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Existence of Common Fixed Points for a Pair of Generalized Geraghty Contractions in Convex Metric Spaces

Research Article

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Abstract: In this paper, we prove the existence of common fixed points of generalized Geraghty contractions for a pair of self maps on a complete convex metric space under the influence of altering distances with out continuity. These results generalize some of the previously known results. Two open problems are also given.

MSC: 47H09, 54H25.

Keywords: Complete convex metric space, Geraghty contraction, common fixed points, altering distances, starshaped set.

1. Introduction and Preliminaries

Fixed point theory is one of the out standing subfields of non linear functional analysis. It has been used in research area of mathematics and non linear sciences. In 1922 Banach [3] proved a fixed point theorem for contraction mappings is one of the pivotal results in analysis. This theorem which has been extended and generalized by several authors. In 1973, Geraghty [7] introduced and studied a generalization of Banach contraction mapping principle in complete metric spaces by taking the class of functions $\Gamma = \{\beta : [0,\infty) \to [0,\infty) |\, \beta(t_n) \to 1 \Rightarrow t_n \to 0\}$. Throughout this paper, $\Gamma = \{\beta : [0,\infty) \to [0,\infty) |\, \beta(t_n) \to 1 \Rightarrow t_n \to 0\}$. $\Psi = \{\psi : \mathbb{R}^+ \to \mathbb{R}^+ \mid \psi(t)$ is a decreasing an $\psi(t) = 0 \Leftrightarrow t = 0\}$. $F(f)$ the set of all fixed points of f.

Definition 1.1 ([7])**.** *Let (X, d) be a metric space. A selfmap $f : X \to X$ is said to be a Geraghty contraction if there exists $\beta \in \Gamma$ such that*

$$d(fx, fy) \le \beta(d(x,y))\, d(x,y) \text{ for all } x, y \in X. \tag{1}$$

Theorem 1.2 ([7])**.** *Let (X, d) be a complete metric space. Let $f : X \to X$ be a self map. If there exists $\beta \in \Gamma$ such that*

$$d(fx, fy) \le \beta(d(x,y))\, d(x,y) \text{ for all } x, y \in X, \tag{2}$$

then f has a unique fixed point.

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Definition 1.3. *A selfmap $f : X \to X$ is said to be a generalized Geraghty contraction if there exists $\beta \in \Gamma$ satisfying, for all $x, y \in X$*

$$d(fx, fy) \leq \beta(M(x,y)) M(x,y),$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,fx), d(y,fy), \frac{1}{2}[d(x,fx) + d(y,fx)]\right\}.$$

The following class of functions namely the class of altering distance functions which we denote by Ψ_1 was introduced by Khan, Swaleh and Sessa [11] as follows. $\Psi = \{\psi : \mathbb{R}^+ \to \mathbb{R}^+ \mid \psi$ is non decreasing, continuous and $\psi(t) = 0 \Leftrightarrow t = 0\}$. The altering distance functions were used by many researchers [6, 12] to obtain fixed points. In 1970, Takahashi [13] introduced the notation of convexity in metric spaces. Guay, Singh and Whitfield [8] and many authors [1, 2] have studied fixed point theorems in convex metric spaces.

Definition 1.4 ([13])**.** *Let (X,d) be a metric space. A continuous mapping $W : X \times X \times [0.1]$ is said to be a convex structure on X for all $x, y \in X$ and $\lambda \in [0,1]$ such that*

$$d(u, W(x,y,\lambda)) \leq \lambda d(u,x) + (1-\lambda) d(u,x) \text{ holds for all } u \in X. \tag{3}$$

The metric space (X,d) together with a convex structure W is called a convex metric space, which is denoted by (X,d,W).

Note: Normed linear spaces are examples of convex matric spaces.

Definition 1.5 ([13])**.** *A subset K of a convex metric space (X,d,W) is said to be a convex set if $W(x,y,\lambda) \in K$* for all $x, y \in K$ and $\lambda \in [0,1]$.

