International Journal of Mathematics And its Applications
Volume 5, Issue 1–C (2017), 327–341.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Some Euler Spaces of Difference Sequences and Matrix Mappings

Research Article

Shadab Ahmad Khan[1*] and Ashfaque A.Ansari[1]

1 Department of Mathematics & Statistics, D. D. U. Gorakhpur University, Gorakhpur, India.

Abstract: Altay and Başar [2] and Altay, Başar and Mursaleen [3] introduced the Euler sequence spaces e_0^r, e_c^r, and e_∞^r, respectively. Polat and Başar [25] introduced the spaces $e_0^r(\Delta^{(m)})$, $e_c^r(\Delta^{(m)})$, and $e_\infty^r(\Delta^{(m)})$ consisting of all sequences whose m^{th} order differences are in the Euler spaces e_0^r, e_c^r, and e_∞^r, respectively. Moreover, the authors give some topological properties and inclusion relations, and determine the α-, β-, γ-, and continuous duals of the spaces e_0^r, e_c^r, e_p^r, e_∞^r, $e_0^r(\Delta^{(m)})$, $e_c^r(\Delta^{(m)})$ and $e_\infty^r(\Delta^{(m)})$, for $1 \leq p < \infty$ and their basis have been constructed. The last section of the article is devoted to the characterization of some matrix mappings on the sequence spaces e_c^r and $e_c^r(\Delta^{(m)})$.

Keywords: The α-, β-, γ-, and continuous duals, basis, matrix mappings and difference sequence spaces of order m.

1. Preliminaries, Background, and Notation

Let ω denote the space of all real or complex valued sequences. Any vector subspace of ω is called as a sequence space. We write ℓ_∞, c, c_0 and bv for the spaces of all bounded, convergent, null and bounded variation sequences, respectively. Also by bs, cs and ℓ_p, we denote the spaces of all bounded, convergent, and p-absolutely summable series, respectively, where $1 \leq p < \infty$. A sequence space λ with a linear topology is called a K-space provided each of the maps $p_i : \lambda \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$, where $\mathbb{C}$ denotes the set of complex numbers and $\mathbb{N} = \{0, 1, 2, \ldots\}$. A K-space λ is called an FK-space provided λ is a complete linear metric space. An FK-space whose topology is normable is called a BK-space. Let λ, μ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where n, $k \in \mathbb{N}$. Then, we say that A defines a matrix mapping from λ into μ and we denote it by writing $A : \lambda \to \mu$, if for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = \{(Ax)_n\}$, the A-transform of x, is in μ, where

$$(Ax)_n = \sum_k a_{nk} x_k \quad (n \in \mathbb{N}). \tag{1}$$

For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. By $(\lambda : \mu)$, we denote the class of all matrices A such that $A : \lambda \to \mu$. Thus, $A \in (\lambda : \mu)$ if and only if the series on the right side of (1) converges for each $n \in \mathbb{N}$ and every $x \in \lambda$, and we have $Ax = \{(Ax)_n\}_{n \in \mathbb{N}} \in \mu$ for all $x \in \lambda$. A sequence x is said to be A-summable to α if Ax converges to α, which is called as the A-limit of x.

* *E-mail: sakhanan15@gmail.com*

The approach constructing a new sequence space by means of the matrix domain of a particular limitation method was recently employed by Wang [27], Ng and Lee [24], Malkowsky [23], Altay, Başar and Mursaleen [3], Altay and Başar [2]. They introduced the sequence space $(\ell_\infty)_{N_q}$ and $c_{N\,q}$ in [27], $(\ell_p)_{c_1} = X_p$ in [24], $(\ell_\infty)_{R^t} = r^t_\infty$, $c_{R^t} = r^t_c$ and $(c_0)_{R^t} = r^t_0$ in [23], and $(\ell_p)_{E^r} = e^r_p$ in [3], and $(c_0)_{E^r} = e^r_0$, $c_{E^r} = e^r_c$ in [2]; where $1 \le p \le \infty$ and N_q denotes the Nörlund mean. The main purpose of this article, which is the natural continuation of [2,3,5] following [27], [24], [23] and [4], is to introduce the spaces $e^r_0(\Delta^{(m)}), e^r_c\left(\Delta^{(m)}\right)$, and $e^r_\infty(\Delta^{(m)})$ of sequences whose m^{th} order difference are null, convergent, and bounded and also to derive some related results that fill up the gap in the existing literature. Furthermore, we construct the basis of the spaces e^r_0, e^r_c, $e^r_0(\Delta^{(m)})$ and $e^r_c\left(\Delta^{(m)}\right)$ and determine the α-, β- and γ-duals of the spaces. Besides this, we essentially characterize the matrix classes $(e^r_c : \ell_p)$, $(e^r_c : c)$, $(e^r_c(\Delta^{(m)}) : \ell_p)$ and $(e^r_c(\Delta^{(m)}) : c)$ and also derive the characterizations of some other classes by means of a given basic lemma, where $1 \le p \le \infty$.

2. Difference Sequences of Order m in Some Euler Spaces

Firstly, we give the definitions of some sequence spaces in the existing literature. The Euler sequence spaces e^r_0 and e^r_c were defined by Altay and Başar [2] and the spaces e^r_p and e^r_∞ were defined by Altay et al. [3], as follows :

$$e^r_0 = \{\ x = (x_k) \in \omega\ :\ \lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k x_k = 0\},$$
$$e^r_c = \{\ x = (x_k) \in \omega : \lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k x_k \ \text{ exists}\},$$
$$e^r_p = \{\ x = (x_k) \in \omega\ :\ \sum_{n} \Big| \sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k x_k \Big|^{p} < \infty\},\quad (1 \le p < \infty),$$
$$e^r_\infty = \{\ x = (x_k) \in \omega\ : \sup_{n\in\mathbb{N}} \sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k x_k < \infty\},$$

where $E^r = (e^r_{nk})$ denotes the Euler means of order r defined by

$$e^r_{nk} = \begin{cases} \binom{n}{k} (1-r)^{n-k} r^k & , (0 \le k \le n)\ , \\ 0 & ,\quad (k > n) \end{cases}$$

for all $k, n \in \mathbb{N}$. It is known that the method E^r is regular for $0 < r < 1$ and E^r is invertible such that $(E^r)^{-1} = E^{1/r}$ with $r \neq 0$. We assume unless stated otherwise that $0 < r < 1$. Altay and Başar [2] gave the inclusion relations between the sequence spaces e^r_0 and e^r_c with the classical sequence spaces, determined the Schauder basis for these spaces. They also calculated the alpha-, beta-, gamma- and continuous duals of the Euler sequence spaces, and characterized some matrix mappings on e^r_0 and e^r_c. The difference spaces $\ell_\infty(\Delta)$, $c\,(\Delta)$, and $c_0(\Delta)$, consisting of all sequences $x = (x_k)$ such that $\Delta^1 x = (x_k - x_{k+1})$ in the sequence spaces ℓ_∞, c, and c_0, were defined by Kizmaz [19]. Let $p = (p_k)$ be an arbitrary bounded sequence of positive reals. Then, the linear spaces $\ell_\infty(p)$, $c(p)$, and $c_0\,(p)$ were defined by Maddox [21] as follows:

$$\ell_\infty\,(p) = \{x = (x_k) \in \omega\ : \sup_{k\in\mathbb{N}} |x_k|^{pk} < \infty\},$$
$$c\,(p) = \{x = (x_k) \in \omega\ :\ \lim_{k\to\infty} |x_k - \ell|^{pk} = 0 \ \text{ for some } \ \ell \in \mathbb{R}\},\ \text{ and}$$
$$c_0\,(p) = \left\{\ x = (x_k) \in \omega\ :\ \lim_{k\to\infty} |x_k|^{pk}\ = 0\right\}.$$

Let v denotes one of the sequence spaces ℓ_∞, c, or c_0. In [1], Ahmad and Mursaleen defined the paranormed spaces of the difference sequences

$$\Delta v\,(p) = \left\{\ x = (x_k) \in \omega\ :\ \Delta^1 x = (x_k - x_{k+1}) \in v\,(p)\right\}.$$

