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On a Subordination Associated with a Certain Subclass of Analytic Functions Defined by Salagean Derivatives

Research Article

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Abstract: In this paper we discuss some subordination results for a subclass of functions analytic in the unit disk U.

Keywords: Subordination, Salagean Derivatives, Subclass of Analytic Functions.

1. Introduction

Let A be the class of functions $f(z)$ analytic in the unit disk $U = \{z : |z| < 1\}$ and let S denote a subclass of A consisting of functions univalent in U and normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

We denote the class of convex functions of order α by $K(\alpha)$, i.e.,

$$K(\alpha) = \left\{ f \in S : Re\left(1 + \frac{zf''}{f'}\right) > \alpha, z \in U \right\}$$

Definition 1.1 (Hadamard product or convolution). *Given two functions $f(z)$ and $g(z)$, where $f(z)$ is defined as in (1) and $g(z)$ is given by*

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

The Hadamard product (or convulation) $f * g$ of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z) \tag{2}$$

Definition 1.2. *Let $f(z)$ and $g(z)$ be analytic in the unit disk U. Then $f(z)$ is said to be subordinate to $g(z)$ in U and we write $f(z) \prec g(z)$, $\quad z \in U$.*

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if there exists a Schwarz function $\omega(z)$, analytic in U with $\omega(0) = 0, |\omega(z)| < 1$ such that

$$f(z) = g(\omega(z)), \quad z \in U \tag{3}$$

In particular, if the function $g(z)$ in univalent in U, then $f(z)$ is surbodinate to $g(z)$ if

$$f(0) = g(0), f(U) \subseteq g(U) \tag{4}$$

Definition 1.3. *A sequence $\{C_n\}_{n=1}^{\infty}$ of complex numbers is said to be a surbodinating factor sequence of $f(z)$ if whenever $f(z)$ of the form (1) is analytic, univalent and convex in U, the surbodination is given by $\sum\limits_{n=1}^{\infty} a_n C_n z^n \prec f(z)\ z \in U,\ a_1 = 1$.*

We have the following theorem

Theorem 1.4 ([1])**.** *The sequence $\{c_k\}_{k=1}^{\infty}$ is a surbodinating factor sequence if and only if*

$$Re\left\{1 + 2\sum_{k=1}^{\infty} c_k z^k\right\} > 0 \quad (z \in U) \tag{5}$$

Let

$$S_n(\alpha) = \left\{f \in A : Re\Big(\frac{(D^{n+1}f(z)}{D^n f(z)}\Big) > \alpha, z \in U\right\} \tag{6}$$

Here $D^n f(z)$ is the Salagean derivatives, $n = 0, 1, 2, \ldots.$ Such that

$$D^o f(z) = f(z)$$
$$D^1 f(z) = Df(z) = zf'(z)$$
$$D^n f(z) = D(D^{n-1}f(z)) = z[D^{n-1}f(z)]'$$

therefore,

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k$$

The class $S_n(\alpha)$ was studied by Salagean [2] and Kadioglu [3]. In [3] the following result was established

Theorem 1.5 ([3])**.** *$f(z) \in S_n(\alpha)$ if and only if*

$$\sum_{k=2}^{\infty} k^n (k - \alpha)|a_k| \le 1 - \alpha \tag{7}$$

where $n \in N,\ 0 \le \alpha < 1$,

It is natural to consider the class $\widetilde{S}_n(\alpha)$ such that

$$\widetilde{S}_n(\alpha) = \left\{f \in A : \sum_{k=2}^{\infty} k^n (k - \alpha)|a_k| \le 1 - \alpha\right\} \tag{8}$$

$n = N \cup [0],\ 0 \le \alpha < 1$,

Remark 1.6 ([4])**.** *If $n = 0$ and α=0 in $\widetilde{S}_n(\alpha)$ we have the class $S_o(0) = \{f \in A : \sum_{k=2}^{\infty} k|a_k| \le 1\}$ which is the subclass of the class of starlike function.*

Remark 1.7 ([5])**.** *If $n = 0$ in $\widetilde{S}_n(\alpha)$ we have the class $S_o(\alpha) = \{f \in A : \sum_{k=2}^{\infty}(k - \alpha)|a_k| \le 1 - \alpha\}$ which is the subclass of class of starlike function of order α.*

Remark 1.8 ([4])**.** *If $n = 1$ and $\alpha = 0$ in $\widetilde{S}_n(\alpha)$ we have the class$S_1(0) = \{f \in A : \sum_{k=2}^{\infty} k^2|a_k| \le 1\}$ which is the subclass of class of convex function.*

Remark 1.9 ([5])**.** *If $n = 1$ in $\widetilde{S}_n(\alpha)$ we have the class$S_1(\alpha) = \{f \in A : \sum_{k=2}^{\infty} k(k - \alpha)|a_k| \le 1 - \alpha\}$ which is the subclass of the class of convex function of order α.*

2. Main Result

Our main result in this paper in the following theorem.