Definition 1.6. *A set M is called q-star shaped with $q \in M$ if the segment $[q,x] = \{W(q,x,\lambda) \mid 0 \leq \lambda \leq 1\}$ is contained in M for all $x \in M$.*

Definition 1.7 ([8])**.** *A convex metric spac (X,d,W) is said to satisfy property (I) if for all $x, y, q \in X$ and $\lambda \in [0,1]$,*

$$d(W(x,q,\lambda),\ W(y,q,\lambda) \leq \lambda d(x,y).$$

Definition 1.8. *For a non-empty subset M of a metric space (X,d) and $x \in X$, an element $y \in M$ is said to be a best approximant of x in M or a best M-approximant to x if $d(x,y) = dist(x,M) = \inf\{d(x,y) : y \in M\}$. The set of all best M-approximants to x is denoted by $P_M(x)$.*

Definition 1.9. *For a convex subset M of a convex metric space (X,d,W) a mapping $f : M \to X$ is said to be affine if for all $x, y \in M, f(W(x,y,\lambda)) = W(fx, fy, \lambda)$ for all $\lambda \in [0,1]$. f is said to be affine with respect to $q \in M$ if $f(W(x,q,\lambda)) = W(fx, fq, \lambda)$ for all $x \in M$ and $\lambda \in [0,1]$.*

Suppose M is a nonempty subset of a metric space (X,d) and f, T are self mappings of M. A point $x \in M$ is a common fixed(coincidence) point of f and T if $x = fx = Tx\,(fx = Tx)$.

Definition 1.10. *The mappings $f, T : M \to M$ are said to satisfy property (P) in M if $d(fx_n, Tx_n) \to 0$ implies $d(f(Tx_n), T(fx_n)) \to 0$ for any sequence $\{x_n\}$ in M.*

Definition 1.11 ([9])**.** *A pair (f,T) of self mappings of a metric space (X,d) is said to be compatible, if $d(f(Tx_n), T(fx_n)) \to 0$ whenever x_n is a sequence in X such that $Tx_n, fx_n \to z \in X$.*

Definition 1.12 ([10]). *A pair (f,T) of self mappings of a metric space (X,d) is said to be weakly compatible if they commute at their coincidence points. i.e., if $fTx = Tfx$ whenever $fx = Tx$.*

In 2013, Chandok and Narang [5] proved the following result.

Theorem 1.13 ([5]). *Let M be a nonempty closed subset of a metric space (X,d). Let $f,T : M \to M$ be self mappings,*

$$q \in F(f) \text{ and } T(M\backslash\{q\}) \subset f(M)\backslash\{q\}.$$

Suppose that there exist $k \in (0,1)$ such that

$$d(Tx,Ty) \le k\max\left\{d(fx,fy), d(fx,Tx), d(fy,Ty), \frac{1}{2}\left[d(fx,Ty)+d(fy,Ty)\right]\right\}$$

for all $x,y \in M$.

If further, f and T are continuous, $cl\left[T(M\backslash\{q\})\right]$ is complete and f and T are weakly compatible on $M\backslash\{q\}$, then $F(f)\bigcap F(T)$ is a singleton. The following Lemma, which we use in the next Section, can be easily established.

Lemma 1.14 ([4]). *Let (X,d) be a metric space. Let $\{x_n\}$ sequence in X such that $d(x_{n+1},x_n) \to 0$ as $n \to \infty$. If $\{x_n\}$ not a Cauchy sequence then there exists an $\epsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k > k$ and $d(x_{mk},x_{nk}) > \epsilon$. For each $k > 0$, corresponding to m_k we can choose n_k to be the smallest integer such that $d(x_{mk},x_{nk}) > \epsilon$ and $d(x_{mk},x_{nk-1}) \le \epsilon$. It can be shown that the following identities are satisfied:*

(1) $\lim\limits_{k\to\infty} d(x_{nk-1},x_{mk+1}) = \epsilon$,

(2) $\lim\limits_{k\to\infty} d(x_{nk},x_{mk}) = \epsilon$,

(3) $\lim\limits_{k\to\infty} d(x_{nk-1},x_{mk}) = \epsilon$,

(4) $\lim\limits_{k\to\infty} d(x_{nk},x_{mk+1}) = \epsilon$.