The idea of difference sequences was generalized by Çolak and Et [14]. They defined the sequence spaces

$$\Delta^m v = \{\ x = (x_k) \in \omega\ : \Delta^m x \in v\},$$

where $\Delta^m x = \Delta^1(\Delta^{m-1}x)$ for $m = 1,\ 2,\ 3,\ \ldots$. In [22], Malkowsky and Parashar defined the sequence spaces

$$\Delta^{(m)} v = \left\{\ x = (x_k) \in \omega\ : \Delta^{(m)} x \in v\right\},$$

where $m \in \mathbb{N}$ and $\Delta^{(m)}x = \Delta^{(1)}(\Delta^{(m-1)}x)$. Başar and Altay [9] recently defined the space of sequences of p-bounded variation, which is the difference spaces of the sequence spaces ℓ_p and ℓ_∞, as follows:

$$bv_p = \{x = (x_k) \in \omega\ :\ \sum_k |x_k - x_{k-1}|^p < \infty\}\ \ (1 \le p \le \infty),$$

and

$$bv_\infty = \{\ x = (x_k) \in \omega\ :\ \sup_{k\in\mathbb{N}} |x_k - x_{k-1}| < \infty\}.$$

Aydin and Başar [6] studied the sequence spaces a_0^r and a_c^r, defined by

$$a_0^r\ = \{\ x = (x_k)\ \in \omega\ :\ \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \left(1 + r^k\right) x_k = 0\},$$

and

$$a_c^r\ = \{\ x = (x_k)\ \in \omega\ :\ \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \left(1 + r^k\right) x_k \text{ exists } \}.$$

More recently, Aydin and Başar [7] introduced the difference sequence spaces $a_0^r\ (\Delta)$ and $a_c^r\ (\Delta)$, defined by

$$a_0^r\ (\Delta) = \{\ x = (x_k)\ \in \omega\ :\ \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \left(1 + r^k\right) (x_k - x_{k-1}) = 0\},$$

and

$$a_c^r\ (\Delta) = \{\ x = (x_k)\ \in \omega\ :\ \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \left(1 + r^k\right) (x_k - x_{k-1})\ \ \text{ exists } \}.$$

Our main focus in this study is on the triangle matrix $\Delta^{(m)} = (\delta_{nk}^{(m)})$ defined by

$$\delta_{nk}^{(m)} = \begin{cases} (-1)^{n-k} \left(\begin{smallmatrix} m \\ n-k \end{smallmatrix} \right) & (\max\ \{0,\ n-m\}\ \le k \le n)\,, \\ 0 & (0 \le k < \max\{0, n-m\} \text{ or } k > n)\,, \end{cases}$$

Altay and Polat [4] defined the Euler sequence spaces with difference operator Δ as follows:

$$e_0^r\ (\Delta) = \{\ x = (x_k) \in \omega\ : \lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{k}\ (1-r)^{n-k} r^k \Delta x_k = 0\},$$

$$e_c^r\ (\Delta) = \{x = (x_k) \in \omega : \lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{k}\ (1-r)^{n-k} r^k \Delta x_k\ \ exists\},$$

$$e_\infty^r\ (\Delta) = \{\ x = (x_k) \in \omega\ : \sup_{n\in\mathbb{N}} |\sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k \Delta x_k| < \infty\}$$

where $\Delta x_k = x_k - x_{k-1}$. Following Altay and Polat [4], Polat and Başar [25] gave the new sequence spaces $e_0^r(\Delta^{(m)})$, $e_c^r(\Delta^{(m)})$ and $e_\infty^r(\Delta^{(m)})$ consisting of all sequences $x = (x_k)$ such that their $\Delta^{(m)}-$transforms are in the Euler spaces e_0^r, e_c^r and e_∞^r, respectively, that is,

$$e_0^r\left(\Delta^{(m)}\right) = \{x = (x_k) \in \omega \ : \Delta^{(m)}x \in e_0^r\},$$
$$e_c^r\left(\Delta^{(m)}\right) = \{x = (x_k) \in \omega \ : \Delta^{(m)}x \in e_c^r\},$$
$$e_\infty^r\left(\Delta^{(m)}\right) = \{x = (x_k) \in \omega \ : \Delta^{(m)}x \in e_\infty^r\}$$

The sequence spaces $e_0^r\left(\Delta^{(m)}\right)$, $e_c^r\left(\Delta^{(m)}\right)$ and $e_\infty^r\left(\Delta^{(m)}\right)$ are reduced in the case $m = 1$ to the spaces $e_0^r(\Delta)$,$e_c^r(\Delta)$ and $e_\infty^r(\Delta)$ of Altay and Polat [4]. Başarir and Kayikci [10] defined the matrix $B^{(m)} = (b_{nk}^{(m)})$ by

$$b_{nk}^{(m)} = \begin{cases} \binom{m}{n-k} r^{m-n+k}s^{n-k} \ , & (\max\{0 \ , n-m\} \leq k \leq n), \\ 0 & , (0 \leq k < \max\{0, n-m\} \text{ or } k > n) \end{cases} \tag{2}$$

For all $k,\ n \in \mathbb{N}$ which is reduced to the mth order difference matrix $\Delta^{(m)}$ in case $r = 1$, $s = -1$, where $\Delta^{(m)} = \Delta\left(\Delta^{(m-1)}\right)$ and $m \in \mathbb{N}$. Kara and Başarir [17] introduced the B^m-Euler difference sequence spaces $e_0^r(B^{(m)})$, $e_c^r(B^{(m)})$ and $e_\infty^r(B^{(m)})$ as the set of all sequences whose B^m-transforms are in the Euler spaces e_0^r, e_c^r and e_∞^r, respectively, that is,

$$e_0^r\left(B^{(m)}\right) = \{x = (x_k) \in \omega \ : B^m x \in e_0^r\},$$
$$e_c^r\left(B^{(m)}\right) = \{x = (x_k) \in \omega \ : B^m x \in e_c^r\},$$
$$e_\infty^r\left(B^{(m)}\right) = \{x = (x_k) \in \omega \ : B^m x \in e_\infty^r\},$$

Karakaya and Polat [18] defined the new paranormed Euler sequence spaces with difference operator Δ as follows:

$$e_0^r(\Delta, p) = \{x = (x_k) \in \omega \ : \lim_{n\to\infty} |\sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k \Delta x_k|^{pn} = 0\},$$
$$e_c^r(\Delta, p) = \{x = (x_k) \in \omega \ : \exists \ell \in \mathbb{C} \ni \lim_{n\to\infty} |\sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k (\Delta x_k - \ell)|^{pn} = 0\},$$
$$e_\infty^r(\Delta, p) = \{x = (x_k) \in \omega \ : \sup_{n\in\mathbb{N}} |\sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k \Delta x_k|^{pn} < \infty\}.$$

The new sequence spaces $e_0^r(\Delta, p)$, $e_c^r(\Delta, p)$ and $e_\infty^r(\Delta, p)$ are reduced to some sequence spaces corresponding to special cases of (p_k). For instance, in the case $p_k = 1$ for all $k \in \mathbb{N}$, the sequence spaces $e_0^r(\Delta, p)$, $e_c^r(\Delta, p)$ and $e_\infty^r(\Delta, p)$ are reduced to the sequence spaces $e_0^r(\Delta)$, $e_c^r(\Delta)$ and $e_\infty^r(\Delta)$ defined by Altay and Polat [4]. Demiriz and Cakan [15] introduced the sequence spaces $e_0^r(u, p)$ and $e_c^r(u, p)$ of non absolute type, as the sets of all sequences such that $E^{r,u}-$ transforms are in the spaces $c_0(p)$ and $c(p)$, respectively, that is,

$$e_0^r(u, p) = \{x = (x_k) \in \omega \ : \lim_{n\to\infty} |\sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k u_k x_k|^{pn} = 0\},$$
$$e_c^r(u, p) = \{x = (x_k) \in \omega \ : \exists \ell \in \mathbb{C} \ni \lim_{n\to\infty} |\sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k (u_k x_k - \ell)|^{pn} = 0\},$$

where $u = (u_k)$ is the sequence of non-zero reals. In the case $(u_k) = (p_k) = e = (1, 1, 1, \ldots)$, the sequence spaces $e_0^r(u, p)$, $e_c^r(u, p)$ and are, respectively, reduced to the sequence spaces e_0^r and e_c^r introduced by Altay and Başarir [2]. Define the sequence $y = \{y_k(r)\}$, which will be frequently used, as the E^r-transform of a sequence $x = (x_k)$, i.e.,

$$y_k(r) = \sum_{j=0}^{k} \binom{k}{j} (1-r)^{k-j} r^j x_j \ , \qquad k \in \mathbb{N} \tag{3}$$

