International Journal of Mathematics And its Applications
Volume 5, Issue 1–C (2017), 343–348.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

On the Approximation of Function by Product Means in the Holder Metric

Research Article

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Abstract: In this paper, we have established a theorem on the approximation of function by product means in the holder metric and generalized an early result of Singh and Mahajan [7].

Keywords: Cesaro matrix, Euler matrix, Holder metric, Degree of approximation.

1. Introduction

Let the transforms

$$\partial_n = \sum_{k=1}^{n} a_{nk} s_k \tag{1}$$

$$\sigma_n = \sum_{k=1}^{n} b_{nk} s_k \tag{2}$$

be two regular methods of summability. Then both transform of a sequence $\{s_n\}$ is given by

$$t_n = \sum_{p=1}^{n} a_{np}\sigma_p = \sum_{p=1}^{n}\sum_{k=1}^{n} a_{np} b_{pk} s_k, \tag{3}$$

the sequence $\{s_n\}$ is said to be summable t_n to the sum s, if

$$\lim_{n\to\infty} t_n = s \tag{4}$$

Let $s(t) \in C_{2\pi}$ be a 2π-periodic analog signal whose Fourier trigonometric expansion be given by

$$s\,(t) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=0}^{\infty} A_n(t) \tag{5}$$

and let $\{s_n(t)\}$ be the sequence of partial sum of (5). Let the $(E,1)$ and $(C,2)$ transform for the sequence $\{s_n\}$ be defined by

$$E_n^1 = \frac{1}{2^n}\sum_{k=0}^{n} \binom{n}{k} s_k(t), \tag{6}$$

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$$C_n^2 = \frac{2}{(n+1)(n+2)} \sum_{k=0}^{n} (n-k+1) s_k(t) \tag{7}$$

The product $(C,2)(E,1)$ transform is given by

$$t_n(s,t) = \frac{2}{(n+1)(n+2)} \sum_{k=0}^{n} (n-k+1) \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} s_v(t) \tag{8}$$

The sequence $\{s_n\}$ is said to be $(E,1)(C,2)$ to the sum s, if

$$\lim_{n\to\infty} t_n(s;t) = s \tag{9}$$

2. Regularity Condition

$$t_n(s,t) = \frac{2}{(n+1)(n+2)} \sum_{k=0}^{n} (n-k+1) \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} s_v(t) = \sum_{k=0}^{\infty} C_{n,k} s_k, \tag{10}$$

where

$$C_{n,k} = \begin{cases} \frac{(n-k+1)2^{1-k}}{(n+1)(n+2)} \sum\limits_{v=0}^{k} \binom{n}{v}, & k \le n; \\ 0, & k > n. \end{cases} \tag{11}$$

Now

(1) $\sum\limits_{k=0}^{\infty} |C_{n,k}| = \sum\limits_{k=0}^{n} \left| \frac{(n-k+1)2^{1-k}}{(n+1)(n+2)} \sum\limits_{v=0}^{k} \binom{k}{v} \right| = 1$

(2) $C_{n,k} = \left(\frac{1}{n+1}\right)(1) \to 0$ as $n \to \infty$ for fixed k

(3) $\sum\limits_{k=0}^{\infty} C_{n,k} = 1$

Thus (C,2)(E,1) method is regular. Singh [7] defined the space H_α by

$$H_\alpha = \{s(t) \in C_{2\pi} \ : |s(t_1) - s(t_2)| = K(|t_1 - t_2|^\alpha)\} \tag{12}$$

The norm $||.||_\alpha$ by

$$||s||_\alpha = ||s||_c + \sup_{t_1,t_2} \{\Delta^\alpha s(t_1,t_2)\} \tag{13}$$

Where

$$||s||_c = \sup_{0=t=2\pi} |s(t)|$$

$$\Delta^\alpha s(t_1,t_2) = \frac{|s(t_1) - s(t_2)|}{|t_1 - t_2|^\alpha}, \quad t_1 \ne t_2 \tag{14}$$

And choosing $\Delta^0 s(t_1,t_2) = 0$. The element of the space H_α are called Holder continuous functions. If D is the collection of all differentiable functions defined on $[\pi,\pi]$ then $C_{2\pi} \supseteq H_\beta \supseteq H_\alpha \supseteq D$ for $0 \le \beta < \alpha \le 1$.

$$H_\alpha = \{s(t) \in C_{2\pi} \ : |s(t_1) - s(t_2)| = K|t_1 - t_2|^\alpha, \quad 0 < \alpha \le 1\} \tag{15}$$

is Banach space and the metric induced by the norm $||.||_\alpha$ on H_α is said to be Holder metric. We write

$$\emptyset_{t_1}(t) = s(t_1 + t) + s(t_1 - t) - 2s(t_1) \tag{16}$$

$$K_n(t) = \frac{1}{p\,(\mathrm{n}+1)(\mathrm{n}+2)} \sum_{k=0}^{n} \left\{ \frac{(n-k+1)}{2^k} \sum_{v=0}^{k} \left[\binom{k}{v} \frac{\sin\left(v+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)} \right] \right\} \tag{17}$$