2. Main Results

Now we prove the existence of common fixed points for generalized Geraghty contractions in complete metric spaces, with property (P), under the influence of altering distances.

Theorem 2.1. *Let M be a nonempty closed subset of a metric space (X,d). Let $f,T : M \to M$ be self mappings, $q \in F(f)$ and $T(M\backslash\{q\}) \subset f(M)\backslash\{q\}$. Suppose that there exist $\beta \in \Gamma$ and $\psi \in \Psi$ such that for all $x,y \in M$*

$$\psi(d(Tx,Ty)) \le \beta(\psi(M(x,y)))\,\psi(M(x,y)), \tag{4}$$

where $M(x,y) = \max\left\{d(fx,fy), d(fx,Tx), d(fy,Ty), \frac{1}{2}\left[d(fx,Ty)+d(fy,Tx)\right]\right\}$. Let $x_0 \in M\backslash\{q\}$. Then $\{fx_n\}$ is a Cauchy sequence in $M\backslash\{q\}$ where $fx_n = Tx_{n-1}$ for $n = 1,2,3,\ldots$.

If further, f and T are continuous, $cl\left[T(M\backslash\{q\})\right]$ is complete and f and T satisfy property (P) then $F(f)\cap F(T)$ is singleton.

Proof. Let $x_0 \in M\backslash\{q\}$. Since $T(M\backslash\{q\}) \subset f(M)\backslash\{q\}$, we can find $x_1 \in M\backslash\{q\}$ such that $y_1 = fx_1 = Tx_0$ with $y_1 \neq q$ and we can find $x_2 \in M\backslash\{q\}$ such that $y_2 = fx_n = Tx_1$ with $y_2 \neq q$. Similarly we can find $x_n \in M\backslash\{q\}$ such that $y_n = fx_n = Tx_{n-1}$ with $y_n \neq q$ for $n \geq 1$. Now,

$$\psi(d(fx_{n+1}, fx_n)) = \psi(d(Tx_n, Tx_{n-1})) \leq \beta(\psi(M(x_n, x_{n-1})))\psi(M(x_n, x_{n-1})) \tag{5}$$

where

$$\begin{aligned} M(x_n, x_{n-1}) &= \max\left\{d(fx_n, fx_{n-1}), d(fx_n, Tx_n), d(fx_{n-1}, Tx_{n-1}), \frac{1}{2}[d(fx_n, Tx_{n-1}) + d(fx_{n-1}, Tx_n)]\right\} \\ &= \max\left\{d(fx_n, fx_{n-1}), d(fx_n, fx_{n+1}), d(fx_{n-1}, fx_n), \frac{1}{2}[d(fx_n, fx_n), d(fx_{n-1}, fx_{n+1})]\right\} \\ &= \max\{d(fx_n, fx_{n-1}), d(fx_{n+1}, fx_n)\} \end{aligned}$$

Suppose $d(fx_n, fx_{n-1}) < d(fx_{n+1}, fx_n)$ for some n, then from (5)

$$\psi(d(fx_{n+1}, fx_n)) \leq \beta(\psi(d(fx_{n+1}, fx_n)))\psi(d(fx_{n+1}, fx_n)) < \psi(d(fx_{n+1}, fx_n)),$$

a contradiction. Therefore $d(fx_{n+1}, fx_n) \leq d(fx_n, fx_{n-1})$, which implies that $\psi(d(fx_{n+1}, fx_{n-1})) \leq \psi(d(fx_n, fx_{n-1}))$. Thus $\{d(fx_{n+1}, fx_{n-1})\}$ is a decreasing sequence and converges to r(say) and $\{\psi(d(fx_{n+1}, fx_n))\}$ is a decreasing sequence and converges to s(say). From (5), we have

$$\psi(d(fx_{n+1}, fx_n)) \leq \beta(\psi(d(fx_n, fx_{n-1})))\psi(d(fx_n, fx_{n-1})).$$

