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Almost Distributive Fuzzy Lattices

Research Article

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Abstract: In this paper, we introduce the concept of an Almost Distributive Fuzzy Lattice (ADFL) as a generalization of Distributive Fuzzy Lattice (DFL), we also characterize ADFL as DFL using properties of ADL. We state and prove some results of an ADFL, too.

Keywords: Almost Distributive Lattice (ADL), Fuzzy Partial Order Relation, Fuzzy poset, Distributive Fuzzy Lattice (DFL) and Almost distributive Fuzzy Lattice (ADFL).

1. Introduction

The concept of an Almost Distributive Lattice (ADL) was introduced by U.M.Swamy and G.C.Rao [5] as a common abstraction of almost all the existing ring theoretic generalizations of a Boolean algebra on one hand and the class of distributive lattices on the other. On the other hand, L. A. Zadeh [6] in 1965 introduced the notion of fuzzy set to describe vagueness mathematically in its very abstractness and tried to solve such problems by assigning to each possible individual in the universe of discourse a value representing its grade of membership in the fuzzy set. In 1971, Zadeh [7] defined a fuzzy ordering as a generalization of the concept of ordering, that is, a fuzzy ordering is a fuzzy relation that is transitive. In particular, a fuzzy partial ordering is a fuzzy ordering that is reflexive and antisymmetric. In 1994, Ajmal and Thomas [1] defined a fuzzy lattice as a fuzzy algebra and characterized fuzzy sublattices. More recently, Chon [3], considering the notion of fuzzy order of Zadeh [7], introduced a new notion of fuzzy lattice and studied the level sets of fuzzy lattices. He also introduced the notions of distributive and modular fuzzy lattices and considered some basic properties of fuzzy lattices. In this paper we give the definition of an ADFL and we give some preliminary results in an ADFL. We characterize ADFL as DFL using the properties of ADL.

2. Preliminaries

First, we recall certain definitions and properties of ADLs, Fuzzy Partial Order Relations and Fuzzy Lattices from [3, 5] that are required in the paper. We also use the concept of lattices and distributive lattices from [2] and [4].

Definition 2.1 ([5]). *An algebra $(R, \vee, \wedge, 0)$ of type $(2, 2, 0)$ is called an Almost Distributive Lattice, abbreviated as ADL, if it satisfies the following axioms:*

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(L1) $a \vee 0 = a$

(L2) $0 \wedge a = 0$

(L3) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$

(L4) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

(L5) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(L6) $(a \vee b) \wedge b = b$

for all $a, b, c \in R$.

Now we give some basic results.

Lemma 2.2 ([5]). *For any* $a \in R$, *we have*

(1). $a \wedge 0 = 0$

(2). $a \wedge a = a$

(3). $a \vee a = a$

(4). $0 \vee a = a$.

Lemma 2.3 ([5]). *For any* $a, b \in R$, *we have*

(1). $(a \wedge b) \vee b = b$

(2). $a \vee (a \wedge b) = a = a \wedge (a \vee b)$

(3). $a \vee (b \wedge a) = a = (a \vee b) \wedge a$

(4). $a \vee b = a$ *if and only if* $a \wedge b = b$

(5). $a \vee b = b$ *if and only if* $a \wedge b = a$.

Definition 2.4 ([5]). *For any* $a, b \in R$, *we say that* a *is less than or equal to* b *and write* $a \leq b$ *is* $a \wedge b = a$ *or equivalently,* $a \vee b = b$.

Lemma 2.5 ([5]). *For any* $a, b, c \in R$, *we have*

(1). $(a \vee b) \wedge c = (b \vee a) \wedge c$

(2). $\wedge$ *is associative in* R

(3). $a \wedge b \wedge c = b \wedge a \wedge c$.

Next, we give some properties and definitions of Fuzzy Partial Order Relation, Fuzzy Lattice and Fuzzy Distributive Lattice.

