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Remainder Term for an Alternating Series

Research Article

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Abstract: In this paper we give a rational approximation to Gregory series, by applying a correction function to the series. The introduction of correction function certainly improves the value of sum of the series and gives a better approximation to it. We also show that the correction function follows an infinite continued fraction.

Keywords: Correction function, error function, remainder term, Gregory series, rational approximation, infinite continued fraction, successive convergents.

1. Introduction

Commenting on the Lilavati rule for finding the value of circumference of a circle from its diameter, the commentator Sankara refers to several important enunciations from the works of earlier and contemporary mathematicians and gives a detailed exposition of various results contained in them. Sankara also refers to various infinite series for computing the circumference from the diameter. One such series attributed to illustrious mathematician Madhava of 14th century is $C = \frac{4d}{1} - \frac{4d}{3} + \frac{4d}{5} - \cdots \pm \frac{4d}{2n-1} \mp \frac{4d\left(\frac{2n}{2}\right)}{(2n)^2+1}$, where + or −, indicates that n is odd or even and C is the circumference of a circle of diameter d or more specifically, $C = \frac{4d}{1} - \frac{4d}{3} + \frac{4d}{5} - \cdots + (-1)^{n-1}\frac{4d}{2n-1} + (-1)^n \frac{4d(2n)/2}{(2n)^2+1}$. The remainder term $(-1)^n 4dG_n$ where $G_n = \frac{(2n)/2}{(2n)^2+1}$ has been augmented to the series for C by Madhava to get a better approximation. The introduction of the remainder term definitely improves the value of C and is very effective in giving a better approximation for it. Sankara has provided two other forms of the multiplier G_n denoted by G'_n and G''_n where $G'_n = \frac{1}{4n}$ and $G''_n = \frac{n^2+1}{[4\left(n^2+1\right)+1]n}$ of which G''_n is found to be more accurate correction function.

2. Rational Approximation of Alternating Series $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3}$

The alternating series $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3}$ satisfies the conditions of alternating series test and so it is convergent. If R_n denotes the remainder term after n terms of the series, then $R_n = (-1)^n G_n$ where G_n is the correction function after n terms of the series.

Theorem 2.1. *The correction function for the alternating series* $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3}$ *is* $G_n = \frac{1}{2n^3+3n^2+\frac{9}{2}n+\frac{7}{4}}$.

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Proof. If G_n denotes the correction function after n terms of the series, it follows that $G_n + G_{n+1} = \frac{1}{(n+1)^3}$. The error function is $E_n = G_n + G_{n+1} - \frac{1}{(n+1)^3}$. We may choose G_n in such a way that $|E_n|$ is a minimum function of n. For a fixed n and for $r_1,\ r_2,\ r_3 \in R$. Choose $G_n(r_1, r_2, r_3) = \frac{1}{\{2n^3+6n^2+6n+2\}-\{(r_1n^2+r_2n+r_3)\}}$. Then error function

$$E_n(r_1, r_2, r_3) = G_n(r_1, r_2, r_3) + G_{n+1}(r_1, r_2, r_3) - r_1 = 3r_2 = \frac{3}{2}r_3 = \frac{1}{4}$$

is a rational function of $r_1,\ r_2,\ r_3$. i.e., $E_n(r_1, r_2, r_3) = \frac{N_n(r_1,r_2,r_3)}{D_n(r_1,r_2,r_3)}$; $D_n(r_1, r_2, r_3) \approx 4n^9$ is a maximum for large n. $|N_n(r_1, r_2, r_3)|$ is a minimum function of n for $r_1 = 3,\ r_2 = \frac{3}{2},\ r_3 = \frac{1}{4}$. So $|E_n(r_1, r_2, r_3)|$ is a minimum function of n for $r_1 = 3,\ r_2 = \frac{3}{2},\ r_3 = \frac{1}{4}$. Thus for $r_1 = 3,\ r_2 = \frac{3}{2},\ r_3 = \frac{1}{4}$, we have both G_n and E_n are functions of a single variable n. Thus the correction function is $G_n = \frac{1}{2n^3+3n^2+\frac{9}{2}n+\frac{7}{4}}$. The corresponding error function is

$$|E_n| = \frac{\frac{873}{4}n^2 + \frac{301}{2}n + \frac{933}{16}}{\left(2n^3 + 3n^2 + \frac{9}{2}n + \frac{7}{4}\right)\left(2n^3 + 9n^2 + \frac{21}{2}n + \frac{45}{4}\right)(n+1)^3}$$

□

Remark 2.2. *Clearly* $G_n < \frac{1}{(n+1)^3}$, *absolute value of* $(n+1)^{th}$ *term.*

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