Suppose $\beta(\psi(d(fx_n, fx_{n-1}))) \to 1$ then by hypothesis $\psi(d(fx_n, fx_{n-1})) \to 0$, which implies that $s = 0$. Now $r \leq d(fx_n, fx_{n+1})$ implies $\psi(r) \leq \psi(d(fx_n, fx_{n+1})) \to 0$. Hence $\psi(r) \leq 0$ so that $r = 0$. Hence without loss of generality we may suppose that $\beta(\psi(d(fx_n, fx_{n-1}))) \nrightarrow 1$, then there exists $0 < \gamma < 1$ such that$\beta(\psi(d(fx_n, fx_{n-1}))) < r$ for infinitely many n. Therefore $\psi(d(fx_{n+1}, fx_n)) \leq \gamma\psi(d(fx_n, fx_{n-1}))$ for infinitely many n. On letting $n \to \infty$, we get $s \leq \gamma s$ which implies that $s = 0$. Therefore $r = 0$. Now we show that $\{fx_n\}$ is Cauchy. Suppose that $\{y_n\} = \{fx_n\}$ is not Cauchy. Then by Lemma 1.14 there exists an $\epsilon > 0$ and sequences of positive integers $n_k > m_k > k$ and $d(ym_k, yn_k) > \epsilon$ and $d(ym_k, yn_{k-1}) \leq \epsilon$. Then the following identities can be established.

(1) $\lim_{k\to\infty} d(y_{mk}, y_{nk}) = \epsilon$,

(2) $\lim_{k\to\infty} d(y_{mk+1}, y_{nk-1}) = \epsilon$,

(3) $\lim_{k\to\infty} d(y_{mk}, y_{nk-1}) = \epsilon$, and

(4) $\lim_{k\to\infty} d(y_{mk+1}, y_{nk}) = \epsilon$.

we have

$$\psi(\in) \leq \psi(d(y_{mk}, y_{nk})) = \psi(d(fx_{mk}, fx_{nk})) = \psi(d(Tx_{mk-1}, Tx_{nk-1})) \leq \beta(\psi(M(x_{mk-1}, x_{nk-1})))\psi(M(x_{mk-1}, x_{nk-1}))$$

where

$$\begin{aligned} M(x_{mk-1}, x_{nk-1}) = \max\{&d(fx_{mk-1}, fx_{nk-1}), d(fx_{mk-1}, Tx_{mk-1}), d(fx_{nk-1}, Tx_{nk-1}), \\ &\left.\frac{1}{2}[d(fx_{mk-1}, Tx_{nk-1}) + d(fx_{nk-1}, Tx_{mk-1})]\right\} \\ = \max&\left\{d(y_{mk-1}, y_{nk-1}), d(y_{mk-1}, y_{mk}), d(y_{nk-1}, y_{nk}), \frac{1}{2}[d(y_{mk-1}, y_{nk}), d(y_{mk-1}, y_{mk})]\right\} \end{aligned}$$

Now $\beta(\psi(M(x_{mk-1}, x_{nk-1}))) \to 1$ implies $\psi(M(x_{mk-1}, x_{nk-1})) \to 0$. From (??), we have $\psi(\epsilon) = 0$, a contradiction. Hence $\beta(\psi(M(x_{mk-1}, x_{nk-1}))) \nrightarrow 1$. Then there exists $0 < \gamma < 1$ such that $\beta(\psi(M(x_{mk-1}, x_{nk-1}))) < \gamma$ for infinitely many k. Then from (??) we have,

$$\psi(\in) \le \psi(d(y_{mk}, y_{nk})) \le \gamma\psi(M(x_{mk-1}, x_{nk-1})). \tag{6}$$

On letting $k \to \infty$ in (6), since $d(y_{mk}, y_{nk}) > \epsilon$ we have,

$$0 < \psi(\in +0) = \lim_{k\to\infty} \psi(d(y_{mk}, y_{nk})) \le \gamma\psi(M(x_{mk-1}, x_{nk-1})) \le \gamma\psi(\in +0),$$