Definition 2.6 ([3]). *Let* X *be a set. A function* $A : X \times X \longrightarrow [0, 1]$ *is called a fuzzy relation in* X. *The fuzzy relation* A *in* X *is reflexive iff* $A(x, x) = 1$ *for all* $x \in X$, A *is transitive iff* $A(x, z) \geq \sup_{y \in X} \min(A(x, y), A(y, z))$, *and* A *is antisymmetric iff* $A(x, y) > 0$ *and* $A(y, x) > 0$ *implies* $x = y$. *A fuzzy relation* A *is fuzzy partial order relation if* A *is reflexive, antisymmetric and transitive. A fuzzy partial order relation* A *is a fuzzy total order relation iff* $A(x, y) > 0$ *or* $A(y, x) > 0$ *for all* $x, y \in R$.*If* A *is a fuzzy partial order relation in a set* X, *then* (X, A) *is called a fuzzy partially ordered set or a fuzzy poset. If* B *is a fuzzy total order relation in a set* X, *then* (X, B) *is called a fuzzy totally ordered set or a fuzzy chain.*

Now we define a fuzzy lattice as a fuzzy partial order relation.

Definition 2.7 ([3]). *Let (X, A) be a fuzzy poset and let $B \subseteq X$. An element $u \in X$ is said to be an upper bound for a subset B iff $A(b, u) > 0$ for all $b \in B$. An upper bound u_0 for B is the least upper bound of B iff $A(u_0, u) > 0$ for every upper bound u for B. An element $v \in X$ is said to be a lower bound for a subset B iff $A(v, b) > 0$ for all $b \in B$. A lower bound v_0 for B is the greatest lower bound of B iff $A(v, v_0) > 0$ for every lower bound v for B.*

We denote the least upper bound of the set {x, y} by $x \vee y$ and denote the greatest lower bound of the set {x, y} by $x \wedge y$.

Definition 2.8 ([3]). *Let (X, A) be a fuzzy poset. (X, A) is a fuzzy lattice iff $x \vee y$ and $x \wedge y$ exist for all $x, y \in X$.*

Definition 2.9 ([3]). *Let (X, A) be a fuzzy lattice. (X, A) is distributive if and only if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $(x \vee y) \wedge (x \vee z) = x \vee (y \wedge z)$.*

3. Almost Distributive Fuzzy Lattice(ADFL)

In this section we give some definitions and develop some properties of an Almost Distributive Fuzzy Lattice.

Definition 3.1. *Let $(R, \vee, \wedge, 0)$ be an algebra of type (2, 2, 0) and (R, A) be a fuzzy poset. Then we call (R, A) is an Almost Distributive Fuzzy Lattice (ADFL) if the following axioms are satisfied:*

(F1) $A(a, a \vee 0) = A(a \vee 0, a) = 1$

(F2) $A(0, 0 \wedge a) = A(0 \wedge a, 0) = 1$

(F3) $A((a \vee b) \wedge c, (a \wedge c) \vee (b \wedge c)) = A((a \wedge c) \vee (b \wedge c), (a \vee b) \wedge c) = 1$

(F4) $A(a \wedge (b \vee c), (a \wedge b) \vee (a \wedge c)) = A((a \wedge b) \vee (a \wedge c), a \wedge (b \vee c)) = 1$

(F5) $A(a \vee (b \wedge c), (a \vee b) \wedge (a \vee c)) = A((a \vee b) \wedge (a \vee c), a \vee (b \wedge c)) = 1$

(F6) $A((a \vee b) \wedge b, b) = A(b, (a \vee b) \wedge b) = 1$

for all $a, b, c \in R$.

It is possible to show with examples that the above six properties are independent.

Example 3.2. *Let R = {0, a, b, c} and define two binary operations $\vee$ and $\wedge$ in R as follows:*

