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Generalized U - H Stability of New n - type of Additive Quartic Functional Equation in Non - Archimedean Orthogonally Space

Research Article

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Abstract: Using direct method, we prove the stability of the orthogonally additive-quartic functional equation of the form $f\left(nx+n^2y+n^3z\right) + f\left(nx-n^2y+n^3z\right) + f\left(nx+n^2y-n^3z\right) + f\left(-nx+n^2y+n^3z\right) = 2\left[f\left(nx+n^2y\right)+f\left(nx-n^2y\right)\right] + 2\left[f\left(n^2y+n^3z\right)+f\left(n^2y-n^3z\right)\right] + 2\left[f\left(nx+n^3z\right)+f\left(nx-n^3z\right)\right] - 3n\left[f\left(x\right)-f\left(-x\right)\right] - 3n^2\left[f\left(y\right)-f\left(-y\right)\right] - 3n^3\left[f\left(z\right)-f\left(-z\right)\right] - 2n^4\left[f\left(x\right)+f\left(-x\right)\right] - 2n^8\left[f\left(y\right)+f\left(-y\right)\right] - 2n^{12}\left[f\left(z\right)+f\left(-z\right)\right]$ for all x, y, z in X with $x \perp y$, $y \perp z$ and $z \perp x$ in Banach non-Archimedean Orthogonally spaces. Here $\perp$ is the orthogonally in the sense of Ratz.

MSC: Additive functional equation, Quartic functional equation, Banach Non-Archimedean Orthogonally spaces.

Keywords: 39B52, 32B72, 32B82.

1. Introduction and Preliminaries

In 1987, Hensel [16] introduced a norm space which does not have the Archimedean property. It turned out that non - Archimedean spaces have many nice applications [8, 19, 20, 26]. A valuation is a function $|.|$ from a field K into $[0, \infty)$ such that 0 is the unique element having the 0 valutions, $|rs| = |r|\,.\,|s|$ and the triangle inequality holds $|r+s| \leq |r| + |s|$, for all $r,\ s \in K$. A field K is called a valuation if K carries a valuation. Throughout this paper, we assume that the base field is valued field, hence call it simply field. The usual absolute values of R and C are the examples of valuations.

Let us consider a valuation which satisfies a stronger condition then the triangle inequality. If the triangle inequality is replaced by $|r+s| = max\left\{|r|\,.\,|s|\right\}$, for all $r, s \in K$. Then the function $|.|$ is called non - Archimedean valuation, and the field is called a non - Archimedean field. Clearly $|1| = |-1|$ and $|n| \leq 1$ for all $n \in N$. A trivial examples of a non - Archimedean valuation is a function $|.|$ by taking everything except for 0 into 1 and $|0| = 0$.

Definition 1.1. *Let X be a vector space over a scalar field K with non - Archimedean valuation $|.|$. A function $\|.\| : X \to [0, \infty)$ is to be a non - Archimedean norm if satisfies the following conditions:*

(i). $\|x\| = 0$ *if and only if* $x = 0$.

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(ii). $\|rx\| = |r| \, \|x\| \, (r \in K, x \in X)$.

(iii). The strong triangle inequality, $\|x+y\| \leq max \{\|x\| . \|y\| : (x, y \in X)\}$

Then $(X, \|.\|)$ *is called a non-Archimedean space.*

Definition 1.2. *Let* $\{x_n\}$ *be a sequence in a non - Archimedean normed space* X*. Then the sequence* $\{x_n\}$ *is called Cauchy if for a given* $\epsilon > 0$*, there exists a positive integer* N *such that* $\|x_n - x_m\| \leq \epsilon$ *for all* $n, m \geq N$.

Definition 1.3. *Let* x_n *be a sequence in a non - Archimedean normed space* X*. Then the sequence* x_n *is called Convergent, if for given* $\epsilon > 0$*, there exists a positive integer* N *such that* $\|x_n - x\| \leq \epsilon$ *for all* $n \geq N$*. Then we call* $x \in X$ *be a limit of the sequence* $\{x_n\}$*, and it is denoted by* $\lim_{n\to\infty} x_n = x$.