a contradiction. Therefore $\{fx_n\}$ is a Cauchy sequence in $M\setminus\{q\}$. Hence $\{Tx_{n-1}\}$ is a Cauchy sequence in $M\setminus\{q\}$. Therefore $Tx_{n-1} \to z\,(\text{say}) \in cl[T(M\setminus\{q\})]$ and consequently $fx_n \to z$. Now $d(fx_n, Tx_n) = d(fx_n, fx_{n+1}) \to 0$ as $n \to \infty$. From property(P) we have $d(Tfx_n, fTx_{n+1}) \to 0$. Since f and T are continuous so that $d(Tz, fz) = 0$. Hence $Tz = fz$. Therefore z is a coincidence points of f and T. Again by property(P), we have $fTz = Tfz$. Now

$$\psi(d(Tfz, Tz)) \le \beta(\psi(M(fz, z)))\psi(M(fz, z))$$

where

$$\begin{aligned} M(fz, z) &= \max\left\{d(ffz, fz), d(ffz, Tfz), d(fz, Tz), \frac{1}{2}[d(ffz, Tz) + d(fz, Tfz)]\right\} \\ &= d(Tfz, Tz) \end{aligned}$$

Therefore, from (??) we have,

$$\psi(d(Tfz, Tz)) \le \beta(\psi(d(Tfz, Tz)))\psi(d(Tfz, Tz)),$$

which implies that $\psi(d(Tfz, Tz)) = 0$. Hence $Tfz = Tz$. Therefore $Tfz = ffz = Tz = fz$. Let z and w be two fixed points of f and T with $w \ne z$. Consider $\psi(d(Tz, Tw)) \le \beta(\psi(M(z, w)))\psi(M(z, w))$, which implies that $\psi(d(Tz, Tw)) \le \beta(\psi(d(z, w)))\psi(d(z, w)) < \psi(d(z, w))$, a contradiction. Therefore $F(f) \cap F(T)$ is singleton. □

Note: We observe that Theorem 1.13 follows from Theorem 2.1 by taking $\psi(T) = kt$ for $t > 0$.

Theorem 2.2. *Let M be a nonempty closed subset of a metric space (X, d). Let $f, T : M \to M$ be a self mappings, $q \in F(f)$ and $T(M\setminus\{q\}) \subset f(M)\setminus\{q\}$. Suppose that there exist $\beta \in \Gamma$ and $\psi \in \Psi$ and a positive integer n such that for all $x, y \in M$*

$$\psi(d(T^n x, T^n y)) \le \beta(\psi(M_1(x, y)))\psi(M_1(x, y)), \tag{7}$$

where $M_1(x, y) = \max\left\{d(fx, fy)\, d(fx, T^n x), d(fy, T^n y), \frac{1}{2}[d(fx, T^n y) + d(fy, T^n x)]\right\}$. Let $x_0 \in M\setminus\{q\}$. Then $\{fxm\}$ is a Cauchy sequence in $M\setminus\{q\}$ where $fx_m = T^n x_{m-1}$, for $m = 1, 2, 3, \ldots$.
If further, f and T^n are continuous, $cl[T(M\setminus\{q\})]$ is complete, and f and T^n satisfy property (P) then $F(f) \cap F(T^n)$ is singleton.

Proof. By using the hypotheses $T(M\setminus\{q\}) \subset f(M)\setminus\{q\}$ we can prove $T^n(M\setminus\{q\}) \subset f(M)\setminus\{q\}$ by induction n. Therefore the proof of Theorem 2.2 follows from Theorem 2.1 by replacing T with T^n. □

Now we extend the existence of common fixed points for generalized Geraghty contractions in convex metric spaces with property (P) under the influence of altering distances.

Theorem 2.3. *Let M be a nonempty closed subset of a convex metric space (X,d,W) with property (I). Let $f,T:M\to M$ be a self mappings. Suppose that M is q-star shaped with $q\in F(f)$ and f is continuous and affine with respect to q. Define $T_\lambda : M\to M$ by $T_\lambda(x)=W(Tx,q,\lambda), 0<\lambda<1$. Suppose f and T_λ have property (P) and $cl[T(M\backslash\{q\})]$ is complete. $cl[T(M)]\subset f(M)\backslash\{q\}, T$ is continuous. Suppose that there exist $\beta\in\Gamma$ and $\psi\in\Psi$ such that for all $x,y\in M$*

$$\psi(d(Tx,Ty))\le\beta(\psi(M_2(x,y)))\psi(M_3(x,y)), \tag{8}$$

where

$$M_2(x,y)=\max\left\{d(fx,fy),d(fx,[Tx,q]),d(fy,[Ty,q]),\frac{1}{2}[d(fx,[Ty,q])+d(fy,[Tx]\,q)]\right\}\quad \textit{and}$$
$$M_3(x,y)=\max\left\{d(fx,fy),d(fx,T_\lambda x),d(fy,T_\lambda y),\frac{1}{2}[d(fx,T_\lambda y)+d(fy,T_\lambda x)]\right\}.$$