∨	0	a	b	c
0	0	a	b	c
a	a	a	b	c
b	b	a	b	b
c	c	c	c	c

and

∧	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	c	b	c
c	0	a	b	c

Define a fuzzy relation $A : R \times R \longrightarrow [0,1]$ as follows: $A(0, 0) = A(a, a) = A(b, b) = A(c, c) = 1$, $A(a, 0) = A(b, 0) = A(c, 0) = A(b, a) = A(b, c) = A(c, a) = A(c, b) = 0$ $A(0, a) = 0.3$, $A(0, b) = 0.5$, $A(0, c) = 0.8$, $A(a, b) = 0.2$ and $A(a, c) = 0.4$. Clearly (R, A) is a fuzzy poset. Here (R, A) is an ADFL since it satisfies the above six axioms of an ADFL. But (R, A) is not DFL. Since $A((a \wedge b) \vee c, (a \vee c) \wedge (b \vee c)) = A(a \vee c, c \wedge b) = A(c, b) = 0$ and $A((a \vee c) \wedge (b \vee c), (a \wedge b) \vee c) = A(c \wedge b, a \vee c) = A(b, c) = 0$. Thus $A((a \wedge b) \vee c, (a \vee c) \wedge (b \vee c)) = A((a \vee c) \wedge (b \vee c), (a \wedge b) \vee c) = 0 \neq 1$. Which implies $\vee$ is not right distributive over $\wedge$.

From the definitions of ADFL and ADL, The following theorem is direct.

Theorem 3.3. *Let (R, A) be a fuzzy poset . Then R is an ADL iff (R, A) is an ADFL.*

From now on words by R we mean an ADL (R, ∨, ∧, 0) unless otherwise mentioned.

Theorem 3.4. *Let (R, A) be an ADFL . Then* $a = b \Leftrightarrow A(a,\ b) = A(b,\ a) = 1$.

Proof. (⇒) Suppose $a = b$. Then $A(a, b) = A(a, a) = 1$ and $A(b, a) = A(a, a) = 1$, since A is reflexive. Therefore $A(a, b) = A(b, a) = 1$.

Conversely, suppose $A(a, b) = A(b, a) = 1$. Then $A(a, b) > 0$ and $A(b, a) > 0$. Which implies $a = b$. □

Definition 3.5. *Let* (R, A) *be an ADFL. Then for any* $a, b \in R$, $a \leq b$ *if and only if* $A(a, b) > 0$.

In view of the above definition, we have the following theorem.

Theorem 3.6. *If (R, A) is an ADFL then* $a \wedge b = a$ *if and only if* $A(a, b) > 0$.

Lemma 3.7. *Let (R, A) be an ADFL and* $a, b \in R$ *such that* $a \neq b$. *If* $A(a, b) > 0$ *then* $A(b, a) = 0$.

Proof. We prove it by contradiction. Suppose $A(b, a) \neq 0$. Then $A(b, a) > 0$. Since $A(a, b) > 0$ from assumption, $a = b$. Which is a contradiction of $a \neq b$. Therefore $A(b, a) = 0$. □

Lemma 3.8. *Let (R, A) be an ADFL. Then for each a and b in R*

(1). $A(a \wedge b,\ b) > 0$ *and* $A(b \wedge a,\ a) > 0$

(2). $A(a,\ a \vee b) > 0$ *and* $A(b,\ b \vee a) > 0$.

Proof. Let (R, A) be an ADFL and a, b ∈ R.

(1). Since from ADL we have $a \wedge b \wedge b = a \wedge b \Rightarrow a \wedge b \leq b \Rightarrow A(a \wedge b, b) > 0$. Similarly $A(b \wedge a, a) > 0$.

(2). Since from ADL we have $a \wedge (a \vee b) = a$ then $a \leq a \vee b$. This implies $A(a, a \vee b) > 0$. Similarly $A(b, b \vee a) > 0$.

□

Lemma 3.9. *Let (R, A) be an ADFL. For any a and b in R,* $A(a \wedge b,\ b \wedge a) = 1$ *whenever* $A(a,\ b) > 0$.

Proof. Suppose $A(a, b) > 0$ which implies $a \leq b$. Then $a \wedge b = a$ or equivalently $a \vee b = b$. Now,

$$\begin{aligned} A(a \wedge b, b \wedge a) &= A(a, b \wedge a) \\ &= A(((a \vee b) \wedge a), b \wedge a) \\ &= A(b \wedge a, b \wedge a) \\ &= 1 \end{aligned}$$

Hence $A(a \wedge b, b \wedge a) = 1$ whenever $A(a, b) > 0$. □

Now we give some basic results of an ADFL.