Define the sequence $b^{(k)}(r)=\{b_n^{(k)}(r)\}_{n\in\mathbb{N}}$ of the elements of the space e_0^r by

$$b_n^{(k)}(r)=\begin{cases}0, & 0\le n<k\\ \binom{n}{k}(r-1)^{n-k}r^{-n}, & n\ge k\end{cases}\tag{4}$$

for every fixed $k\in\mathbb{N}$. Then:

Let us define the sequence $y=(y_k)$, which will be frequently used, as the $B(m,r)$-transform of a sequence $x=(x_k)$, that is,

$$y_k=(E^r\Delta^{(m)}x)_k=\sum_{j=0}^{k}\sum_{i=j}^{k}\binom{k}{i}\binom{m}{i-j}(-1)^{i-j}r^i(1-r)^{n-i}x_k\ ;\qquad (k,\ m\in\mathbb{N})\,.\tag{5}$$

Here and after by $B(m,r)$, we denote the matrix $B(m,r)=(b_{nk}(m,r))$ defined by

$$b_{nk}(m,r)=\begin{cases}\sum\limits_{i=k}^{n}\binom{m}{i-k}\binom{n}{i}(-1)^{i-k}r^i(1-r)^{n-i}, & (0\le k\le n)\\ 0, & (k>n)\end{cases}$$

for all $k,\ m,\ n\in\mathbb{N}$. Now, we may begin with the following theorem which is essential in the text.

Theorem 2.1. *The sets e_0^r and e_c^r are the linear spaces with the co-ordinate wise addition and scalar multiplication which are the BK-spaces with the norm $\| x\|_{e_0^r}=\| x\|_{e_c^r}=\| E^r x\|_{\ell_\infty}$.*

Theorem 2.2. *The Euler sequence spaces e_0^r and e_c^r of nonabsolute type are linearly isomorphic to the spaces c_0 and c, respectively, ie., $e_0^r\cong c_0$ and $e_c^r\cong c$.*

Proof. To prove this, we should show the existence of a linear bijection between the spaces e_0^r and c_0. Consider the transformation T defined, with the notation of (3), from e_0^r to c_0 by $x\mapsto y=Tx$. The linearity of T is clear. Further, it is trivial that $x=\theta$ whenever $Tx=\theta$ and hence T is injective, where $\theta=(0,0,0,\ldots)$. Let $y\in c_0$ and define the sequence $x=\{x_k(r)\}$ by $x_k(r)=\sum_{j=0}^{k}\binom{k}{j}(r-1)^{k-j}r^{-k}y_j$, $k\in\mathbb{N}$. Then, we have

$$\lim_{n\to\infty}(E^r x)_n=\lim_{n\to\infty}\left[\sum_{k=0}^{n}\binom{n}{k}(1-r)^{n-k}r^k\sum_{j=0}^{k}\binom{k}{j}(r-1)^{k-j}r^{-k}y_j\right]=\lim_{n\to\infty}y_n=0$$

which says us that $x\in e_0^r$. Additionally, we observe that

$$\begin{aligned}\| x\|_{e_0^r}&=\sup_{n\in\mathbb{N}}\Big|\sum_{k=0}^{n}\binom{n}{k}(1-r)^{n-k}r^k\sum_{j=0}^{k}\binom{k}{j}(r-1)^{k-j}r^{-k}y_j\Big|\\&=\sup_{n\in\mathbb{N}}|y_n|=\| y\|_{c_0}<\infty.\end{aligned}$$

Consequently, we see from here that T is surjective and is norm preserving. Hence, T is a linear bijection which therefore says us that the spaces e_0^r and c_0 are linearly isomorphic. It is clear here that if the spaces e_0^r and c_0 are respectively replaced by the spaces e_c^r and c, then we obtain the fact that $e_c^r\cong c$. This completes the proof. ☐

Theorem 2.3. *Although the inclusions $c_0\subset e_0^r$ and $c\subset e_c^r$ strictly hold, neither of the spaces e_0^r and ℓ_∞ includes the other one.*

Proof. Let us take any $s\in c_0$. Then, bearing in mind the regularity of the Euler means of order r, we immediately observe that $E^r s\in c_0$ which means that $s\in e_0^r$. Hence, the inclusion $c_0\subset e_0^r$ holds. Furthermore, let us consider the sequence $u=\{u_k(r)\}$ defined by $u_k(r)=(-r)^{-k}$ for all $k\in\mathbb{N}$. Then, since $E^r u=\{(-r)^k\}\in c_0$, u is in e_0^r but not in c_0. By the similar discussion, one can see that the inclusion $c\subset e_c^r$ also holds. To establish the second part of theorem, consider that sequence $u=\{u_k(r)\}$ defined above, and $x=e=(1,1,1,\ldots)$ Then, u is in e_0^r but not in ℓ_∞ and x is in ℓ_∞ but not in e_0^r. Hence, the sequence spaces e_0^r and ℓ_∞ overlap but neither contains the other. This completes the proof. ☐

Theorem 2.4. *If $0 < t < r < 1$, then $e_0^r \subset e_0^t$ and $e_c^r \subset e_c^t$.*

Proof. Let us take $x = (x_k) \in e_0^r$. Then, for all $k \in \mathbb{N}$, we observe that

$$\mathcal{Z}_k = \sum_{i=0}^{k} e_{ki}^t x_i = \sum_{i=0}^{k} e_{ki}^t \left(\sum_{j=0}^{i} e_{ij}^{1/r} y_j \right) = \sum_{j=0}^{k} e_{kj}^{t/r} y_j.$$

Since $0 < \frac{t}{r} < 1$, the method $E^{t/r}$ is regular which implies that $\mathcal{Z} = \mathcal{Z}_k \in c_0$ whenever $y = (y_k) \in c_0$ and we thus see that $x = (x_k) \in e_0^t$. This means that the inclusion $e_0^r \subset e_0^t$ holds. Now, one can show in the similar way that the inclusion $e_c^r \subset e_c^t$ also holds and so we leave the detail to the reader. □

Theorem 2.5. *The Euler sequence space e_p^r of non-absolute type is linearly isomorphic to the space ℓ_p, i.e., $e_p^r \cong \ell_p$; where $1 \leq p \leq \infty$.*

Theorem 2.6. *The inclusion $\ell_p \subset e_p^r$ strictly holds for $1 \leq p < \infty$.*

Theorem 2.7. *Neither of the spaces e_p^r and ℓ_∞ includes the other one, where $1 \leq p < \infty$.*

Theorem 2.8. *Let $\lambda \in \{e_0^r, e_c^r, e_\infty^r\}$. Then the set $\lambda(\Delta^{(m)})$ becomes a linear space with the coordinatewise addition and scalar multiplication which is the BK-space with the norm $\|x\|_{\lambda(\Delta^{(m)})} = \|\Delta^{(m)} x\|_\lambda$.*

Theorem 2.9. *The spaces $e_0^r(\Delta^{(m)})$, $e_c^r(\Delta^{(m)})$, and $e_\infty^r(\Delta^{(m)})$ are linearly isomorphic to the spaces c_0, c and ℓ_∞, respectively; that is, $e_0^r\left(\Delta^{(m)}\right) \cong c_0$, $e_c^r\left(\Delta^{(m)}\right) \cong c$,* and *$e_\infty^r\left(\Delta^{(m)}\right) \cong \ell_\infty$.*