Let $x_0\in M\backslash\{q\}$. Then for $0<\lambda<1$, f and T_λ have unique fixed point x_λ. i.e., $F(f)\cap F(T_\lambda)$ is singleton.

Proof. Suppose $0<\lambda<1$, f and define $T_\lambda x=W(Tx,q,\lambda)$ for all $x\in M$. Let $x_0\in M\backslash\{q\}$. Since $T_\lambda(M\backslash\{q\})\subset f(M)\backslash\{q\}$. We can find $x_1\in M\backslash\{q\}$ such that $y_1=fx_1=T_\lambda x_0$ with $y_1\ne q$ and we can find $x_2\in M\backslash\{q\}$ such that $y_2=fx_2=T_\lambda x_1$ with $y_2\ne q$. Similarly we can find $x_n\in M\backslash\{q\}$ such that $y_n=fx_n=T_\lambda x_{n-1}$ with $y_n\ne q$ for $n\ge 1$. Now

$$\begin{aligned}\psi(d(fx_{n+1},fx_n))&=\psi(d(T_\lambda x_n,T_\lambda x_{n-1}))=\psi(W(T_n,q,\lambda),W(T_{n-1},q,\lambda))\\&\le\psi(\lambda d(Tx_n,Tx_{n-1}))\le\psi(d(Tx_n,Tx_{n-1}))\\&\le\beta(\psi(M_2(x_n,x_{n-1})))\psi(M_3(x_n,x_{n-1}))\end{aligned}\tag{9}$$

where

$$M_2(x_n,x_{n-1})=\max\left\{d(fx_n,fx_{n-1}),d(fx_n,[Tx_n,q]),d(fx_{n-1},[Tx_{n-1},q]),\frac{1}{2}[d(fx_n,[Tx_{n-1},q])+d(fx_{n-1},[Tx_n,q])]\right\}$$

and

$$\begin{aligned}M_3(x_n,x_{n-1})=\max\{&d(fx_n,fx_{n-1}),d(fx_n,T_\lambda x_n),\ d(fx_n,T_\lambda x_n),d(fx_{n-1},T_\lambda x_{n-1}),\\&\frac{1}{2}[d(fx_n,T_\lambda x_{n-1})+d(fx_{n-1},T_\lambda x_n)]\}\\=\max\{&d(fx_n,fx_{n-1}),d(fx_n,fx_{n+1})\}\end{aligned}$$

Suppose $d(fx_n,fx_{n-1})<d(fx_{n+1},fx_n)$ for some n, then from (9)

$$\psi(d(fx_{n+1},fx_n))\le\beta(\psi(M_2(fx_n,fx_{n-1})))\psi(d(fx_{n+1},fx_n))<\psi(d(fx_{n+1},fx_n)),$$

a contradiction. Therefore $d(fx_{n+1},fx_n)\le d(fx_n,fx_{n-1})$ for all n, which implies that $\psi(d(fx_{n+1},fx_n))\le\psi(d(fx_n,fx_{n-1}))$ for all n. Thus $\{d(fx_{n+1},fx_n)\}$ is a decreasing sequence and converges to r (say) and $\{\psi(d(fx_{n+1},fx_n))\}$ is a decreasing sequence and converges to s(say). We know that $r\le d(fx_n,fx_{n+1})$ which implies that $\psi(r)\le\psi(d(fx_n,fx_{n+1}))$ so that $\psi(r)\le r$. Now $d(fx_n,fx_{n-1})\le M_2(x_n,x_{n-1})$ so that $\psi(d(fx_n,fx_{n-1}))\le\psi(M_2(x_n,x_{n-1}))$. Suppose that $\beta(\psi(M_2(x_n,x_{n-1})))\to 1$ implies $\psi(M_2(x_n,x_{n-1}))\to 0$ which implies that $\psi(d(fx_n,fx_{n-1}))\to 0$. Therefore $s=\lim_{n\to\infty}\psi(d(fx_n,fx_{n-1}))=0$. Therefore $\psi(r)\le s$ implies $\psi(r)=0$ so that $r=0$.