Lemma 3.10. *Let (R, A) be an ADFL . Then for any a and b in R, we have*

(1). $A(0,\ a \wedge 0) > 0$;

(2). $A(a,\ a \wedge a) > 0$;

(3). $A((a \wedge b) \vee\ b,\ b) = 1$;

(4). $A(a \vee\ a,\ a) > 0$;

(5). $A(a,\ 0 \vee\ a) > 0$.

Proof. Suppose (R, A) is an ADFL and $a, b \in R$.

(1). $A(0, a \wedge 0) = A(0, (a \vee 0) \wedge 0) = A(0, 0) = 1$ Hence $A(0, a \wedge 0) > 0$.

(2). $A(a, a \wedge a) = A(a, (a \vee 0) \wedge (a \vee 0)) = A(a, a \vee (0 \wedge 0)) = A(a, a \vee 0) = A(a, a) = 1$. Hence $A(a, a \wedge a) > 0$.

(3). From (F6) we have $A((a \vee b) \wedge b, b) = 1$. Which implies $A((a \wedge b) \vee (b \wedge b), b) = 1$. Then $A((a \wedge b) \vee b, b) = 1$. Hence $A((a \wedge b) \vee b, b) = 1$.

(4). From (3) above $A((a \wedge b) \vee b, b) = 1 \Rightarrow A((a \wedge a) \vee a, a) = 1 \Rightarrow A(a \vee a, a) = 1$. Hence $A(a \vee a, a) > 0$.

(5). $A(a, 0 \vee a) = A(a, (0 \wedge a) \vee a) = A(a, a) = 1$. Hence $A(a, 0 \vee a) > 0$.

□

Lemma 3.11. *Let $(R,\ A)$ be an ADFL. Then for any a and b in R, we have*

(1). $A(a,\ a \wedge (a \vee\ b)) = 1$;

(2). $A(a \vee (a \wedge\ b),\ a) = 1$;

(3). $A(a \vee (a \wedge\ b),\ a \wedge (a \vee\ b)) = 1$;

(4). $A(a,\ (a \vee\ b) \wedge\ a) = 1$;

(5). $A(a \vee (b \wedge\ a),\ a) = 1$;

(6). $A(a \vee (b \wedge\ a),\ (a \vee\ b) \wedge\ a) = 1$.

Proof. Suppose (R, A) is an ADFL and $a, b \in R$.

(1).

$$\begin{aligned} A(a, a \wedge (a \vee b)) &= A(a, (a \vee 0) \wedge (a \vee b)) \\ &= A(a, a \vee (0 \wedge b)) \\ &= A(a, a \vee 0) \\ &= A(a, a) \\ &= 1. \end{aligned}$$

Therefore $A(a, a \wedge (a \vee b)) = 1$.

(2).

$$\begin{aligned} A(a \vee (a \wedge b), a) &= A((a \vee a) \wedge (a \vee b), a) \\ &= A(a \wedge (a \vee b), a) \\ &= A(a, a) \\ &= 1. \end{aligned}$$

Therefore $A(a \vee (a \wedge b), a) = 1$.

(3).
$$\begin{aligned} A(a \vee (a \wedge b), a \wedge (a \vee b)) &\geq \sup_{b \in R} \min(A(a \vee (a \wedge b), b), A(b, a \wedge (a \vee b))) \\ &\geq \min(A(a \vee (a \wedge b), a), A(a, a \wedge (a \vee b))) \\ &= \min(1, 1) \\ &= 1. \end{aligned}$$

Therefore $A(a \vee (a \wedge b), a \wedge (a \vee b)) = 1$.

(4).
$$\begin{aligned} A(a, (a \vee b) \wedge a) &= A(a, (a \vee b) \wedge (a \vee 0)) \\ &= A(a, a \vee (b \wedge 0)) \\ &= A(a, a \vee 0) \\ &= A(a, a) \\ &= 1. \end{aligned}$$

Therefore $A(a, (a \vee b) \wedge a) = 1$.

(5).
$$\begin{aligned} A(a \vee (b \wedge a), a) &= A((a \vee b) \wedge (a \vee a), a) \\ &= A((a \vee b) \wedge a, a) \\ &= A(a, a) \\ &= 1. \end{aligned}$$

Therefore $A(a \vee (b \wedge a), a) = 1$.