Theorem 2.10. *Let $0 < t < r < 1$. Then the inclusions $e_0^r(\Delta^{(m)}) \subset e_0^t(\Delta^{(m)})$, $e_c^r\left(\Delta^{(m)}\right) \subset e_c^t\left(\Delta^{(m)}\right)$, and $e_\infty^r(\Delta^{(m)}) \subset e_\infty^t(\Delta^{(m)})$ hold.*

Proof. Let $x \in e_0^r(\Delta^{(m)})$. Consider the equality

$$\begin{aligned} u_n &= \sum_{k=0}^{n} [\sum_{i=k}^{n} \binom{n}{i} \binom{m}{i-k} (-1)^{i-k} t^i (1-t)^{n-i}] x_k \\ &= \sum_{k=0}^{n} e_{nk}^t (\sum_{i=0}^{k} e_{ki}^{1/r} y_i) = \sum_{k=0}^{n} e_{nk}^{t/r} y_k;\ (n \in \mathbb{N}). \end{aligned}$$

Then, as $0 < t/r < 1$ and $E_{t/r}$ is a T-matrix one can easily see that $u = (u_n) \in c_0$ whenever $y = (y_n) \in c_0$ which means that $e_0^r(\Delta^{(m)}) \subset e_0^t(\Delta^{(m)})$, as desired. □

Theorem 2.11. *The inclusions $e_0^r(\Delta^{(m)}) \subset e_0^r(\Delta^{(m+1)})$, $e_c^r\left(\Delta^{(m)}\right) \subset e_c^r\left(\Delta^{(m+1)}\right)$, $e_\infty^r(\Delta^{(m)}) \subset e_\infty^r(\Delta^{(m+1)})$, $e_0^r(\Delta^{(m)}) \subset e_c^r\left(\Delta^{(m)}\right)$, and $e_c^r\left(\Delta^{(m)}\right) \subset e_\infty^r(\Delta^{(m)})$ strictly hold.*

Proof. Let $x \in e_0^r(\Delta^{(m)})$. Then as the inequality

$$|(E_r\Delta^{(m+1)}x)_k| = |E_r\Delta^{(m)}(\Delta x)_k| = |(E_r\Delta^{(m)}x)_k - \left(E_r\Delta^{(m)}x\right)_{k-1}| \leq |((E_r\Delta^{(m)}x)_k| + |(E_r\Delta^{(m)}x)_{k-1}|$$

trivially holds and tends to zero as $k \to \infty$, $x \in e_0^r\left(\Delta^{(m+1)}\right)$. This shows that the inclusion $e_0^r\left(\Delta^{(m)}\right) \subset e_0^r\left(\Delta^{(m+1)}\right)$ holds. Moreover, let us consider the sequence $x = \{x_n\,(m+1, r)\}$ defined by $x_n\,(m+1, r) = \binom{m+n}{n}$ for all $n \in \mathbb{N}$. Then, as $y = E^r\Delta^{(m+1)}x = ((1-r)^n) \in c_0$ and $z = E^r\Delta^{(m)}x = e \notin c_0$ we immediately observe that x is in $e_0^r(\Delta^{(m+1)})$ but not in $e_0^r(\Delta^{(m)})$. Because there is at least one sequence in $e_0^r(\Delta^{(m+1)}) \setminus e_0^r\,(\Delta^{(m)})$, the inclusion $e_0^r(\Delta^{(m+1)}) \subset e_0^r(\Delta^{(m)})$ is strict. The validity of the inclusions $e_0^r(\Delta^{(m)}) \subset e_c^r(\Delta^{(m)}) \subset e_\infty^r\left(\Delta^{(m)}\right)$ is easily seen by combining the definition of the sequence spaces $e_0^r(\Delta^{(m)})$, $e_c^r\left(\Delta^{(m)}\right)$ and $e_\infty^r(\Delta^{(m)})$ with the well-known strict inclusions $c_0 \subset c \subset \ell_\infty$. □

3. The Basis for the Spaces $e_0^r, e_c^r, e_0^r\left(\Delta^{(m)}\right)$ and $e_c^r(\Delta^{(m)})$

In the present section, we give the sequences of the points of the spaces e_0^r, e_c^r, $e_0^r\left(\Delta^{(m)}\right)$ and $e_c^r\left(\Delta^{(m)}\right)$ which form the Schauder basis for those spaces. Firstly, we define the Schauder basis of a normad space. If a normed sequence space λ contains a sequence (b_n) with the property that, for every $x \in \lambda$, there is a unique sequence of scalars (α_n) such that

$$\lim_{n\to\infty} \|x - (\alpha_0 b_0 + \alpha_1 b_1 + \cdots + \alpha_n b_n)\| = 0,$$

then (b_n) is called a Schauder basis (or briefly basis) for λ. The series $\sum \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n), and written as $x = \sum \alpha_k b_k$.

4. The $\alpha-, \beta-, \gamma-$, and Continuous Duals of the Spaces $e_0^r, e_c^r, e_0^r\left(\Delta^{(m)}\right), e_c^r(\Delta^{(m)})$ and $e_\infty^r(\Delta^{(m)})$

In this section, we state and prove the theorems determining the α-, β-, γ- and continuous duals of the sequence spaces e_0^r, e_c^r, $e_0^r(\Delta^{(m)})$, $e_c^r\left(\Delta^{(m)}\right)$ and $e_\infty^r(\Delta^{(m)})$. For the sequence spaces λ and μ, define the set $S(\lambda,\mu)$ by

$$S(\lambda,\mu) = \{z = (z_k) \in w : \ xz = (x_k z_k) \in \mu \ \text{ for all } \ x \in \lambda\}. \tag{6}$$

is called the multiplier space of the sequence spaces λ and μ. One can easily observe for a sequence space v with $\lambda \supset v \supset \mu$ that the inclusion $S(\lambda,\mu) \subset S(v,\mu)$ and $S(\lambda,\mu) \subset S(\lambda, v)$ hold. With the notation of (6), the α-, β- and γ-duals of a sequences pace λ, which are respectively denoted by $\lambda^\alpha, \lambda^\beta$ and λ^γ, are defined by

$$\lambda^\alpha = S(\lambda,\ell_1), \ \lambda^\beta = S(\lambda, cs) \ \text{ and } \ \lambda^\gamma = S(\lambda, bs).$$

Now, it is immediate for the sequence spaces λ and μ that $\lambda^\alpha \subseteq \lambda^\beta \subseteq \lambda^\gamma$ and $\lambda^\eta \supset \mu^\eta$ whenever $\lambda \subset \mu$, where $\eta \in \{\alpha,\beta,\gamma\}$. It is well known that

$$(\ell_p)^\beta = \ell_q \ \text{ and } \ (\ell_\infty)^\beta = \ell_1, \tag{7}$$

where $1 \le p < \infty$ and $p^{-1} + q^{-1} = 1$. We shall throughout denote the collection of all finite subsets of $\mathbb{N}$ by $\mathcal{F}$. The continuous dual of a normed space X is defined as the space of all bounded linear functionals on X and is denoted by X^*.