Now suppose that $\beta(\psi(M_2(x_n, x_{n-1}))) \nrightarrow 1$, that there exists $0 < \alpha < 1$ such that $\beta(\psi(M_2(x_n, x_{n-1}))) < \alpha$ for infinitely many n. Therefore

$$\begin{aligned}\psi(r) \leq \mathrm{s} \leq \psi(d(fx_n, fx_{n+1})) &\leq \beta(\psi(M_2(x_n, x_{n-1})))\,\psi(d(fx_n, fx_{n-1}))\\ &\leq \alpha\psi(d(fx_n, fx_{n-1}))\end{aligned}$$

On letting $n \to \infty$, we get $s \leq \alpha s$ which implies that $s = 0$. Therefore $r = 0$. Now we show that $\{fx_n\}$ is Cauchy. Suppose $\{y_n\} = \{fx_n\}$ is not Cauchy. Then by Lemma 1.14 there exists an $\epsilon > 0$ and sequences of positive integers $n_k > m_k > k$ and $d(y_{mk}, y_{nk}) > \in$ and $d(y_{mk}, y_{nk-1}) \leq \epsilon$. We have

$$\begin{aligned}\psi(\in) \leq \psi(d(y_{mk}, y_{nk})) = \psi(d(fx_{mk}, fx_{nk})) &= \psi(d(T_\lambda x_{mk-1}, T_\lambda x_{nk-1}))\\ &= \psi(d(W(Tx_{mk-1}, q, \lambda), W(Tx_{nk-1}, q, \lambda)))\\ &\leq \psi(\lambda d(Tx_{mk-1}, Tx_{nk-1}))\\ &\leq \psi(d(Tx_{mk-1}, Tx_{nk-1}))\\ &\leq \beta(\psi(M_2(x_{mk-1}, x_{nk-1})))\,\psi(M_3(x_{mk-1}, x_{nk-1}))\end{aligned} \tag{10}$$

where

$$\begin{aligned}M_2(x_{mk-1}, x_{nk-1}) = \max\{&d(fx_{mk-1}, fx_{nk-1}),\ d(fx_{mk-1}, [Tx_{mk-1}, q]), d(fx_{nk-1}, [Tx_{nk-1}, q]),\\ &\frac{1}{2}[d(fx_{mk-1}, [Tx_{nk-1}, q]) + d(fx_{nk-1}, [Tx_{mk-1}, q])]\} \quad \text{and}\\ M_3(x_{mk-1}, x_{nk-1}) = \max\{&d(fx_{mk-1}, fx_{nk-1}),\ d(fx_{mk-1}, T_\lambda x_{mk-1}), d(fx_{nk-1}, T_\lambda x_{nk-1}),\\ &\frac{1}{2}[d(fx_{mk-1}, T_\lambda x_{nk-1}) + d(fx_{nk-1}, T_\lambda x_{mk-1})]\}\\ = \max&\left\{d(y_{mk-1}, y_{nk-1}), d(y_{mk-1}, y_{mk}), d(y_{nk-1}, y_{nk}), \frac{1}{2}[d(y_{mk-1}, y_{nk}) + d(y_{nk-1}, y_{mk})]\right\}.\end{aligned}$$

Now $\beta(\psi(M_2(x_{mk-1}, x_{nk-1}))) \to 1$ implies $\psi(M_2(x_{mk-1}, x_{nk-1})) \to 0$, which implies that $d(y_{mk-1}, y_{nk-1}) \to 0$ so that $\epsilon = 0$, a contradiction. Now suppose that $\beta(\psi(M_2(x_{mk-1}, x_{nk-1}))) \nrightarrow 1$, that there exists $0 < \gamma < 1$ such that $\beta(\psi(M_2(x_{mk-1}, x_{nk-1}))) < \gamma$ for infinitely many k. Therefore

$$\leq \psi(d(y_{mk}, y_{nk})) \leq \gamma\psi(M_3(x_{mk-1}, x_{nk-1})) \leq \gamma\psi(d(y_{mk-1}, y_{nk-1})). \tag{11}$$