3. Known Result

Singh and Mahajan [7] established the following theorems:

Theorem 3.1. *Let $w(t)$ defined in (12) be such that*

$$\int_t^{\pi} \frac{w(u)}{u^2} du = O\left\{H\left(t\right)\right\}, \quad H(t) \geq 0 \tag{18}$$

$$\int_0^t H\left(u\right) du = O\left\{tH\left(t\right)\right\}, \quad as \quad t \to 0^+ \tag{19}$$

Then for $0 \leq \beta < \eta \leq 1$ and $s \in H_w$ we have

$$\left\|t_n\left(s;t_1\right) - s\right\|_{w^*} = O\left\{\left(\left(n+1\right)^{-1} H\left(\frac{\pi}{n+1}\right)\right)^{1-\frac{\beta}{\eta}}\right\} \tag{20}$$

Theorem 3.2. *Let $w(t)$ defined in (12) and for $0 \leq \beta < \eta \leq 1$ and $s \in H_w$, we have*

$$\left\|t_n\left(s;t_1\right) - s\right\|_{w^*} = O\left\{\left(w\left(\frac{\pi}{n}+1\right)\right)^{1-\frac{\beta}{\eta}} + \left(\left(n+1\right)^{-1} \sum_{k=1}^{n+1} w\left(\frac{1}{K+1}\right)\right)^{1-\frac{\beta}{\eta}}\right\} \tag{21}$$

4. Main Result

In this paper we proved the following theorems:

Theorem 4.1. *Let for $0 \leq \beta < \alpha \leq 1$ and $s \in H_\alpha$, then*

$$\left\|t_n\left(s;t_1\right) - s\right\|_{\beta} = O\left[\left(n+1\right)^{-\alpha+\beta} \log\left(n+1\right)^{\frac{\beta}{\alpha}}\right] \tag{22}$$

5. Lemma

For the proof of our theorems following lemmas are required.

Lemma 5.1 ([7]). *Let $\emptyset_{t_1}(t)$ be defined in (16) then for $s \in H_\alpha$ and $0 < \alpha \leq 1$ we have*

$$\left|\emptyset_{t_1}\left(t\right) - \emptyset_{t_2}\left(t\right)\right| = 4K\left(\left|t_1 - t_2\right|^{\alpha}\right) \tag{23}$$

$$\left|\emptyset_{t_1}\left(t\right) - \emptyset_{t_2}\left(t\right)\right| = 4K\left(\left|t\right|^{\alpha}\right) \tag{24}$$

Lemma 5.2 ([2]). *For $0 \leq t \leq \frac{\pi}{n+1}$, $\sin nt \leq n \sin t$, we have*

$$\left|K_n\left(t\right)\right| = O(n+1) \tag{25}$$

Lemma 5.3 ([2]). *For $\frac{\pi}{n+1} \leq t \leq \pi, \sin\frac{t}{2} \geq \frac{t}{\pi}$ and $\sin nt \leq 1$, we have*

$$\left|K_n\left(t\right)\right| = O\left(\frac{1}{t}\right) \tag{26}$$

6. Proof of the Theorem

Proof of the Theorem 4.1: BY Zygmund [1], we have

$$s_n(t_1) - s = \frac{1}{2\pi}\int_0^\pi \emptyset_{t_1}(t)\frac{\sin\left(n+\frac{1}{2}\right)t}{sin\frac{t}{2}}dt \tag{27}$$

The $(E,1)$ transform is given by

$$E_n^1(t_1) - s = \frac{1}{\pi 2^{n+1}}\int_0^\pi \emptyset_{t_1}(t)\left\{\sum_{k=0}^{n}\binom{n}{k}\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\right\}dt \tag{28}$$

The $(C,2)(E,1)$ transform is given by

$$|t_n(s;t_1) - s| = \frac{1}{p\,(\mathrm{n}+1)\,(\mathrm{n}+2)}\sum_{k=0}^{n}\left\{\frac{(n-k+1)}{2^k}\int_0^p \frac{\emptyset_{t_1}(t)}{sin\frac{t}{2}}\left[\sum_{v=0}^{k}\binom{k}{v}\sin\left(v+\frac{1}{2}\right)t\right]\mathrm{dt}\right\} = \int_0^\pi \emptyset_{t_1}(t)\,K_n(t)\,dt \tag{29}$$

Now

$$\begin{aligned} E_n(t) &= \int_0^\pi \emptyset_{t_1}(t)\,K_n(t)\,dt \\ E_n(t_1,t_2) = |E_n(t_1) - E_n(t_2)| &= \left(\int_0^{\frac{\pi}{n+1}} + \int_{\frac{\pi}{n+1}}^{\pi}\right)\left|\emptyset_{t_1}(t) - \emptyset_{t_2}(t)\right|\,\left|K_n(t)\right|\,dt \\ &= I_1 + I_2 \end{aligned} \tag{30}$$