Definition 1.4. *If every Cauchy sequence in* X *converges, then the non - Archimedean normed space* x *is called a non - Archimedean Banach space.*

Assume that X is Real Inner product space and $f : X \to R$ is a solution of the orthogonal Cauchy functional equation

$$f(x+y) = f(x) + f(y), \langle x, y \rangle = 0.$$

By a Pythagorean Theorem, we have, $f(x) = \|x\|^2$ is a solution of Cauchy functional equation. Of course, this functional does not satisfy the additive equation everywhere. Thus orthogonal Cauchy functional equation is not equivalent to the classic Cauchy functional equation on the whole inner product space. G. Pinsker characterized orthogonally additive functional on an inner product space. When the orthogonality is the ordinary one in such spaces. K. Sundaresan [46] generalized this result to arbitary Banach spaces equipped with the Bikhoff - James orthogonally. The orthogonal Cauchy functional equation,

$$f(x+y) = f(x) + f(y), \quad x \perp y, \tag{1}$$

in which $\perp$ is on abstract orthogonality relation, was first investigated by S. Gudder and D. Strawther [15]. They defined $\perp$ by a system consisting of five axioms and describe the general semi-continuous real-valued solution of conditional Cauchy functional equation. In 1985 J. Ratz [42] introduced a new definition of orthogonality by using more restrictive axioms than of S. Gudder and D. Strawther [15]. Moreover, he investigated the structure of orthogonally additive mappings. J. Ratz and Gy. Szabo [43] investigated the problem in a rather more general framework. Let us recall the orthogonality in the sense of J. Ratz. et. al., [42]. Suppose that X is a Real vector space with dim $X \geq 2$ and $\perp$ is a binary relation on X with the following properties:

(O_1) Totally of $\perp$ for zero: $x \perp 0$, $0 \perp x$ for all $x \in X$.

(O_2) Independence: if $x, y \in X - 0$, $x \perp y$, then x, y are linearly independent.

(O_3) Homogeneity: if $x, y, z \in X$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in R$.

(O_4) The Thalesian property: if p is a 3-dimensional subspace of X. That is, $x \in P$ and $\lambda \in P$,

which is the set of non - negative Real numbers, then there exists $y_0 \in P$ such that $x \perp y_0$ and $x + y_0 \perp \lambda x - y_0$. The pair $(x, \perp)$ is called the orthogonality space. By an orthogonality normed space we mean an orthogonality space having a normed structure. Some interesting examples are, (1). The trivial orthogonality on a vector space defined by (O_1) and for non-zero elements $x, y, z \in X$, $x \perp y$, $y \perp z$ and $z \perp x$ if and only if x, y, z are linearly independent.

(2). The ordinary orthogonality on an inner product space $(X, \langle ., . \rangle)$ given by $x, y, z \in X$, $x \perp y$, $y \perp z$ and $z \perp x$ if and only if $\langle x, y, z \rangle = 0$.

(3). The Birkhoff - James orthogonality on a normed space $(X, \|.\|)$ defined by $x, y, z \in X$, $x \perp y$, $y \perp z$ and $z \perp x$ if and only if $\|x + \lambda y\| \geq \|x\|$ for all λP.

(4). The relation $\perp$ is called a symmetric if $x \perp y$ implies $y \perp x$ and $y \perp z$ implies $z \perp y$, $z \perp x$ implies $x \perp z$ for all $x, y, z \in X$.

Clearly examples (1) and (2) are symmetric but example (3) is not. It is remarkable to note, however, that a real normed space of dimension greater than 3 is an inner product space if and if only if the Birkhoff-James orthogonality is symmetric. There are several orthogonality notations on a real normed space such as Birkhoff - james, Singer, Carlesson, Unitary-Boussouis, Roberts, Pythagorean, Isosceles and Diminnie [1, 4, 9, 11, 17]. The stability problem of functional equations originated from the following question of Ulam [47]; under what condition does there exist an additive mapping near an approximately additive mapping. In 1941, Hyers [17] gave a partial affirmative answer to the question of Ulam in the content of Banach spaces. In 1978 Th. M. Rassias [38] extended the Theorem of Hyers by considering the unbounded Cauchy difference. Considered the following quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{2}$$

is related to a symmetric bi-additive function [14]. Jun and Kim [22] introduced the following cubic functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x) \tag{3}$$

and they established the general solution and generalized Hyers-Ulam-Rassias stability for the functional equation. The function $f(x) = cx^3$ satisfies the functional equation (3), then the functional equation (3) is called a cubic functional equation. Every solution of the cubic functional is said to be cubic function. A functional $f : X \to Y$ is said to be quartic functional equation, then

$$f(3x+y) + f(x+3y) = 64f(x) + 64f(y) + 24f(x+y) - 6f(x-y) \tag{4}$$

it is easy to show that the function $f(x) = x^4$ satisfies the functional equation (4), which is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic mapping. Subsequently P.K. Sahoo and J.K. Chung [44], modified the J.M. Rassias equation as

$$f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 24f(x) - 6f(y). \tag{5}$$