Theorem 2.1. *Let* $f(z) \in \widetilde{S}_n(\alpha)$, *then*

$$\frac{2^n(2-\alpha)}{2[(1-\alpha)+2^n(2-\alpha)]}(f*g)(z) \prec g(z) \tag{9}$$

where $n \in N \cup [0]$, $0 \leq \alpha < 1$, $g(z)$ *is a convex function. and*

$$Re(f(z)) > -\frac{(1-\alpha)+2^n(2-\alpha)]}{2^n(2-\alpha)} \tag{10}$$

The constant factor $\frac{2^n(2-\alpha)}{2[(1-\alpha)+2^n(2-\alpha)]}$ *cannot be replaced by a larger one*

Proof. Let $f(z) \in \widetilde{S}_n(\alpha)$ and suppose that $g(z) = z + \sum\limits_{k=2}^{\infty} b_k z^k \in K(\alpha)$ i.e. $g(z)$ is a convex function of order α. Then by definition,

$$\begin{aligned} \frac{2^n(2-\alpha)}{2[(1-\alpha)+2^n(2-\alpha)]}(f*g)_{(z)} &= \frac{2^n(2-\alpha)}{2[(1-\alpha)+2^n(2-\alpha])}(z+\sum_{k=1}^{\infty} a_k b_k z^k) \\ &= \sum_{k=1}^{\infty} \frac{2^n(2-\alpha)}{2[(1-\alpha)+2^n(2-\alpha)} a_k b_k z^k, \quad a_1 = 1 \end{aligned} \tag{11}$$

Hence, by Definition 1.3, to show subordination (9) it is enough to prove that

$$\left\{ \frac{2^n(2-\alpha)}{2[(1-\alpha)+2^n(2-\alpha)]} a_k \right\}_{k=1}^{\infty}. \tag{12}$$

is a surbodinating factor sequence with $a_1 = 1$. Therefore by Theorem 1.1 , it is sufficient to show that

$$\text{Re}\left\{1+2\sum_{k=1}^{\infty} \frac{2^n(2-\alpha)}{2[(1-\alpha)+2^n(2-\alpha)]} a_k z^k\right\} > 0, \quad (z \in U) \tag{13}$$

Now,

$$\begin{aligned} \text{Re}\left\{1+2\sum_{k=1}^{\infty} \frac{2^n(2-\alpha)}{2[(1-\alpha)+2^n(2-\alpha)]} a_k z^k\right\} &= \text{Re}\left\{1+\frac{2^n(2-\alpha)z}{1-\alpha+2^n(2-\alpha)}+\frac{2n(2-\alpha)z}{1-\alpha+2^n(2-\alpha)} \times \sum_{k=2}^{\infty} a_k z^k\right\} \\ &> \text{Re}\left\{1-\frac{2^n(2-\alpha)r}{1-\alpha+2^n(2-\alpha)}-\frac{1}{1-\alpha+2^n(2-\alpha)} \times \sum_{k=2}^{\infty} k^n(k-\alpha)|a_k|r\right\} \\ &> \text{Re}\left\{1-\frac{2^n(2-\alpha)r}{1-\alpha+2^n(2-\alpha)}-\frac{(1-\alpha)r}{1-\alpha+2^n(2-\alpha)}\right\} \\ &= 1-r > 0 \end{aligned} \tag{14}$$

Since $(|z| = r < 1)$. Therefore, we obtain

$$\text{Re}\left\{1+2\sum_{k=1}^{\infty} \frac{2^n(2-\alpha)}{2[(1-\alpha)+2^n(2-\alpha)]} a_k z^k\right\} > 0, \quad (z \in U)$$

which is (13) that we are to established. We now show that

$$Re(f(z)) > -\frac{2(1-\alpha)+2^n(2-\alpha)}{2^n(2-\alpha)}$$

Taking $g(z) = \frac{z}{1-z}$ which is a convex function (9) becomes

$$\frac{2^n(2-\alpha)}{2[(1-\alpha)+2^n(2-\alpha)]} f(z) * \frac{z}{1-z} \prec \frac{z}{1-z}$$

and note that $f(z) * \frac{z}{1-z}$. Since

$$\mathrm{Re}\left(\frac{z}{1-z}\right) > -\frac{1}{2}, \quad |z| = r \tag{15}$$

which implies that

$$\mathrm{Re}\left\{\frac{2^n(2-\alpha)}{2[(1-\alpha)+2^n(2-\alpha)]} f(z) * \frac{z}{1-z}\right\} > -\frac{1}{2} \tag{16}$$

Hence, we have

$$\mathrm{Re}(f(z)) > -\frac{(1-\alpha)+2^n(2-\alpha)}{2^n(2-\alpha)}$$

which is the (10). To show the sharpness of the constant factor $\frac{2^n(2-\alpha)}{2[(1-\alpha)+2^n(2-\alpha)]}$ we consider the function:

$$f_1(z) = \frac{z(2^n(2-\alpha))+(1-\alpha)z^2}{2^n(2-\alpha)} \tag{17}$$

Applying (10) with $g(z) = \frac{z}{1-z}$ and $f(z) = f_1(z)$ we have

$$\frac{z(2^n(2-\alpha))+(1-\alpha)z^2}{2[(1-\alpha)+2^n(2-\alpha)]} \prec \frac{z}{1-z} \tag{18}$$