Lemma 4.1. *$A \in (c_0 : \ell_1) = (c \ : \ell_1)$ if and only if*

$$\sup_{K\in\mathcal{F}} \sum_n \left| \sum_{k\in K} a_{nk} \right| < \infty.$$

Lemma 4.2. *$A \in (c_0 : c)$ if and only if*

$$\lim_{n\to\infty} a_{nk} = \alpha_k, k \in \mathbb{N}, \tag{8}$$

$$\sup_{n\in\mathbb{N}} \sum_k a_{nk} < \infty. \tag{9}$$

Lemma 4.3. *$A \in (c_0 : \ell_\infty)$ if and only if (9) holds.*

Lemma 4.4. *$A \in (c : c)$ if and only if (8), (9) hold, and $\lim_{n\to\infty} \sum_k a_{nk}$ exists.*

Lemma 4.5. *$A \in (c_0 \ : \ell_\infty) = (c \ : \ell_\infty) = (\ell_\infty \ : \ \ell_\infty)$ if and only if (9) holds.*

Theorem 4.6. *The α-dual of the spaces e_0^r, e_c^r is*

$$b_r = \{a = (a_k) \in \omega : \sup_{K \in \mathcal{F}} \sum_n \mid \sum_{k \in K} \binom{n}{k} (r-1)^{n-k} r^{-n} a_n \mid < \infty\}.$$

Proof. Let $a = (a_n) \in \omega$ and define the matrix B^r whose rows are the product of the rows of the matrix $E^{1/r}$ and the sequence $a = (a_n)$. Bearing in mind the relation (3), we immediately derive that

$$a_n x_n = \sum_{k=0}^{n} \binom{n}{k} (r-1)^{n-k} r^{-n} a_n y_k = (B^r y)_n, \; n \in \mathbb{N} \tag{10}$$

We therefore observe by (10) that $ax = (a_n x_n) \in \ell_1$ whenever $x \in e_0^r$ or e_c^r if and only if $B^r y \in \ell_1$ whenever $y \in c_0$ of c. Then we derive by Lemma 4.1 that

$$\sup_{k \in \mathcal{F}} \sum_n \mid \sum_{k \in K} \binom{n}{k} (r-1)^{n-k} r^{-n} a_n \mid < \infty.$$

which yields the consequence that $\{e_0^r\}^\alpha = \{e_c^r\}^\alpha = b_r$. □

Theorem 4.7. *Define the sets a_q^r and a_∞^r as follows:*

$$a_q^r = \{a = (a_k) \in \omega \;\; : \sup_{K \in \mathcal{F}} \sum_k \mid \sum_{n \in K} \binom{n}{k} (r-1)^{n-k} r^{-n} a_n \mid^q < \infty\}$$

and

$$a_\infty^r = \{a = (a_k) \in \omega \;\; : \sup_{k \in \mathbb{N}} \;\; \sum_n \mid \binom{n}{k} (r-1)^{n-k} r^{-n} a_n \mid < \infty\}.$$

Then, $(e_1^r)^\alpha = e_\infty^r$ and $(e_p^r)^\alpha = a_q^r$; where $1 < p \leq \infty$.

Proof. Let us define the matrix B^r whose rows are the product of the rows of the matrix $E^{1/r}$ with the sequence $a = (a_n)$. Therefore, we easily obtain by bearing in mind the relation (3) that

$$a_n x_n = \sum_{k=0}^{n} \binom{n}{k} (r-1)^{n-k} r^{-n} a_n y_k = (B^r y)_n (k \in \mathbb{N}) \tag{11}$$

Thus, we observe by (11) that $ax = (a_n x_n) \in \ell_1$ whenever $x \in e_p^r$ if and only if $B^r y \in \ell_1$ whenever $y \in \ell_p$. This means that $a = (a_n) \in (e_p^r)^\alpha$ if and only if $B^r \in (\ell_p \; : \; \ell_1)$. Then B^r instead of A that

$$\sup_{K \in \mathcal{F}} \sum_k \mid \sum_{n \in k} \binom{n}{k} (r-1)^{n-k} r^{-n} a_n \mid^q < \infty.$$

This yields the desired consequence that $(e_p^r)^\alpha = a_q^r$. □

Theorem 4.8. *Define the sets d_1^r, d_2^r, d_3^r and b_q^r by*

$$d_1^r = \{a = (a_k) \in \omega \;\; : \;\; \sum_{j=k}^{\infty} \binom{j}{k} (r-1)^{j-k} r^{-j} a_j \;\; \textit{exists for each } k \in \mathbb{N}\},$$

$$d_2^r = \{a = (a_k) \in \omega \;\; : \sup_{n,k \in \mathbb{N}} \;\; \mid \sum_{j=k}^{n} \binom{j}{k} (r-1)^{j-k} r^{-j} a_j \mid < \infty\},$$

$$d_3^r = \{a = (a_k) \in \omega \;\; : \;\; \lim_{n \to \infty} \sum_k \mid \sum_{j=k}^{n} \binom{j}{k} (r-1)^{j-k} r^{-j} a_j \mid \; = \sum_k \mid \sum_{j=k}^{\infty} \binom{j}{k} (r-1)^{j-k} r^{-j} a_j \mid\}$$

and

$$b_q^r = \{a = (a_k) \in \omega \;\; : \sup_{n \in \mathbb{N}} \sum_{k=0}^{n} \mid \sum_{j=k}^{n} \binom{j}{k} (r-1)^{j-k} r^{-j} a_j \mid^q < \infty\} \;\; (1 \; < q < \infty).$$

Then, $(e_1^r)^\beta = d_1^r \cap d_2^r$, $(e_p^r)^\beta = d_1^r \cap b_q^r$ and $(e_\infty^r)^\beta = d_1^r \cap d_3^r$; where $(1 < p < \infty)$.

Theorem 4.9. *Define the set b_1^r by*

$$b_1^r = \{a = (a_k) \in \omega \ : \sup_{n\in\mathbb{N}} \sum_{k=0}^{n} | \sum_{j=k}^{n} \binom{j}{k} (r-1)^{j-k} r^{-j} a_j \mid < \infty\}.$$

Then, $(e_1^r)^\gamma = d_2^r$, $(e_p^r)^\gamma = b_q^r$ and $(e_\infty^r)^\gamma = b_1^r$; where $1 < p < \infty$.

Proof. Let $a = (a_k) \in b_q^r$ and $x = (x_k) \in e_p^r$. Then, we obtain by applying the Hölder's inequality that

$$|\sum_{k=0}^{n} a_k x_k| = |\sum_{k=0}^{n} \left[\sum_{j=0}^{k} \binom{k}{j} (r-1)^{k-j} r^{-k} y_j \right] a_k | = | \sum_{k=0}^{n} t_{nk}^r y_k | \le \left(\sum_{k=0}^{n} | t_{nk}^r |^q \right)^{1/q} \left(\sum_{k=0}^{n} | y_k |^p \right)^{1/p}$$

which gives us by taking supremum over $n \in \mathbb{N}$ that

$$\sup_{n\in\mathbb{N}} |\sum_{k=0}^{n} a_k x_k| \le \sup_{n\in\mathbb{N}} \left[\left(\sum_{k=0}^{n} |t_{nk}^r|^q \right)^{1/q} \left(\sum_{k=0}^{n} |y_k|^p \right)^{1/p} \right] \le \|y\|_{\ell_p} . \left(\sup_{n\in\mathbb{N}} \sum_{k=0}^{n} |t_{nk}^r|^q \right)^{1/q} < \infty.$$

This means that $a = (a_k) \in (e_p^r)^\gamma$. Hence,

$$b_q^r \subset (e_p^r)^\gamma. \tag{12}$$

Conversely, let $a = (a_k) \in (e_p^r)^\gamma$ and $x = (x_k) \in e_p^r$. Then, one can easily see that $(\sum_{k=0}^{n} t_{nk}^r y_k)_{n\in\mathbb{N}} \in \ell_\infty$ whenever $(a_k x_k) \in bs$. This shows that the triangle matrix $T^r = (t_{nk}^r)$, is in the class $(\ell_p : \ell_\infty)$. Hence, $A \in (\ell_p : c)$ iff $\sup_{n\in\mathbb{N}} \sum_k |a_{nk}|^q < \infty (1 < p < \infty)$ holds with t_{nk}^r instead of a_{nk} which yields that $a = (a_k) \in b_q^r$. That is to say that

$$(e_p^r)^\gamma \subset b_q^r. \tag{13}$$

Therefore, by combining the inclusions (12) and (13), we get $(e_p^r)^r = b_q^r$. $\square$

Theorem 4.10. *Define the sets d_1^r, d_2^r, and d_3^r by*

$$d_1^r = \left\{ a = (a_k) \in \omega : \sup_{n\in\mathbb{N}} \sum_{k=0}^{n} | \sum_{j=k}^{n} \binom{j}{k} (r-1)^{j-k} r^{-j} a_j | < \infty \right\},$$

$$d_2^r = \left\{ a = (a_k) \in \omega : \sum_{j=k}^{\infty} \binom{j}{k} (r-1)^{j-k} r^{-j} a_j \ \textit{exists for each}\ k \in \mathbb{N} \right\}, \ \textit{and}$$

$$d_3^r = \left\{ a = (a_k) \in \omega : \lim_{n\to\infty} \sum_{k=0}^{n} \sum_{j=k}^{n} \binom{j}{k} (r-1)^{j-k} r^{-j} a_j \ \textit{exists} \right\}.$$

Then $\{e_0^r\}^\beta = d_1^r \cap d_2^r$ and $\{e_c^r\}^\beta = d_1^r \cap d_2^r \cap d_3^r$.

