$

above equation have negative roots if it follows Hurwitz condition $b_1 > 0$, $b_3 > 0$ and $b_1b_2 > b_3$.

6. Persistence

Biologically, persistence means the survival of all populations in future time. Mathematically, persistence of a system means that strictly positive solutions do not have omega limit points on the boundary of a non-negative cone.

Theorem 6.1. *If the system of equations (1)-(4) holds the conditions (19)-(??) then the system persists.*

Proof. From the system of equations. First equation (1) of the system

$$\frac{dB}{dt} = B\left(1 - \frac{B}{K(I,P)}\right) - S_2B^2I - S_1B^2P + kF$$
$$\frac{dB}{dt} \geq B\left(1 - \frac{B}{K(0,0)}\right) - S_2B^2I_{\max} - S_1B^2P_{\max}$$
$$\frac{dB}{dt} \geq B - B^2\left\{\frac{1}{K_0} + S_2I_{\max} + S_1P_{\max}\right\}$$

let $\left\{\frac{1}{K_0} + S_2I_{\max} + S_1P_{\max}\right\} = \pi$

$$\frac{dB}{dt} \geq B\left(1 - B\pi\right)$$

after solving above equation, we get

$$\lim_{t\to\infty} \inf B \geq \frac{1}{\pi} \tag{19}$$

Now from equation (2) of system,

$$\frac{dP}{dt} = P\left(1 - \frac{P}{L}\right) + S_{10}B^2P$$
$$\frac{dP}{dt} \geq P\left(1 - \frac{P}{L}\right)$$

after solving this equation, we obtain

$$\lim_{t\to\infty} \inf P \geq L \tag{20}$$

Now from equation (3),

$$\frac{dI}{dt} = \mu\theta P + S_{20}B^2I - QI,$$
$$\frac{dI}{dt} \geq \mu\theta P_{min} - QI,$$

so we obtain

$$\lim_{t\to\infty} \inf I \geq \frac{\mu\theta L}{Q}, \tag{21}$$

Now from equation (4)

$$\frac{dF}{dt} = \left(k_1S_1B^2P + k_2S_2B^2I\right) - k_0F$$
$$\frac{dF}{dt} \geq \left(k_1S_1P_{min} + k_2S_2I_{min}\right)B^2_{min} - k_0F$$

we get,

$$\lim_{t\to\infty} \inf F \geq \frac{\left(k_1S_1P_{min} + k_2S_2I_{min}\right)B^2_{min}}{k_0}$$
$$\lim_{t\to\infty} \inf F \geq \frac{Qk_1S_1L + k_2S_2\mu\theta L}{\pi^2k_0Q}$$

□

7. Numerical Simulation

We solve the system numerically by using these values of parameters $K_0 = 10$, $K_1 = 0.2$, $c_1 = 0.001$, $c_2 = 0.001$, $S_2 = 0.06$, $S_1 = 0.008$, $k = 0.1$, $L = 5$, $S_{10} = 0.02$, $\mu = 0.5$, $\theta = 2$, $S_{20} = 0.2$, $Q = 0.75$, $k_1 = 0.002$, $k_2 = 0.02$ and $k_0 = 0.02$.For the above set of parameters the equilibrium (1.3068,5.1706,4.7389,0.5558) obtain. By using mathematical software we obtain following figures,