(6).
$$\begin{aligned} A(a \vee (b \wedge a), (a \vee b) \wedge a) &\geq \sup_{c \in R} \min(A(a \vee (b \wedge a), c), A(c, (a \vee b) \wedge a)) \\ &\geq \min(A(a \vee (b \wedge a), a), A(a, (a \vee b) \wedge a)) \\ &= \min(1, 1) \\ &= 1. \end{aligned}$$

Therefore $A(a \vee (b \wedge a), (a \vee b) \wedge a) = 1$.

□

Corollary 3.12. *Let (R, A) be an ADFL. Then for any a and b in R*

(1). $A(a \vee b, a) > 0$ *if and only if* $A(b, a \wedge b) > 0$.

(2). $A(a \vee b, b) > 0$ *and* $A(b, a \vee b) > 0$ *iff* $A(a \wedge b, a) > 0$ *and* $A(a, a \wedge b) > 0$.

In the above Lemma, we mentioned absorption laws that are valid in an ADFL in general. Regarding the remaining absorption laws we have the following.

Theorem 3.13. *Let (R, A) be an ADFL. Then for any a,b in R the following are equivalent.*

(1). $A((a \wedge b) \vee a, a) > 0$ *and* $A(a, (a \wedge b) \vee a) > 0$.

(2). $A(a \wedge (b \vee a), a) > 0$ *and* $A(a, a \wedge (b \vee a)) > 0$.

(3). $A(a \vee b, b \vee a) > 0$ *and* $A(b \vee a, a \vee b) > 0$.

(4). The infimum of a and b exists in R and equals $a \wedge b$.

(5). $A(a \wedge b, b \wedge a) > 0$ *and* $A(b \wedge a, a \wedge b) > 0$.

(6). The supremum of a and b exists in R and equals $a \vee b$.

(7). There exists $x \in R$ such that $A(a, x) > 0$ and $A(b, x) > 0$.

(8). $A((b \wedge a) \vee b, b) > 0$ and $A(b, (b \wedge a) \vee b) > 0$.

(9). $A(b \wedge (a \vee b), b) > 0$ and $A(b, b \wedge (a \vee b)) > 0$.

Using the absorption laws and the above corollary, we have the following.

Lemma 3.14. *Let (R, A) be an ADFL. Then for any a, b in R, we have*

(1). $A(a \vee b, a \vee (b \vee a)) = 1$.

(2). $A((a \vee b) \vee a, a \vee b) = 1$.

(3). $A((a \vee b) \vee a, a \vee (b \vee a)) = 1$.

Proof. Suppose (R, A) is an ADFL and a, b $\in$ R.

(1).
$$\begin{aligned} A(a \vee b, a \vee (b \vee a)) &= A(a \vee (b \wedge (b \vee a)), a \vee (b \vee a)) \\ &= A((a \vee b) \wedge [a \vee (b \vee a)], a \vee (b \vee a)) \\ &= A([(a \vee b) \wedge a] \vee [(a \vee b) \wedge (b \vee a)], a \vee (b \vee a)) \\ &= A([(a \vee b) \wedge a] \vee ([(a \vee b) \wedge b] \vee [(a \vee b) \wedge a]), a \vee (b \vee a)) \\ &= A(a \vee (b \vee a), a \vee (b \vee a)) \\ &= 1. \end{aligned}$$

Therefore $A(a \vee b, a \vee (b \vee a)) = 1$.

(2). From properties of an ADL we have $(a \vee b) \wedge a = a$. Which implies $(a \vee b) \vee a = a \vee b$. Hence $A((a \vee b) \vee a, a \vee b) = A(a \vee b, a \vee b) = 1$.

(3).
$$\begin{aligned} A((a \vee b) \vee a, a \vee (b \vee a)) &\geq \sup_{c \in R} \min(A((a \vee b) \vee a, c), A(c, a \vee (b \vee a))) \\ &\geq \min(A((a \vee b) \vee a, a \vee b), A(a \vee b, a \vee (b \vee a)) \\ &= \min(1, 1) \\ &= 1. \end{aligned}$$

Therefore $A((a \vee b) \vee a, a \vee (b \vee a)) = 1$.