In this paper, the authors introduce a new n type of additive - quadratic functional equation

$$\begin{aligned} f\left(nx + n^2y + n^3z\right) + f\left(nx - n^2y + n^3z\right) + f\left(nx + n^2y - n^3z\right) + f\left(-nx + n^2y + n^3z\right) &= 2\left[f\left(nx + n^2y\right) + f\left(nx - n^2y\right)\right] \\ &+ 2\left[f\left(n^2y + n^3z\right) + f\left(n^2y - n^3z\right)\right] + 2\left[f\left(nx + n^3z\right) + f\left(nx - n^3z\right)\right] \\ &- 3n\left[f(x) - f(-x)\right] - 3n^2\left[f(y) - f(-y)\right] - 3n^3\left[f(z) - f(-z)\right] \\ &- 2n^4\left[f(x) + f(-x)\right] - 2n^8\left[f(y) + f(-y)\right] - 2n^{12}\left[f(z) + f(-z)\right] \end{aligned} \tag{6}$$

where n is a positive integer with $n \geq 3$.

This paper is organized as follows. In this section 2, we prove the General solution and Generalized Hyers - Ulam stability of the orthogonally additive - quartic functional equation (6). In section 3, we prove the Hyers - Ulam stability of the orthogonally additive - quartic functional equation (6) in non - Archimedean orthogonality spaces for an odd mapping. In the section 4, we prove the Hyers - Ulam stability of the orthogonally additive - quartic functional equation (6) in non - Archimedean orthogonality spaces for an even mapping and mixed case. Throughout this paper, we assume that $(X, \perp)$ is a non - Archimedean orthogonally space and that $(Y, \|.\|)$ is a Real non - Archimedean Banach spaces. Assume that $|z| \neq 1$.

2. General Solution of the Additive Quartic Functional Equation (6)

In this section, we deal with the solution of the additive quartic functional equation. Throughout this section, let us consider X and Y be a Real vector spaces.

Theorem 2.1. *If the mapping $f : X \to Y$ satisfies the functional equation (6) for all $x, y, z \in X$, then $f : X \to Y$ satisfies the functional equation (1) for all $x, y \in X$.*

Theorem 2.2. *If the mapping $f : X \to Y$ satisfies the functional equation (6) for all $x, y, z \in X$, then $f : X \to Y$ satisfies the functional equation (5) for all $x, y \in X$.*

3. Stability of the Orthogonally Additive - Quartic Functional Equation (6): Odd Mapping Case

In this section, we deal with the stability problem for the orthogonality additive quartic functional equation

$$\begin{aligned} Df(x,y,z) &= f\left(nx+n^2y+n^3z\right)+f\left(nx-n^2y+n^3z\right)+f\left(nx+n^2y-n^3z\right) \\ &+f\left(-nx+n^2y+n^3z\right)-2\left[f\left(nx+n^2y\right)+f\left(nx-n^2y\right)\right]-2\left[f\left(n^2y+n^3z\right)\right. \\ &\left.+\,f\left(n^2y-n^3z\right)\right]-2\left[f\left(nx+n^3z\right)+f\left(nx-n^3z\right)\right]+3n\left[f(x)-f(-x)\right] \\ &+3n^2\left[f(y)-f(-y)\right]+3n^3\left[f(z)-f(-z)\right]+2n^4\left[f(x)+f(-x)\right] \\ &+2n^8\left[f(y)+f(-y)\right]+2n^{12}\left[f(z)+f(-z)\right]=0 \end{aligned}$$

for all $x, y, z \in X$ with $x \perp y$, $y \perp z$ and $z \perp x$ in Banach non - Archimedean orthogonally spaces: An odd mapping case.