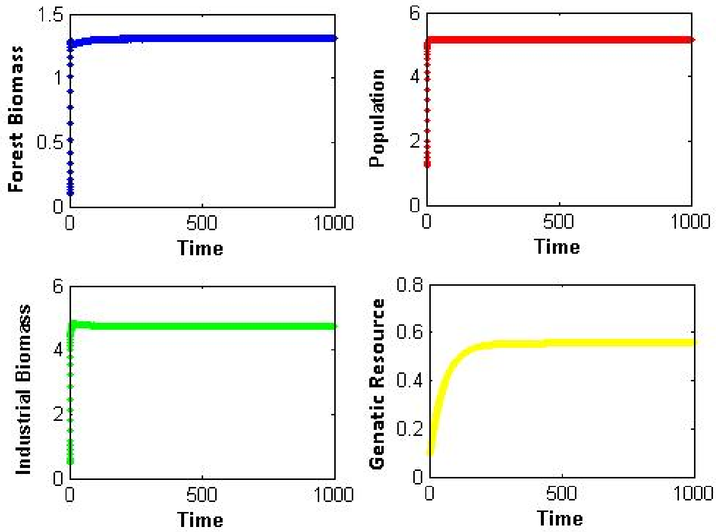

Figure 1. **Figure of B,P,I and F with respect to time.**

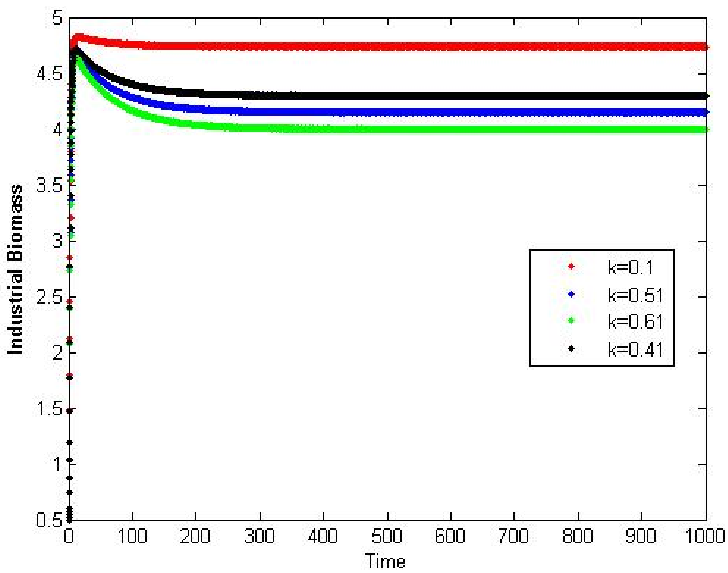

Figure 2. **Behavior of Industrial Biomass with varying the value of Genetic Resources on Forestry Biomass.**

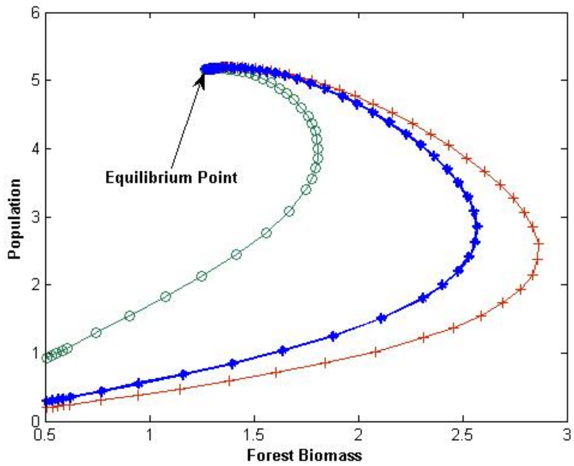

Figure 3. **Phase plot of Forestry Biomass and Population.**

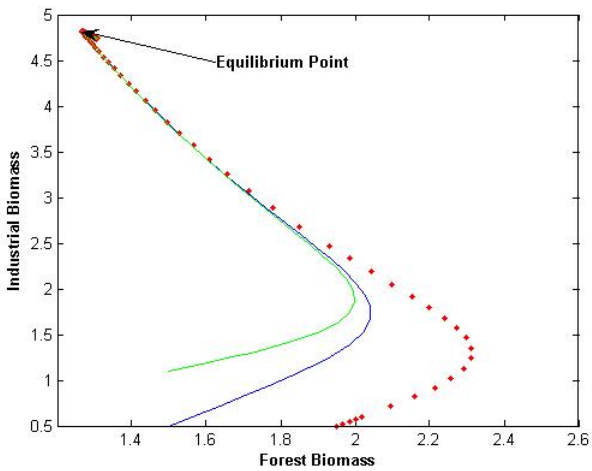

Figure 4. **Phase plot of Forestry Biomass and Industrial Biomass.**

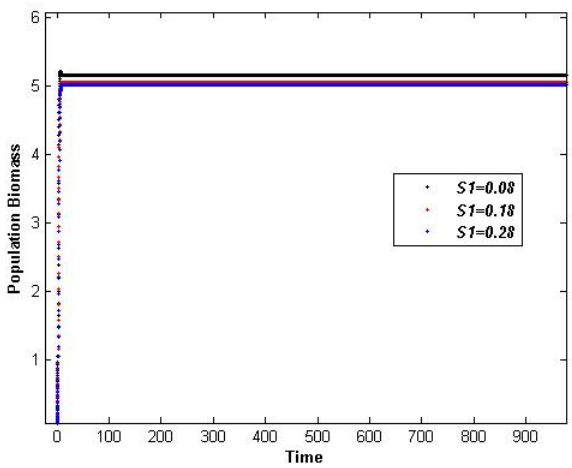

Figure 5. **Behavior of Population Biomass at different values of Depletion rate of Forestry Biomass.**