On letting $k \to \infty$ in (11), we have

$$\psi(\in +0) = \lim_{k\to\infty} \psi(d(y_{mk}, y_{nk})) \leq \gamma\overline{\lim}\,\psi(M_3(x_{mk-1}, y_{nk-1})).$$

a contradiction. Therefore $\{fx_n\}$ is Cauchy sequence in $M\backslash\{q\}$. Hence $\{T_\lambda x_{n-1}\}$ is Cauchy sequence in $M\backslash\{q\}$. Therefore $T_\lambda x_{n-1} \to z\,(\text{say}) \in cl\,[T(M\backslash\{q\})]$ and consequently $fx_n \to z$. Now $d(fx_n, T_\lambda x_n) = d(fx_n, fx_{n+1}) \to 0$ as $n \to \infty$. From property(P) we have $d(T_\lambda fx_n, fT_\lambda x_n) \to 0$. Since f and T_λ are continuous so that $d(T_\lambda z, fz) \to 0$. Hence $T_\lambda z = fz$. Therefore z is a coincidence point of f and T_λ. Again by property (P), we have $fT_\lambda z = T_\lambda fz$. Now

$$\psi(d(Tfz, Tz)) \leq \beta(\psi(M_2(fz, z)))\,\psi(M_3(fz, z)) \tag{12}$$

where

$$M(fz,z) = \max\left\{d(ffz,fz), d(ffz,T_\lambda fz), d(fz,T_\lambda fz), \frac{1}{2}[d(ffz,T_\lambda fz) + d(fz,T_\lambda fz)]\right\}$$
$$= d(T_\lambda fz, T_\lambda fz).$$

Therefore, from (12) we have,

$$\begin{aligned}\psi(d(T_\lambda fz, T_\lambda fz)) &= \psi(d(W(T_\lambda fz,q,\lambda), W(T_\lambda z,q,\lambda)))\\ &\le \psi(\lambda d(Tfz,Tz))\\ &\le \psi(d(Tfz,Tz))\\ &\le \beta(\psi(M_2(fz,z)))\psi(M_3(fz,z))\\ &= \beta(\psi(M_2(fz,z)))\psi(d(T_\lambda fz, T_\lambda z)),\end{aligned}$$

which implies that $\psi(d(T_\lambda fz, T_\lambda fz)) = 0$. Hence $T_\lambda fz = T_\lambda z$. Therefore $T_\lambda fz = ffz = T_\lambda z = fz$. Suppose z and w are two fixed points of f and T_λ with $w \neq z$. Consider

$$\begin{aligned}\psi(d(z,w)) = \psi(d(fz,fw)) = \psi(d(T_\lambda z, T_\lambda w)) &\le \psi(\lambda d(Tz,Tw))\\ &\le \psi(d(Tz,Tw))\\ &\le \beta(\psi(M_2(z,w)))\psi(M_3(z,w))\\ &< \psi(M_3(z,w)) = \psi(d(z,w)),\end{aligned}$$

a contradiction. Therefore $F(f)\bigcap F(T_\lambda)$ is singleton. □

Open Problem 2.4. *If f and T have property(P) then is it true that f and T_λ have property(P)? (Here we observe that if $\{x_n\}$ is such that $d(fx_n, Tx_n) \to 0$ then $d(fT_\lambda x_n, T_\lambda fx_n) \to 0$).*

Open Problem 2.5. *If p is a common fixed point of f and T then is it true that $p_\lambda = W(p,q,\lambda)$ a fixed point of f and T_λ? (We observe that p_λ is a fixed point of f, since f is affine with respect to q. i.e., $f(p_\lambda) = f(W(p,q,\lambda)) = W(fp,fq,\lambda) = W(p,q,\lambda) = p_\lambda$).*

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