By using the fact that

$$|Re(z)| \le |z| \tag{19}$$

We show that

$$\min_{z\in U}\left\{Re\frac{z(2^n(2-\alpha))+(1-\alpha)z^2}{2[(1-\alpha)+2^n(2-\alpha)]}\right\} = -\frac{1}{2} \tag{20}$$

We have that

$$\begin{aligned}
\left|Re\frac{z(2^n(2-\alpha))+(1-\alpha)z^2}{2[(1-\alpha)+2^n(2-\alpha)]}\right| &\le \left|\frac{z(2^n(2-\alpha))+(1-\alpha)z^2}{2[(1-\alpha)+2^n(2-\alpha)]}\right| \\
&= \left|\frac{z[(2^n(2-\alpha))+(1-\alpha)z]}{2[(1-\alpha)+2^n(2-\alpha)]}\right| = \frac{|z[(2^n(2-\alpha))-(1-\alpha)z]|}{|2[(1-\alpha)+2^n(2-\alpha)]|} \\
&\le \frac{|z||2^n(2-\alpha)-(1-\alpha)z|}{|2[(1-\alpha)+2^n(2-\alpha)]|} \le \frac{|2^n(2-\alpha)-(1-\alpha)z|}{2[(1-\alpha)+2^n(2-\alpha)]} \\
&\le \frac{|2^n(2-\alpha)+(1-\alpha)z|}{2[(1-\alpha)+2^n(2-\alpha)]} \le \frac{2^n(2-\alpha)+(1-\alpha)}{2[(1-\alpha)+2^n(2-\alpha)]} = \frac{1}{2}, \quad (|z|=1),
\end{aligned} \tag{21}$$

This implies that

$$\left|Re\frac{z(2^n(2-\alpha))-(1-\alpha)z^2}{2[(1-\alpha)+2^n(2-\alpha)]}\right| \le \frac{1}{2} \tag{22}$$

$$\text{ie.,}\quad -\frac{1}{2} \le \left|Re\frac{z(2^{n+1})-(1-\alpha)z^2}{2[(1-\alpha)+2^n(2-\alpha)]}\right| \le \frac{1}{2} \tag{23}$$

Hence, we have

$$\min_{z\in U}\left\{Re\frac{z(2^n(2-\alpha))-(1-\alpha)z^2}{2[(1-\alpha)+2^n(2-\alpha)]}\right\} \ge -\frac{1}{2}$$

$$\text{ie.,}\quad \min_{z\in U}\left\{Re\frac{2^n(2-\alpha)}{2[(1-\alpha)+2^n(2-\alpha)]}(f_1 * \frac{z}{1-z})\right\} \ge -\frac{1}{2}$$

which completes the proof of Theorem 1.3. □

3. Some Applications

Taking $n = 0$ in Theorem 2.1, we obtain the following:

Corollary 3.1. *If the function* $f(z)$ *defined by* (1) *is in* $\widetilde{S}_n(\alpha)$ *then* $\frac{2-\alpha}{6-4\alpha}(f*g)(z) \prec g(z)$, $(z \in U;\ g \in K(\alpha))$ *and*

$$Re\,(f(z)) > -\frac{3-2\alpha}{2-\alpha} \tag{24}$$

which is a result of [6].

Taking $n = 0$ and $\alpha = 0$ in Theorem 2.1, we obtain the following:

Corollary 3.2. *If the function* $f(z)$ *defined by* (1) *in* $\widetilde{S}_n(\alpha)$ $f(z)$ *then* $\frac{1}{3}(f*g)(z) \prec g(z)$, $(z \in U;\ g \in K(\alpha))$ *and*

$$Re\,(f(z)) > -\frac{3}{2} \tag{25}$$

which is a result of [7].

Taking $n = 1$ in Theorem 2.1, we obtain the following:

Corollary 3.3. *If the function* $f(z)$ *defined by* (1) *in* $\widetilde{S}_n(\alpha)$ $f(z)$ *then* $\frac{2-\alpha}{5-3\alpha}(f*g)(z) \prec g(z)$, $(z \in U;\ g \in K(\alpha))$ *and*

$$Re\,(f(z)) > -\frac{5-3\alpha}{4-2\alpha} \tag{26}$$

which is the result generalized by [7].

Taking $n = 1$ and $\alpha = 0$ in Theorem 2.1, we obtain the following:

Corollary 3.4. *If the function* $f(z)$ *defined by (1.1) in* $\widetilde{S}_n(\alpha)$ $f(z)$ *then* $\frac{2}{5}(f*g)(z) \prec g(z)$, $(z \in U;\ g \in K(\alpha))$ *and*

$$Re\,(f(z)) > -\frac{5}{4} \tag{27}$$

which is the result generalized by [4].

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