□

Lemma 3.15. *Let (R, A) be an ADFL. Then for any $a, b, c \in R$, $A((a \vee b) \wedge c, (b \vee a) \wedge c) > 0$ and $A((b \vee a) \wedge c, (a \vee b) \wedge c) > 0$.*

Proof. Suppose (R, A) is an ADFL and $a, b, c \in R$. By Lemma 14 (1) we have, $A(a \wedge c, c) > 0$ and $A(b \wedge c, c) > 0$. Hence by Theorem 19, $A((a \wedge c) \vee (b \wedge c), (b \wedge c) \vee (a \wedge c)) > 0$ and $A((b \wedge c) \vee (a \wedge c), (a \wedge c) \vee (b \wedge c)) > 0$. Hence by right distributive property of $\wedge$ we have $A((a \vee b) \wedge c, (b \vee a) \wedge c) > 0$ and $A((b \vee a) \wedge c, (a \vee b) \wedge c) > 0$. □

Lemma 3.16. *Let (R, A) be an ADFL. Then for any a, b, $c \in R$. $A(a \wedge (b \wedge c), (a \wedge b) \wedge c) = 1$.*

Lemma 3.17. *Let (R, A) be an ADFL. Then for any a, b, $c \in R$. $A(a \wedge b \wedge c, b \wedge a \wedge c) > 0$ and $A(b \wedge a \wedge c, a \wedge b \wedge c) > 0$.*

Proof. Suppose (R, A) is an ADFL and $a, b, c \in R$. By lemma 14, (1) we have $A(a \wedge c, c) > 0$ and $A(b \wedge c, c) > 0$. Hence by Theorem 19, $A((a \wedge c) \wedge (b \wedge c), (b \wedge c) \wedge (a \wedge c)) > 0$ and $A((b \wedge c) \wedge (a \wedge c), (a \wedge c) \wedge (b \wedge c)) > 0$. Hence from associative property of $\wedge$ we have $A(a \wedge [c \wedge (b \wedge c)], b \wedge [c \wedge (a \wedge c)]) > 0$ and $A(b \wedge [c \wedge (a \wedge c)], a \wedge [c \wedge (b \wedge c)]) > 0$. We know that from ADL $c \vee (b \wedge c) = c$. Hence by corollary (18), Lemma (14) and antisymmetry property of A we have $c \wedge (b \wedge c) = b \wedge c$. Similarly $c \wedge (a \wedge c) = a \wedge c$. Hence $A(a \wedge b \wedge c, b \wedge a \wedge c) > 0$ and $A(b \wedge a \wedge c, a \wedge b \wedge c) > 0$. □

More generally, we have

Lemma 3.18. *Let (R, A) be an ADFL. If $a_1, \dots, a_n, b \in R$ and $(i_1, \dots, i_n)$ is any permutation of (1, 2, ., ., ., n). Then $A(a_{i_1} \wedge \cdots \wedge a_{i_n} \wedge b,\ a_1 \wedge \cdots \wedge a_n \wedge b) > 0$ and $A(\ a_1 \wedge \cdots \wedge a_n \wedge b,\ a_{i_1} \wedge \cdots \wedge a_{i_n} \wedge b) > 0$.*

Proof. We prove it by induction. It is true for $n = 2$. Suppose it is true for $n = k$. i.e $A(a_{i_1} \wedge \cdots \wedge a_{i_k} \wedge b,\ a_1 \wedge \cdots \wedge a_k \wedge b) > 0$ and $A(a_1 \wedge \cdots \wedge a_k \wedge b,\ a_{i_1} \wedge \cdots \wedge a_{i_k} \wedge b) > 0$. Since A is antisymmetry we have $a_{i_1} \wedge \cdots \wedge a_{i_k} \wedge b = a_1 \wedge \cdots \wedge a_k \wedge b$.
claim: A$(a_{i_1} \wedge \cdots \wedge a_{i_k} \wedge a_{i_{k+1}} \wedge b,\ a_1 \wedge \cdots \wedge a_k \wedge a_{k+1} \wedge b) > 0$ and A$(a_1 \wedge \cdots \wedge a_k \wedge a_{k+1} \wedge b,\ a_{i_1} \wedge \cdots \wedge a_{i_k} \wedge a_{i_{k+1}} \wedge b) > 0$. Now,