Proof. Because of the proof may also be obtained for the space e_c^r in the similar way, we omit it and give the proof only for the space e_0^r. Consider the equation

$$\sum_{k=0}^{n} a_k x_k = \sum_{k=0}^{n} \left[\sum_{j=0}^{n} \binom{j}{k} (r-1)^{k-j} r^{-k} y_j \right] a_k = \sum_{k=0}^{n} \left[\sum_{j=k}^{n} \binom{j}{k} (r-1)^{j-k} r^{-j} a_j \right] y_k = (T^r y)_n, \tag{14}$$

where $T^r = (t_{nk}^r)$ is defined by

$$t_{nk}^r = \begin{cases} \sum_{j=k}^{n} \binom{j}{k} (r-1)^{j-k} r^{-j} a_j, & 0 \le k \le n,\ k, n \in \mathbb{N} \\ 0, & k > n. \end{cases} \tag{15}$$

Thus , we deduce from Lemma 4.2 with (14) that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in e_0^r$ if and only if $T^r y \in c$ wherever $y = (y_k) \in c_0$. Therefore, we derive from (8) and (9) that

$$\lim_{n\to\infty} t_{nk}^r \text{ exists for each } k \in \mathbb{N} \ \sup_{n\in\mathbb{N}} \sum_{k=0}^{n} |t_{nk}^r| < \infty \tag{16}$$

which shows that $\{e_0^r\}^\beta = d_1^r \cap d_2^r$. □

Theorem 4.11. *The γ-dual of the spaces e_0^r and e_c^r is d_1^r.*

Theorem 4.12. *$\{e_c^r\}^*$ and $\{e_0^r\}^*$ are isometrically isomorphic to ℓ_1.*

Theorem 4.13. *Consider the set D*

$$D = \left\{ a = (a_k) \in \omega : \sup_{K\in\mathcal{F}} \sum_n \Big| \sum_{k\in K} d_{nk}(m,r) \Big| < \infty \right\},$$

where $D(m,r) = (d_{nk}(m,r))$ is defined by

$$d_{nk}(m,r) = \begin{cases} \sum\limits_{j=k}^{n} \binom{m+n-j-1}{n-j} \binom{j}{k} r^{-j}(r-1)^{j-k} a_n, & (0 \le k \le n) \\ 0, & (k > n). \end{cases}$$

for all $k, m, n \in \mathbb{N}$. Then, $[e_0^r(\Delta^{(m)})]^\alpha = [e_c^r(\Delta^{(m)})]^\alpha = [e_\infty^r(\Delta^{(m)})]^\alpha = D$.

Proof. Let λ denotes any one of the spaces e_0^r, e_c^r and e_∞^r. Consider the equation

$$a_n x_n = \sum_{k=0}^{n} \left[\sum_{j=k}^{n} \begin{pmatrix} m+n-j-1 \\ n-j \end{pmatrix} \begin{pmatrix} j \\ k \end{pmatrix} r^{-j}(r-1)^{j-k} a_n \right] y_k = \sum_{k=0}^{n} d_{nk}(m,r)\, y_k = \{D(m,r)\, y\}_n. \tag{17}$$

Thus, we deduce from Lemma 4.1 and (17) that $ax = (a_n x_n) \in \ell_1$ whenever $x = (x_k) \in \lambda\left(\Delta^{(m)}\right)$ if and only if $D(m,r)\, y \in \ell_1$ whenever $y = (y_k) \in \lambda$. This means that $a = (a_n) \in [\lambda\left(\Delta^{(m)}\right)]^\alpha$ if and only if $D(m,r) \in (\lambda : \ell_1)$. This leads us to the desired consequence $[\lambda\left(\Delta^{(m)}\right)]^\alpha = D$. □

Theorem 4.14. *Define the sets B_1, B_2, B_3 and B_4 by*

$$B_1 = \left\{ a = (a_k) \in \omega : \sup_{n\in\mathbb{N}} \sum_k |c_{nk}(m,r)| < \infty \right\},$$

$$B_2 = \left\{ a = (a_k) \in \omega : \lim_{n\to\infty} c_{nk}(m,r) \ \text{ exists for all } k \in \mathbb{N} \right\},$$

$$B_3 = \left\{ a = (a_k) \in \omega : \lim_{n\to\infty} \sum_k c_{nk}(m,r) \ \text{ exists} \right\} \ \text{and}$$

$$B_4 = \left\{ a = (a_k) \in \omega : \lim_{n\to\infty} \sum_k |c_{nk}(m,r)| = \sum_k \Big| \lim_{n\to\infty} c_{nk}(m,r) \Big| \right\}$$

where $C(m,r) = (c_{nk}(m,r))$ is defined by

$$c_{nk}(m,r) = \begin{cases} \sum\limits_{i=k}^{n} [\sum_{j=k}^{i} \binom{m+i-j-1}{i-j} \binom{j}{k} r^{-j}(r-1)^{j-k}] a_i, & (0 \le k \le n),\ (n,\ k \in \mathbb{N}) \\ 0, & (k > n). \end{cases}$$

Then, $[e_0^r(\Delta^{(m)})]^\beta = B_1 \cap B_2$, $[e_c^r(\Delta^{(m)})]^\beta = B_1 \cap B_2 \cap B_3$ and $[e_\infty^r(\Delta^{(m)})]^\beta = B_2 \cap B_4$.

Proof. Consider the equation obtained by using the relation (5)

$$\sum_{k=0}^{n} a_k x_k \sum_{k=0}^{n}\sum_{i=0}^{k}\sum_{j=i}^{k}\binom{m+k-j-1}{k-j}\binom{j}{i} r^{-j}(r-1)^{j-i} a_k y_i = \sum_{k=0}^{n}\left\{\sum_{i=k}^{n}\left[\sum_{j=k}^{i}\binom{m+i-j-1}{i-j}\binom{j}{k} r^{-j}(r-1)^{j-k}\right] a_i\right\} y_k$$
$$= \{C(m,r)\, y\}_n; \quad (n\in\mathbb{N}). \tag{18}$$

Thus, we deduce from Lemma 4.2 and (18) that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in e_0^r(\Delta^{(m)})$ if and only if $C(m,r)\, y \in c$ whenever $y = (y_k) \in c_0$. That is to say that $a = (a_k) \in [e_0^r(\Delta^{(m)})]^{\beta}$ if and only if $C(m,r) \in (c_0 : c)$ which yields us $[e_0^r(\Delta^{(m)})]^{\beta} = B_1 \cap B_2$.

As the reader can easily show the facts about the β-duals of the sequence spaces $e_c^r(\Delta^{(m)})$ and $e_\infty^r(\Delta^{(m)})$ in the similar manner, we omit their proofs. □

Theorem 4.15. *The γ-dual of the spaces $e_0^r(\Delta^{(m)})$, $e_c^r(\Delta^{(m)})$ and $e_\infty^r(\Delta^{(m)})$ is the set B_1.*

Proof. This is obtained in a similar manner used in the proof of Theorem 4.9 with Lemma 4.5 instead of Lemma 4.2 and so we leave the detail to the reader. □

5. Some Matrix Mappings Related to the Sequence Spaces e_c^r and $e_c^r(\Delta^{(m)})$

In this section, we characterize the matrix mappings from e_c^r and $e_c^r(\Delta^{(m)})$ into some of the known sequence spaces and into the Euler, difference, Riesz, Cesaro sequence spaces. We directly prove the theorems characterizing the classes $(e_c^r : \ell_p)$, $(e_c^r : c)$, $e_c^r(\Delta^{(m)} : \ell_p)$ and $e_c^r(\Delta^{(m)} : c)$ and derive the other characterizations from them by means of a given basic lemma, where $1 \le p \le \infty$. We shall write throughout for brevity that $a(n,k) = \sum_{j=0}^{n} a_{jk}$ and $\tilde{a}_{nk} = \sum_{j=k}^{n}\binom{j}{k}(r-1)^{j-k} r^{-j} a_{nj}$. We shall write throughout for brevity that $T(m,r) = (t_{nk}(m,r))$ by $t_{nk}(m,r) = \sum_{j=k}^{\infty}\binom{m+n-j-1}{n-j}\binom{j}{k} r^{-j}(r-1)^{j-k} a_{nj}$, for all $k,m,n \in \mathbb{N}$.

