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Bivariate Cubic Spline Interpolation

Research Article

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Abstract: Using partial derivative bivariate cubic spline interpolation formula is derived and illustrated using an example.

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1. Introduction

The process of constructing a function say g(x) which fits the given set of data is called interpolation. If $g(x)$ is a polynomial then it is called polynomial interpolation [4]. In many cases it is seen that polynomial oscillates varyingly but the function varies smoothly [1]. To overcome this spline function is considered which is a function of polynomial bits joined together . The cubic spline procedure has sufficient flexibility due to the four constants involved in a general cubic polynomial which ensures the condition that the interpolant is continuously differentiable in the interval and has continuous second derivative [3]. Thus because of their smoothness conditions the most frequently used spline interpolation is the cubic spline interpolation. A two variable cubic spline interpolation of a function $z = f(x, y)$ is the fitting of a unique series of cubic splines for a given set of data points (x_i, y_j, z_{ij}). The points (x, y) at which $f(x, y)$ are known lie on a grid in the $x - y$ plane. In order to derive a two variable natural cubic spline the existence of continuity condition of the spline function and its partial derivatives at the edge of each grid are assumed.

2. Two Variable Natural Cubic Spline

Consider the division rectangle $I = [a, b] \times [c, d]$. Let $a = x_1 < x_2 < \cdots < x_n = b$ and $c = y_1 < y_2 < \cdots < y_m = d$ be the set of data points satisfied by $f(x_i, y_j) = z_{ij}$ for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$ [2]. A two variable cubic spline $S_{ij}(x, y)$ is a unique function coinciding with z_{ij} in each rectangular grid $I_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$ for all $i = 1, 2, \ldots, (n-1)$ and $j = 1, 2, \ldots, (m-1)$. Since $S_{ij}(x, y)$ is a cubic spline in two variables its all second order partial derivatives should be linear and continuous. Here we are considering the second order partial derivative with respect to y. Let

$$\frac{\partial^2 S_{ij}}{\partial y^2} = \frac{M_i (x_{i+1} - x)}{h_i} + \frac{M_{i+1} (x - x_i)}{h_i} + \frac{N_j (y_{j+1} - y)}{k_j} + \frac{N_{j+1} (y - y_j)}{k_j} \tag{1}$$

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Let

$$x_{i+1} - x_i = h_i, \quad i = 1, 2, \ldots, n-1$$

$$y_{j+1} - y_j = k_j, \quad j = 1, 2, \ldots, m-1$$

Integrating (1) with respect to y,

$$\frac{\partial S_{ij}}{\partial y} = \frac{M_i\left(x_{i+1}-x\right)\left(y-y_j\right)}{h_i} + \frac{M_{i+1}\left(x-x_i\right)\left(y_{j+1}-y\right)}{h_i} - \frac{N_j\left(y_{j+1}-y\right)^2}{2k_j} + \frac{N_{j+1}\left(y-y_j\right)^2}{2k_j} + A\left(x_{i+1}-x\right) + B \tag{2}$$

Integrating (2) with respect to y,

$$\begin{aligned} S_{ij} = {} & \frac{M_i\left(x_{i+1}-x\right)\left(y-y_j\right)^2}{2h_i} - \frac{M_{i+1}\left(x-x_i\right)\left(y_{j+1}-y\right)^2}{2h_i} + \frac{N_j(y_{j+1}-y)^3}{6k_j} + \frac{N_{j+1}(y-y_j)^3}{6k_j} \\ & + A\left(x_{i+1}-x\right)\left(y-y_j\right) + B\left(y_{j+1}-y\right) + C\left(x-x_i\right) + D \end{aligned} \tag{3}$$

Since the spline interpolates at the knots:

(i) $S_{ij}\left(x_{i+1}, y_j\right) = z_{i+1,j}$

(ii) $S_{ij}\left(x_i, y_{j+1}\right) = z_{i,j+1}$

(iii) $S_{ij}\left(x_i, y_j\right) = z_{i,j}$

(iv) $S_{ij}\left(x_{i+1}, y_{j+1}\right) = z_{i+1,j+1}$

Applying the conditions (i), (ii), (iii), (iv) to (3)

$$z_{i+1,j} = -\frac{M_{i+1}k_j^2}{2} + \frac{N_j k_j^2}{6} + Bk_j + Ch_i + D \tag{4}$$

$$z_{i,j+1} = \frac{M_i k_j^2}{2} + \frac{N_{j+1}k_j^2}{6} + Ah_i k_j + D \tag{5}$$

$$z_{i,j} = \frac{N_j k_j^2}{6} + Bk_j + D \tag{6}$$

$$z_{i+1,j+1} = \frac{N_{j+1}{k_j}^2}{6} + Ch_i + D \tag{7}$$

Solving (4), (5), (6), (7) we get the following constants

$$\begin{aligned} A &= \frac{\left(z_{i,j+1}-z_{ij}\right)}{h_i k_j} - \frac{\left(z_{i+1,j+1}-z_{i+1,j}\right)}{h_i k_j} + \frac{\left(M_{i+1}-M_i\right)k_j}{2h_i} \\ B &= \frac{M_{i+1}k_j}{2} + \frac{\left(N_{j+1}-N_j\right)k_j}{6} - \frac{\left(z_{i+1,j+1}-z_{i+1,j}\right)}{k_j} \\ C &= \frac{\left(z_{i+1,j}-z_{ij}\right)}{h_i} + \frac{M_{i+1}k_j^2}{2h_i} \\ D &= \left(z_{i+1,j+1}-z_{i+1,j}\right) + z_{ij} - \left(\frac{N_{j+1}}{6} + \frac{M_{i+1}}{2}\right)k_j^2 \end{aligned}$$

So from (3) the two variable cubic spline is

$$\begin{aligned} S_{ij} = {} & \frac{M_i\left(x_{i+1}-x\right)\left(y-y_j\right)^2}{2h_i} - \frac{M_{i+1}\left(x-x_i\right)\left(y_{j+1}-y\right)^2}{2h_i} + \frac{N_j(y_{j+1}-y)^3}{6k_j} + \frac{N_{j+1}(y-y_j)^3}{6k_j} \\ & + \left\{\frac{\left(z_{i,j+1}-z_{ij}\right)}{h_i k_j} - \frac{\left(z_{i+1,j+1}-z_{i+1,j}\right)}{h_i k_j} + \frac{\left(M_{i+1}-M_i\right)k_j}{2h_i}\right\}\left(x_{i+1}-x\right)\left(y-y_j\right) \\ & + \left\{\frac{M_{i+1}k_j}{2} + \frac{\left(N_{j+1}-N_j\right)k_j}{6} - \frac{\left(z_{i+1,j+1}-z_{i+1,j}\right)}{k_j}\right\}\left(y_{j+1}-y\right) \\ & + \left\{\frac{\left(z_{i+1,j}-z_{ij}\right)}{h_i} + \frac{M_{i+1}{k_j}^2}{2h_i}\right\}\left(x-x_i\right) + \left\{\left(z_{i+1,j+1}-z_{i+1,j}\right) + z_{ij} - \left(\frac{N_{j+1}}{6} + \frac{M_{i+1}}{2}\right){k_j}^2\right\}, \\ & \forall (x,y) \in \left[x_i, x_{i+1}\right] \times \left[y_j,\ y_{j+1}\right],\ \forall\ i = 1, 2, \ldots, (n-1),\ \forall\ j = 1, 2, \ldots, (m-1) \end{aligned} \tag{8}$$