$$\begin{aligned}
a_{i_1} \wedge \cdots \wedge a_{i_k} \wedge a_{i_{k+1}} \wedge b &= a_{i_1} \wedge \cdots \wedge a_{i_k} \wedge a_{i_{k+1}} \wedge b \wedge b \\
&= a_{i_1} \wedge \cdots \wedge a_{i_k} \wedge b \wedge a_{i_{k+1}} \wedge b \\
&= a_1 \wedge \cdots \wedge a_k \wedge b \wedge a_{k+1} \wedge b \\
&= a_1 \wedge \cdots \wedge a_k \wedge a_{k+1} \wedge b \wedge b \\
&= a_1 \wedge \cdots \wedge a_k \wedge a_{k+1} \wedge b
\end{aligned}$$

Hence $A(a_{i_1} \wedge \cdots \wedge a_{i_k} \wedge a_{i_{k+1}} \wedge b,\ a_1 \wedge \cdots \wedge a_k \wedge a_{k+1} \wedge b) > 0$ and $A(a_1 \wedge \cdots \wedge a_k \wedge a_{k+1} \wedge b,\ a_{i_1} \wedge \cdots \wedge a_{i_k} \wedge a_{i_{k+1}} \wedge b) > 0$. Therefore $A(a_{i_1} \wedge \cdots \wedge a_{i_n} \wedge b,\ a_1 \wedge \cdots \wedge a_n \wedge b) > 0$ and $A(a_1 \wedge \cdots \wedge a_n \wedge b,\ a_{i_1} \wedge \cdots \wedge a_{i_n} \wedge b) > 0$. □

Definition 3.19. *The fuzzy poset (R, A) is directed above if and only if the poset (R, $\leq$) is directed above.*

Here, we characterize ADFL as DFL.

Theorem 3.20. *Let (R, A) be an ADFL. Then the following are equivalent.*

(1). (R, A) is DFL.

(2). The fuzzy poset (R, A) is directed above.

(3). $A(a \vee b,\ b \vee a) > 0$ and $A(b \vee a,\ a \vee b) > 0$.

(4). $A((a \wedge b) \vee c,\ (a \vee c) \wedge (b \vee c)) > 0$ and $A((a \vee c) \wedge (b \vee c),\ (a \wedge b) \vee c) > 0$.

(5). $A(a \wedge b,\ b \wedge a) > 0$ and $A(b \wedge a,\ a \wedge b) > 0$.

(6). The relation $\theta = \{(a,\ b) \in R \times R \mid A(a,\ b \wedge a) > 0\}$ is antisymmetric.

Proof. (1) ⇒ (2). Suppose (R, A) is DFL
Claim: The fuzzy poset (R, A) is directed above.
Since (R, A) is DFL, for any $a, b \in R$ we have $A(a, a \vee b) > 0$ and $A(b, a \vee b) > 0$. Take $c = a \vee b$. Which implies $A(a, c) > 0$ and $A(b, c) > 0 \Rightarrow a \leq c$ and $b \leq c$ for $c \in R$ implies R is directed above. Therefore The fuzzy poset (R, A) is directed above.
(2) ⇒ (3). Suppose the fuzzy poset (R, A) is directed above then $(R, \leq)$ is directed above. Hence there exists c in R such

that $a \leq c$ and $b \leq c$ for $c \in R$ and hence $A(a,c) > 0$ and $A(b,c) > 0$. Implies $A(a, a \vee b) > 0$ and $A(b, a \vee b) > 0$ for $c = a \vee b$ and $A(a, b \vee a) > 0$ and $A(b, b \vee a) > 0$ for $c = b \vee a$. Implies $\sup\{a,b\} = a \vee b$ and $\sup\{b,a\} = b \vee a$. Since $\sup\{a,b\} = \sup\{b,a\}$ then we have $a \vee b = b \vee a$. Therefore $A(a \vee b, b \vee a) > 0$ and $A(b \vee a, a \vee b) > 0$.