Lemma 5.1. *The matrix mappings between the BK-spaces are continuous.*

Lemma 5.2. *$A \in (c : \ell_p)$ if and only if*

$$\sup_{F\in\mathcal{F}} \sum_n \Big|\sum_{k\in F} a_{nk}\Big|^p < \infty, \quad (1 \le p < \infty). \tag{19}$$

Lemma 5.3. *$A \in (e_c^r : \ell_p)$ if and only if*

1. *for 1≤ $p < \infty$,*

$$\sup_{F\in\mathcal{F}} \sum_n \Big|\sum_{k\in F} \tilde{a}_{nk}\Big|^p < \infty, \tag{20}$$

$$\tilde{a}_{nk} \quad \textit{exists of all } k,n\in\mathbb{N}, \tag{21}$$

$$\sum_k \tilde{a}_{nk} \quad \textit{converges for all } n\in\mathbb{N}, \tag{22}$$

$$\sup_{n\in\mathbb{N}} \sum_{k=0}^{m} \Big|\sum_{j=k}^{m} \binom{j}{k}(r-1)^{j-k} r^{-j} a_{nj}\Big| < \infty, \ \ n\in\mathbb{N} \tag{23}$$

2. *for $p = \infty$, (21) and (23) hold, and*

$$\sup_{n\in\mathbb{N}} \sum_k |\tilde{a}_{nk}| < \infty. \tag{24}$$

Proof. Suppose conditions (20)-(23) hold and take any $x \in e_c^r$. Then, $\{a_{nk}\}_{k\in\mathbb{N}} \in \{e_c^r\}^\beta$ for all $n \in \mathbb{N}$ and this implies that Ax exists. Let us define the matrix $B = (b_{nk})$ with $b_{nk} = \tilde{a}_{nk}$ for all $k, n \in \mathbb{N}$. Then, since (19) is satisfied for that matrix B, we have $B \in (c : \ell_p)$. Let us now consider the following equality obtained from the m-th partial sum of the series $\sum_k a_{nk}x_k$:

$$\sum_{k=0}^{m} a_{nk}x_k = \sum_{k=0}^{m}\sum_{j=k}^{m}\binom{j}{k}(r-1)^{j-k}r^{-j}a_{nj}y_k, \quad m, n \in \mathbb{N}. \tag{25}$$

Therefore, we derive form (25) an $m \to \infty$ that

$$\sum_k a_{nk}x_k = \sum_k \tilde{a}_{nk}x_k, \ n \in \mathbb{N} \tag{26}$$

which yields by taking ℓ_p-norm that

$$\|Ax\|_{\ell_p} \leq \|By\|_{\ell_p} < \infty.$$

This means that $A \in (e_c^r : \ell_p)$.

Conversely, suppose that $A \in (e_c^r : \ell_p)$. Then, since e_c^r and ℓ_p are the BK-spaces, we have from Lemma 5.1 that there exists some real constant $K > 0$ such that

$$\|Ax\|_{\ell_p} \leq K\|x\|_{e_c^r} \tag{27}$$

for all $x \in e_c^r$. Since inequality (27) also holds for the sequence $x = (x_k) = \sum_{k\in F} b^{(k)}(r)$ belonging to the space e_c^r, where $b^{(k)}(r) = \{b_n{}^{(k)}(r)\}$ is defined by (4), we thus have for any $F \in \mathcal{F}$ that

$$\|Ax\|_{\ell_p} = \left(\sum_n \left|\sum_{k\in F}\tilde{a}_{nk}\right|^p\right)^{1/p} \leq K\|x\|_{e_c^r} = K$$

which shows the necessity of (20). Since A is applicable to the space e_c^r by the hypothesis, the necessity of conditions (21)-(23) is trivial. This completes the proof of the part (i) of theorem. Since the part (ii) may also be proved in the similar way that of the part (i), we leave the detailed proof to the reader. □

Theorem 5.4. *$A \in (e_c^r : c)$ if and only if (21), (23) and (24) hold*

$$\lim_{n\to\infty}\tilde{a}_{nk} = \alpha_k \ \textit{for each } k \in \mathbb{N} \ \textit{and} \tag{28}$$

$$\lim_{n\to\infty}\sum_k \tilde{a}_{nk} = \alpha. \tag{29}$$

Theorem 5.5.

(i) Let $1 \leq p < \infty$. Then, $A \in \left(e_c^r\left(\Delta^{(m)}\right) : \ell_p\right)$ if and only if

$$\sup_{s\in\mathbb{N}}\sum_k |t_{nk}^s(m,r)| < \infty \ \textit{for any } n \in \mathbb{N}, \tag{30}$$

$$\lim_{s\to\infty}\sum_k t_{nk}^s(m,r) \ \textit{exists for any } n \in \mathbb{N}, \tag{31}$$

$$\lim_{s\to\infty} t_{nk}^s(m,r) = t_{nk}(m,r) \ \textit{ exists for any } k, n \in \mathbb{N}, \tag{32}$$

$$\sup_{F\in\mathcal{F}}\sum_n \left|\sum_{k\in F} t_{nk}(m,r)\right|^p < \infty, \tag{33}$$

where $t_{nk}^s(m,r) = \sum\limits_{j=k}^{s}\binom{m+n-j-1}{n-j}\binom{j}{k}r^{-j}(r-1)^{j-k}a_{nj}$.

(ii) $A \in (e_c^r\left(\Delta^{(m)}\right) : \ell_\infty)$ *if and only if* (30)-(32) *hold and*

$$\sup_{n\in\mathbb{N}} \sum_k |t_{nk}(m,r)| < \infty. \tag{34}$$

Proof. (i) Suppose that the conditions (17)-(33) hold and take any $x \in e_c^r(\Delta^{(m)})$. Then, $\{a_{nk}\}_{k\in\mathbb{N}} \in [e_c^r\left(\Delta^{(m)}\right)]^\beta$ for any $n \in \mathbb{N}$ and this implies that Ax exists. In this situation, as (19) is satisfied, $T(m,r) = (t_{nk}(m,r)) \in (c : \ell_p)$. Let us consider the following equality obtained from the s^{th} partial sum of the series $\sum_k a_{nk}x_k$ by using the relation (5)

$$\sum_{k=0}^{s} a_{nk}x_k = \sum_{k=0}^{s} t_{nk}^s(m,r)\, y_k, \quad (m,\ n,\ s \in \mathbb{N}) \tag{35}$$

Therefore, passing to limit in (12) as $s \to \infty$ we derive that

$$\sum_k a_{nk}x_k = \sum_k t_{nk}(m,r)y_k, \quad (m,n \in \mathbb{N}), \tag{36}$$

which yields the fact that $A \in \left(e_c^r\left(\Delta^{(m)}\right) : \ell_p\right)$, as $T(m,r)\,y \in \ell_p$ whenever $y \in c$.
Conversely, suppose that $A \in \left(e_c^r\left(\Delta^{(m)}\right) : \ell_p\right)$. Then, as $e_c^r\left(\Delta^{(m)}\right)$ and ℓ_p are the BK-spaces, we have from Lemma 5.1 that there exists some real constant $K > 0$ such that

$$\|Ax\|_{\ell_p} \le K.\|x\|_{e_c^r(\Delta^{(m)})}, \tag{37}$$

for all $x \in e_c^r\left(\Delta^{(m)}\right)$. As the inequality (37) is also satisfied for the sequence $x = (x_k) = \sum_{k\in F} b^{(k)}(r)$ belonging to the space $e_c^r\left(\Delta^{(m)}\right)$, where $b^{(k)}(r) = \{b_n^{(k)}(r)\}$ is defined by (4), we have for any $F \in \mathcal{F}$ that

$$\|Ax\|_{\ell_p} = \left(\sum_n \Big|\sum_{k\in F} t_{nk}(m,r)\Big|^p\right)^{1/p} \le K.\|x\|_{e_c^r(\Delta^{(m)})},$$

which shows the necessity of (33). Because A is applicable to the space $e_c^r\left(\Delta^{(m)}\right)$ by the hypothesis, the necessities of (30)-(32) are trivial. This completes the proof of the part (i) of Theorem.
As the part (ii) may also be proved in the similar manner that of the part (i), we leave the detailed proof to the reader. □