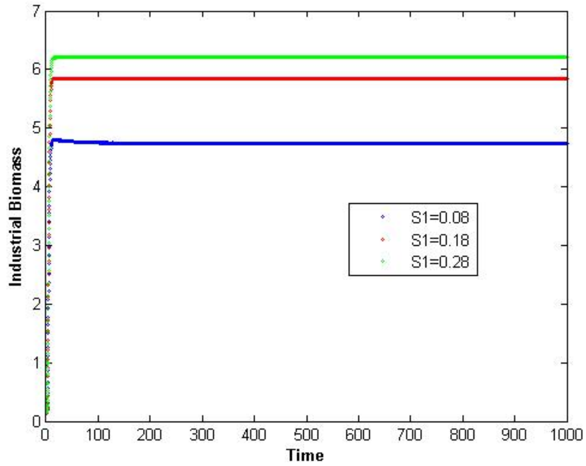

Figure 6. **Behavior of Industrial effect at different values of Depletion rate of Forestry Biomass.**

In the above figures we see variation of B, P, I and F with respect to time. In figure 1 see that as value of the time increases, B,P,I and F tends to their equilibrium points. In Figure 2, as genetic effect on forestry biomass increase the equilibrium of industrial biomass decrease. Figure 3 and 4 represent the phase plot of forestry biomass, population biomass and industrial biomass. Figure 5 and figure 6 described the behavior of population biomass and industrial biomass as Variation of depletion rate of forestry biomass. As value of depletion rate of forestry biomass increase the value of population biomass decrease and value of industrial biomass increase.

8. Conclusion

Forestry resources are very important part of our life, as we know that forestry resources decrease as increase in population. In this paper, a nonlinear mathematical model is proposed and analyzed to study the depletion of forestry resources by the human population and industries. In this system we assumed to be four variables namely, biomass of forestry recourse, density of population, industrial biomass and genetic resources. By the numerical simulation we observed that as effect of genetic resources on forestry biomass are increased the effect of depletion decrease. So if we use the genetic plants for plantation of the forests the effect of depletion of forestry resources can be reduce.

Acknowledgments

The second author thankfully acknowledges the Science and Engineering Research Board (SERB), New Delhi, India for the financial assistance in the form of Junior Research Fellowship (SR/S4/MS: 793/12).

References

[1] M.Agarwal, T.Fatimaa and H.I.Freedman, *Depletion of forestry resource biomass due to industrialization pressure: A ratio-dependent mathematical model*, J. Biol. Dyn., 4(2010), 381-396.

[2] M.Agarwal and S.Devi, *A resource-dependent competition model: Effect of population pressure augmented industrialization*, Int. J. Model. Simul. Sci. Comput. 3(2)(2012), 1250003-1250022.

[3] M.Agarwal and R.Pathak, *Conservation of forestry biomass with the use of alternative resource* , Open J. of Ecology,

5(2015), 87-109.

[4] B.Dubey, S.Sharma, P.Sinha and J.B.Shukla, *Modelling the depletion of forestry resources by population and population pressure augmented industrialization*, Appl. Math. Model., 33(2009), 3002-3014.

[5] B.Dubey and A.S.Narayana, *Modelling effect of industrialization, population and pollution on renewable resource, Non-linear Analysis*, Real world application, 11(2010), 2833-2848.

[6] A.Mishra, K.Lata and J.B.Shukla, *A mathematical model for depletion of forestry resources due to population and polulation pressure augmebted industrialization*, Int. J. Model. Simul. Comput., 5(1)(2014), 1-16.

[7] J.B.Shulka, H.I.Freedman, V.N.Pal, O.P.Mishra, M.Agarwal and A.Shulka, *Degradation and subsequent regeneration of forestry resource: a mathematical model*, Ecol. Model, 44(1989), 219-229.

[8] J.B.Shulka and B.Dubey, *Modelling the depletion and conservation of forestry resource: effect of population and pollution*, J. Math. Bio., 35(1997), 71-94.