(3) $\Rightarrow$ (4). Suppose $A(a \vee b, b \vee a) > 0$ and $A(b \vee a, a \vee b) > 0$. Then by antisymmetry property of A we have $a \vee b = b \vee a$. Now,

$$\begin{aligned} A((a \wedge b) \vee c, (a \vee c) \wedge (b \vee c)) &= A(c \vee (a \wedge b), (a \vee c) \wedge (b \vee c)) \\ &= A((c \vee a) \wedge (c \vee b), (a \vee c) \wedge (b \vee c)) \\ &= A((a \vee c) \wedge (b \vee c), (a \vee c) \wedge (b \vee c)) \\ &= 1. \end{aligned}$$

Therefore $A((a \wedge b) \vee c, (a \vee c) \wedge (b \vee c)) > 0$. Similarly $A((a \vee c) \wedge (b \vee c), (a \wedge b) \vee c) > 0$.

(4) $\Rightarrow$ (5). Suppose $A((a \wedge b) \vee c, (a \vee c) \wedge (b \vee c)) > 0$ and $A((a \vee c) \wedge (b \vee c), (a \wedge b) \vee c) > 0$.

$$\begin{aligned} A(a \wedge b, b \wedge a) = A(a \wedge b, b \wedge [(0 \wedge b) \vee a]) &= A(a \wedge b, b \wedge [(0 \vee a) \wedge (b \vee a)]) \\ &= A(a \wedge b, b \wedge (a \wedge (b \vee a))) \\ &= A(a \wedge b, b \wedge [(a \wedge b) \vee (a \wedge a)]) \\ &= A(a \wedge b, b \wedge ((a \wedge b) \vee a)) \\ &= A(a \wedge b, [b \wedge (a \wedge b)] \vee (b \wedge a)) \\ &= A(a \wedge b, (a \wedge b) \vee (b \wedge a)) \\ &= A(a \wedge b, [a \vee (b \wedge a)] \wedge [b \vee (b \wedge a)]) \\ &= A(a \wedge b, a \wedge b) \\ &= 1. \end{aligned}$$

Therefore $A(a \wedge b, b \wedge a) > 0$. Similarly, $A(b \wedge a, a \wedge b) > 0$.

(5) $\Rightarrow$ (6). Suppose $A(a \wedge b, b \wedge a) > 0$ and $A(b \wedge a, a \wedge b) > 0$. Then by antisymmetry property of A $a \wedge b = b \wedge a$.

Claim: $\theta = \{(a,b) \in R \times R | A(a, b \wedge a) > 0\}$ is antisymmetric.

Let $(a,b), (b,a) \in \theta$.

WTS: a = b. $(a,b), (b,a) \in \theta$ implies

$$A(a, b \wedge a) > 0 \text{ and } A(b, a \wedge b) > 0 \qquad (\star)$$

and from Lemma 14(1) we have

$$A(b \wedge a, a) > 0 \text{ and } A(a \wedge b, b) > 0 \qquad (\star\star)$$

From $(\star)$ and $(\star\star)$ and antisymmetry property of A, $a = b \wedge a$ and $b = a \wedge b$. Since $a \wedge b = b \wedge a \Rightarrow b = a$. Therefore $\theta = \{(a,b) \in R \times R | A(a, b \wedge a) > 0\}$ is antisymmetric.

(6) $\Rightarrow$ (1). Suppose $\theta = \{(a,b) \in R \times R | A(a, b \wedge a) > 0\}$ is antisymmetric. For $a, b \in R$. Since $a \wedge b \wedge b \wedge a = a \wedge b \wedge a = b \wedge a \wedge a = b \wedge a$. Which implies

$$(a \wedge b, b \wedge a) \in \theta \qquad (\star)$$

Similarly, $b \wedge a \wedge a \wedge b = a \wedge b$. Which implies

$$(b \wedge a, a \wedge b) \in \theta \qquad (\star\star)$$

from $(\star)$, $(\star\star)$ and θ is antisymmetry, we have $a \wedge b = b \wedge a \Rightarrow \wedge$ is commutative. Hence by Theorem 19 and antisymmetry property of A, $\vee$ is also commutative. Therefore (R, A) is a DFL. □

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