Theorem 5.6. $A \in \left(e_c^r\left(\Delta^{(m)}\right) : c\right)$ *if and only if* (30)-(32) *and* (34) *hold, and*

$$\lim_{n\to\infty} t_{nk}(m,r) = \alpha_k \ \textit{for each}\ k \in \mathbb{N}, \tag{38}$$

$$\lim_{n\to\infty} \sum_k t_{nk}(m,r) = \alpha \tag{39}$$

Proof. Suppose that A satisfies the conditions (30)-(32), (34), (38), and (39). Let us consider any $x = (x_k)$ in $e_c^r\left(\Delta^{(m)}\right)$. Then, Ax exists and it is trivial that the sequence $y = (y_k)$ connected with the sequence $x = (x_k)$ by the relation (5) is in c such that $y_k \to \ell$ as $k \to \infty$. At this stage, we observe from (38) and (34) that

$$\sum_{j=0}^{k} |\alpha_j| \le \sup_{n\in\mathbb{N}} \sum_j |t_{nj}(m,r)| < \infty$$

holds for every $k \in \mathbb{N}$. This leads us to the fact that $(\alpha_k) \in \ell_1$. Considering (36), let us write

$$\sum_k a_{nk}x_k = \sum_k t_{nk}(m,r)(y_k - \ell) + \ell \sum_k t_{nk}(m,r). \tag{40}$$

In this situation, by letting $n \to \infty$ in (40), one can see that the first term on the right side tends to $\sum_k \alpha_k(y_k - \ell)$ by (34) and (38) and the second term tends to $\ell\alpha$ by (39). Thus, we have that

$$(Ax)_n \to \sum_k \alpha_k(y_k - \ell) + \ell\alpha,$$

which shows that $A \in (e_c^r\left(\Delta^{(m)}\right) : c)$.

Conversely, suppose that $A \in (e_c^r\left(\Delta^{(m)}\right) : c)$. Then, as the inclusion $c \subset \ell_\infty$ holds, the necessities of (30)-(32) and (34) are immediately obtained from Theorem 5.3. To prove the necessity of (38), consider the sequence $x = x^{(k)} = \{x_n^{(k)}(m,r)\}_{n\in\mathbb{N}} \in e_c^r\left(\Delta^{(m)}\right)$ defined by

$$x_n^{(k)}(m,r) = \begin{cases} \sum_{j=k}^n \binom{m+n-j-1}{n-j}\binom{j}{k}(r-1)^{j-k}, & (n \geq k) \\ 0, & (n < k). \end{cases}$$

for each $k \in \mathbb{N}$. As Ax exists and is in c for every $x \in e_c^r\left(\Delta^{(m)}\right)$, one can easily see that $Ax^{(k)} = \{t_{nk}(m,r)\}_{n\in\mathbb{N}} \in c$ for each $k \in \mathbb{N}$ which shows the necessity of (38).

Similarly by putting $x = e$ in (36), we also obtain that $Ax = \{\sum_k t_{nk}(m,r)\}_{n\in\mathbb{N}}$ belongs to the space c and this shows the necessity of (39). □

References

[1] Z.U.Ahmad and Mursaleen, *Köthe-Toeplitz duals of some new sequence spaces and their matrix maps*, Publ. Inst. Math (Beograd), 42(56)(1987), 57-61.

[2] B.Altay and F.Başar, *Some Euler sequence spaces of non-absolute type*, Ukrainian Math. J., 57(1)(2005), 1-17.

[3] B.Altay, F.Başar and Mursaleen, *On the Euler sequence spaces which include the spaces ℓ_p and ℓ_∞-I*, Inform Sci, 176(10)(2006), 1450-1462.

[4] B.Altay and H.Polat, *On some new Euler difference sequence spaces*, Southeast Asian Bull. Math., 30(2006), 209-220.

[5] B.Altay and F.Başar, *The matrix domain and the fine spectrum of the difference operator Δ on the sequence space ℓ_p*, $(0 < p < 1)$, Commun Math. Anal., 2(2)(2007), 1-11.

[6] C.Aydin and F.Başar, *On the new sequence spaces which include the spaces c_0 and c*, Hokkaido Math. J., 33(2)(2004), 383-398.

[7] C.Aydin and F.Başar, *Some new difference sequence spaces*, Appl. Math. Comput, 157(3)(2004), 667-693.

[8] C.Aydin and F.Başar, *Some new sequence spaces which include the spaces ℓ_p and ℓ_∞*, (under communication).

[9] F.Başar and B.Altay, *On the space of sequences of $p-$bounded variation and related matrix mappings*, Ukranian Math. J., 55(1)(2003), 136-147.

[10] M.Başarir and Kayikçi, *On the generalized B^m-Riesz sequence space and β-proprety*, J. Inequal. Appl., 2009(2009), Article ID 385029.

[11] F.Başar and B.Altay, *On the Euler sequence spaces which include the spaces ℓ_p and ℓ_∞-II*, (under communication).

[12] M.Başarir, *On some new sequence space and related matrix transformations*, Indian J. Pure Appl. Math., 26(10)(1995), 1003-1010.

[13] B.Choudhary and S.Nanda, *Functional analysis with applications*, John Wiley & Sons Inc., (1989).

[14] R.Çolak and M.Et, *On some generalized difference sequence spaces and related matrix transformations*, Hokkaido Math. J., 26(3)(1997), 483-492.

[15] S.Demiriz and Çakan, *On some new paranormed Euler sequence spaces and Euler core*, Acta Math. Sin. (Engl. Ser.), 26(7)(2010), 1207-1222.

[16] P.K.Kampthan and M.Gupta, *Sequence spaces and Series*, Marcel Dekker Inc., New York, (1981).

[17] E.E.Kara and M.Başarir, *On compact operators and some Euler $B^{(m)}$-difference sequence spaces*, J. Math. Anal. Appl., 379(2)(2011), 499-511.

[18] V.Karakaya and H.Polat, *Some new paranormed sequence spaces defined by Euler and difference operators*, Acta Sci. Math. (Szeged), 75(2010), 87-100.

[19] H.Kizmaz, *On certain sequence spaces*, Canadian Math. Bull., 24(2)(1981), 169-176.

[20] I.J.Maddox, *Elments of Functional Analysis*, Cambridge, Univ. Press, (1970).

[21] I.J.Maddox, *Continuous and Köthe-Toeplitz duals of certain sequence spaces*, Proc. Cambridge Phil. Soc., 65(1969), 431-435.

[22] E.Malkowsky and S.D.Parashar, *Matrix transformations in space of bounded and convergent difference sequence of order m*, Analysis, 17(1997), 87-97.

[23] E.Malkowsky, *Recent results in the theory of matrix transformations in sequence spaces*, Mat. Vesnik, 49(1997), 187-196.

[24] P-N.Ng and P-Y.Lee, *Cesàro sequence spaces of non-absolute type*, Comment Math. Prace. Mat., 20(2)(1978), 429-433.

[25] H.Polat and F.Başar, *Some Euler spaces of difference sequences of order m*, Acta Mathematica Scientia, 27B(2)(2007), 254-266.

[26] M.Şengönül and F.Başar, *Some new Cesàro sequence spaces of non-absolute type which include the spaces c_0 and c*, Soochow J. Math., 31(1)(2005), 107-119.

[27] C-S.Wang, *On Nörlund sequence spaces*, Tamkang J. Math., 9(1979), 269-274.