In two variable spline function there exist a unique tangent plane at the two surfaces in every node. So corresponding to the node (x_i, y_{j+1}) we have $\frac{\partial S_{ij}}{\partial y}(x_i, y_{j+1}) = \frac{\partial S_{i,j+1}}{\partial y}(x_i, y_{j+1})$

$$\begin{aligned}
\frac{\partial S_{ij}}{\partial y}(x,y) &= \frac{M_i (x_{i+1}-x)(y-y_j)}{h_i} + \frac{M_{i+1}(x-x_i)(y_{j+1}-y)}{h_i} - \frac{N_j (y_{j+1}-y)^2}{2k_j} + \frac{N_{j+1}(y-y_j)^2}{2k_j} \\
&+ \left\{ \frac{(z_{i,j+1}-z_{ij})}{h_i k_j} - \frac{(z_{i+1,j+1}-z_{i+1,j})}{h_i k_j} + \frac{(M_{i+1}-M_i)k_j}{2h_i} \right\}(x_{i+1}-x) \\
&- \left\{ \frac{M_{i+1}k_j}{2} + \frac{(N_{j+1}-N_j)k_j}{6} - \frac{(z_{i+1,j+1}-z_{i+1,j})}{k_j} \right\} \\
\frac{\partial S_{ij}}{\partial y}(x_i, y_{j+1}) &= M_i k_j + \frac{N_{j+1}k_j}{2} + \frac{z_{i,j+1}-z_{ij}}{k_j} + \frac{M_i k_j}{2} - \frac{N_{j+1}k_j}{6} + \frac{N_j k_j}{6} \qquad (9) \\
\frac{\partial S_{i,j+1}}{\partial y}(x_i, y_{j+1}) &= -\frac{N_{j+1}k_{j+1}}{2} + \left\{ \frac{(z_{i,j+2}-z_{i,j+1})}{k_{j+1}} - \frac{(z_{i+1,j+2}-z_{i+1,j+1})}{k_{j+1}} + \frac{(M_{i+1}-M_i)k_{j+1}}{2} \right\} \\
&- \left\{ \frac{M_{i+1}k_{j+1}}{2} + \frac{(N_{j+2}-N_{j+1})k_{j+1}}{6} - \frac{(z_{i+1,j+2}-z_{i+1,j+1})}{k_{j+1}} \right\} \qquad (10)
\end{aligned}$$

Equating (9) and (10)

$$\frac{(z_{i,j+2}-z_{i,j+1})}{k_{j+1}} - \frac{(z_{i,j+1}-z_{ij})}{k_j} = \frac{N_{j+1}}{3}(k_j+k_{j+1}) + \frac{M_i}{2}(k_j+k_{j+1}) + \frac{N_j k_j}{6} + \frac{N_{j+2}k_{j+1}}{6} \qquad (11)$$

Assume $\frac{z_{i,j+1}-z_{ij}}{k_j} = \gamma_{i,j}$, where $i = 1, 2, \ldots, (n-1)$ and $j = 1, 2, \ldots, (m-1)$

$$\frac{z_{i,j+2}-z_{i,j+1}}{k_{j+1}} = \gamma_{i,j+1} \qquad (12)$$

Assume

$$\left.\begin{aligned}
\alpha_j &= \tfrac{k_{j+1}}{k_j+k_{j+1}} \\
\beta_j &= 1-\alpha_j = \tfrac{k_j}{k_j+k_{j+1}} \\
e_{ij} &= \tfrac{6\left(\gamma_{i,j+1}-\gamma_{i,j}\right)}{k_j+k_{j+1}}
\end{aligned}\right\} \qquad (13)$$

where $i = 1, 2, \ldots, (n-2)$ and $j = 1, 2, \ldots, (m-2)$. Therefore equation (11) will become

$$6\left[\gamma_{i,j+1}-\gamma_{i,j}\right] = 2N_{j+1}(k_j+k_{j+1}) + 3M_i(k_j+k_{j+1}) + N_j k_j + N_{j+2}k_{j+1} \qquad (14)$$