Now

$$I_1 = \int_0^{\frac{\pi}{n+1}}\left|\emptyset_{t_1}(t) - \emptyset_{t_2}(t)\right|\,\left|K_n(t)\right|\,dt$$

Using Lemma 5.2 and Equation (24)

$$\begin{aligned} I_1 &= O(n+1)\int_0^{\frac{\pi}{n+1}} t^\alpha dt \\ &= O(n+1)\left(\frac{1}{(n+1)^{\alpha+1}}\right) \\ &= O\left((n+1)^{-\alpha}\right) \end{aligned} \tag{31}$$

Now

$$I_2 = \int_{\frac{\pi}{n+1}}^{\pi}\left|\emptyset_{t_1}(t) - \emptyset_{t_2}(t)\right|\,\left|K_n(t)\right|\,dt$$

Using Lemma 5.3 and Equation (24)

$$\begin{aligned} I_2 &= O\int_{\frac{\pi}{n+1}}^{\pi} t^\alpha\left(\frac{1}{t}\right)dt \\ I_2 &= O\left((n+1)^{-\alpha}\right) \end{aligned} \tag{32}$$

Again

$$\begin{aligned} I_1 &= \int_0^{\frac{\pi}{n+1}}\left|\emptyset_{t_1}(t) - \emptyset_{t_2}(t)\right|\,\left|K_n(t)\right|\,dt \\ &= O\left(|t_1 - t_2|^\alpha\right)\int_0^{\frac{\pi}{n+1}}\left|K_n(t)\right|\,dt \\ &= O\left(|t_1 - t_2|^\alpha\ (n+1)\right) \end{aligned} \tag{33}$$

Now

$$\begin{aligned} I_2 &= \int_{\frac{\pi}{n+1}}^{\pi} \left|\emptyset_{t_1}(t) - \emptyset_{t_2}(t)\right| \left|K_n(t)\right| \, dt \\ &= O\left(|t_1 - t_2|^{\alpha}\right) \int_{\frac{\pi}{n+1}}^{\pi} \left|K_n(t)\right| dt \\ &= O\left(|t_1 - t_2|^{\alpha}\right) \int_{\frac{\pi}{n+1}}^{\pi} \frac{1}{t} dt \\ &= O\left(|t_1 - t_2|^{\alpha} \log(n+1)\right) \end{aligned} \tag{34}$$

Now

$$I_r = {I_r}^{1-\frac{\beta}{\alpha}} {I_r}^{\frac{\beta}{\alpha}}, \; r = 1, 2 \tag{35}$$

From (31) and (33),

$$\begin{aligned} I_1 &= O\left[\left\{(n+1)^{-\alpha}\right\}^{1-\frac{\beta}{\alpha}} \left(|t_1 - t_2|^{\alpha} \; (n+1)\right)^{\frac{\beta}{\alpha}}\right] \\ &= O\left[(n+1)^{-\alpha+\beta} |t_1 - t_2|^{\beta} \; (n+1)^{\frac{\beta}{\alpha}}\right] \\ &= O\left[(n+1)^{-\alpha+\beta+\frac{\beta}{\alpha}} |t_1 - t_2|^{\beta}\right] \end{aligned} \tag{36}$$

From (32) and (34),

$$\begin{aligned} I_2 &= O\left[\left\{(n+1)^{-\alpha}\right\}^{1-\frac{\beta}{\alpha}} \left(|t_1 - t_2|^{\alpha} \log(n+1)\right)^{\frac{\beta}{\alpha}}\right] \\ &= O\left[(n+1)^{-\alpha+\beta} |t_1 - t_2|^{\beta} \; \log(n+1)^{\frac{\beta}{\alpha}}\right] \end{aligned} \tag{37}$$

From (36) and (37), we have

$$\begin{aligned} |s(t_1) - s(t_2)| &= O\left[(n+1)^{-\alpha+\beta+\frac{\beta}{\alpha}} |t_1 - t_2|^{\beta}\right] + O\left[(n+1)^{-\alpha+\beta} |t_1 - t_2|^{\beta} \; \log(n+1)^{\frac{\beta}{\alpha}}\right] \\ &= O\left[(n+1)^{-\alpha+\beta} |t_1 - t_2|^{\beta} \; \log(n+1)^{\frac{\beta}{\alpha}}\right] \end{aligned}$$

From (14)

$$\begin{aligned} \Delta^{\beta} s(t_1, t_2) &= \frac{|s(t_1) - s(t_2)|}{|t_1 - t_2|^{\beta}}, \quad t_1 \neq t_2 \\ &= O\left[(n+1)^{-\alpha+\beta} \log(n+1)^{\frac{\beta}{\alpha}}\right] \end{aligned} \tag{38}$$

Now

$$\|s\|_c = O\left((n+1)^{-\alpha}\right) \tag{39}$$

Combining (38) and (39)

$$\|t_n(s; t_1) - s\|_{\beta} = O\left[(n+1)^{-\alpha+\beta} \log(n+1)^{\frac{\beta}{\alpha}}\right]$$

This completes the proof of Theorem 4.1.

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