Dividing equation (14) by $k_j + k_{j+1}$ we get

$$2N_{j+1} + 3M_i + \beta_j N_j + \alpha_j N_{j+2} = e_{ij}, \qquad (15)$$

where $i = 1, 2, \ldots, (n-2)$ and $j = 1, 2, \ldots, (m-2)$. So the bivariate natural cubic spline is

$$\begin{aligned}
S_{ij} &= \frac{M_i (x_{i+1}-x)(y-y_j)^2}{2h_i} - \frac{M_{i+1}(x-x_i)(y_{j+1}-y)^2}{2h_i} + \frac{N_j (y_{j+1}-y)^3}{6k_j} + \frac{N_{j+1}(y-y_j)^3}{6k_j} \\
&+ \left\{ \frac{(z_{i,j+1}-z_{ij})}{h_i k_j} - \frac{(z_{i+1,j+1}-z_{i+1,j})}{h_i k_j} + \frac{(M_{i+1}-M_i)k_j}{2h_i} \right\}(x_{i+1}-x)(y-y_j) \\
&+ \left\{ \frac{M_{i+1}k_j}{2} + \frac{(N_{j+1}-N_j)k_j}{6} - \frac{(z_{i+1,j+1}-z_{i+1,j})}{k_j} \right\}(y_{j+1}-y) \\
&+ \left\{ \frac{(z_{i+1,j}-z_{ij})}{h_i} + \frac{M_{i+1}{k_j}^2}{2h_i} \right\}(x-x_i) + \left\{ (z_{i+1,j+1}-z_{i+1,j}) + z_{ij} - \left(\frac{N_{j+1}}{6} + \frac{M_{i+1}}{2} \right){k_j}^2 \right\}, \\
&\forall\ (x,y) \in [x_i, x_{i+1}] \times [y_j, y_{j+1}],\ \forall\ i = 1, 2, \ldots, (n-1),\ \forall\ j = 1, 2, \ldots, (m-1)
\end{aligned}$$

where $2N_{j+1} + 3M_i + \beta_j N_j + \alpha_j N_{j+2} = e_{ij}$, $i = 1, 2, \ldots, (n-2)$ and $j = 1, 2, \ldots, (m-2)$. For natural spline $M_1 = M_n = N_1 = N_m = 0$.

3. Illustration

Consider the two variable function $f(x,y)=e^{x^2y}$. The following table gives the function values for x taking values 0, 0.1, 0.2 and y taking values 0, 0.1, 0.2

X \ Y	0	0.1	0.2
0	1	1	1
0.1	1	1.001	1.002
0.2	1	1.004	1.008

$h_1=h_2=0.1$, $k_1=k_2=0.1$. Using (12) and (13),

$$\gamma_{11}=0 \qquad \alpha_1=0.5$$
$$\gamma_{12}=0 \qquad \beta_1=0.5$$
$$\gamma_{21}=0.01 \qquad e_{11}=0$$
$$\gamma_{22}=0.01 \qquad e_{21}=0$$

Using (15)

$$\left.\begin{aligned} 2N_2+3M_1+\beta_1N_1+\alpha_1N_3&=e_{11}\\ 2N_2+3M_2+\beta_1N_1+\alpha_1N_3&=e_{21}\end{aligned}\right\} \tag{16}$$

For a natural cubic spline $M_1=M_3=N_1=N_3=0$. Solving (16), $M_2=0$ and $N_2=0$. Using (8) the two variable cubic splines are

$$S_{ij}(x,y)=\begin{cases} -y\,(0.1-x)^2-0.2x\,(0.1-y)+0.01y+1, (x,y)\in I_{11}\\ (0.2-y)\,x^2-0.2x\,(0.2-y)-0.02\,(0.1-x)+1.002, (x,y)\in I_{12}\\ -y\,(0.2-x)^2-0.2\,(0.1-y)\,(x-0.1)-0.02\,(0.2-x)+0.02y+1.002, (x,y)\in I_{21}\\ (0.2-y)\,(x-0.1)^2-0.25\,(0.2-y)\,(x-0.1)-0.06\,(0.2-x)+0.01\,(y-0.1)+1.007, (x,y)\in I_{22}\end{cases}$$

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