Definition 3.1. *An odd mapping $f : X \to Y$ is called an orthogonally additive mapping if*

$$\begin{aligned} f\left(nx+n^2y+n^3z\right)&+f\left(nx-n^2y+n^3z\right)+f\left(nx+n^2y-n^3z\right)+f\left(-nx+n^2y+n^3z\right) \\ &=2\left[f\left(nx+n^2y\right)+f\left(nx-n^2y\right)\right]+2\left[f\left(n^2y+n^3z\right)+f\left(n^2y-n^3z\right)\right]+2\left[f\left(nx+n^3z\right)\right. \\ &\left.+\,f\left(nx-n^3z\right)\right]-3n\left[f(x)-f(-x)\right]-3n^2\left[f(y)-f(-y)\right]-3n^3\left[f(z)-f(-z)\right] \end{aligned}$$

for all $x, y, z \in X$ with $x \perp y$, $y \perp z$ and $z \perp x$.

Theorem 3.2. *Let $\phi : X^3 \to [0, \infty)$ be a function such that,*

$$\phi(x,y,z) := \sum_{k=0}^{\infty} \frac{1}{|n|^k}\,\phi\left(n^k x, n^k y, n^k z\right) < +\infty$$

for all $x, y, z \in X$ with $x \perp y, y \perp z$ and $z \perp x$. Let $f : X \to Y$ be an odd mapping satisfying

$$\|Df(x, y, z)\| \leq \phi(x, y, z) \tag{7}$$

for all $x, y, z \in X$ with $x \perp y, y \perp z$ and $z \perp x$. Then there exists a unique orthogonally additive mapping $A : X \to X$ such that

$$\|f(x) - A(x)\| \leq \frac{1}{6|n|^{k+1}} \phi\left(n^k x, 0, 0\right) \tag{8}$$

for all $x \in X$.

Proof. Replacing (x, y, z) by $(x, 0, 0)$ in (7), we get

$$\|6nf(x) - 6f(nx)\| \leq \phi(x, 0, 0) \tag{9}$$

for all $x \in X$. Since $x \perp 0$, dividing by $6n$ in (9), we get

$$\left\|f(x) - \frac{nx}{n}\right\| \leq \frac{1}{6|n|} \phi(x, 0, 0) \tag{10}$$

for all $x \in X$. Replacing x by $n^k x$ (10), that

$$\left\|f\left(n^k x\right) - \frac{1}{n} f\left(n.n^k x\right)\right\| \leq \frac{1}{6|n|} \phi\left(n^k x, 0, 0\right) \tag{11}$$

for all $n \geq 0$ and all $x \in X$. Since $n^k x \perp 0$, so,

$$\left\|\frac{1}{n^k} f\left(n^k x\right) - \frac{1}{n^{k+1}} f\left(n^{k+1} x\right)\right\| \leq \frac{1}{6|n|^{k+1}} \phi\left(n^k x, 0, 0\right) \tag{12}$$

for all $k \geq 0$ and all $x \in X$. Now we define a mapping g such that,

$$g(k, x) = \frac{1}{n^{k+1}} f\left(n^{k+1} x\right) - \frac{1}{n^k x} \tag{13}$$

for all $k \geq 0$ and all $x \in X$. Then

$$\|g(k, x)\| \leq \frac{1}{6|n|^{k+1}} \phi\left(n^k x, 0, 0\right) \tag{14}$$

for all $n \geq 0$ and all $x \in X$, so

$$\begin{aligned}
\left\|\frac{1}{n^m} f(n^m x) - \frac{1}{n^k} f\left(n^k x\right)\right\| &= \left\|\sum_{i=k}^{m-1} g(k, x)\right\| \\
&\leq max\left\{\|g(k, x)\|, ..., \|g((m-1), x)\|\right\} \\
&\leq \sum_{i=k}^{m-1} \|g(k, x)\| \\
&\leq \sum_{i=k}^{m-1} \frac{1}{6|n|^{k+1}} \phi\left(n^k x, 0, 0\right)
\end{aligned} \tag{15}$$

which tends to zero as $k \to \infty$, for all $m \geq k \geq 0$ and all $x \in X$. Thus the sequence $\left\{\frac{1}{n^k} f\left(n^k x\right)\right\}$ is Cauchy sequence. Since Y is a non - Archimedean Banach space. The sequence

$$\left\{\frac{1}{n^k} f\left(n^k x\right)\right\}$$

converges. So we can define a mapping $A : X \to Y$ such that,

$$\lim_{k\to\infty} \frac{1}{n^k} f\left(n^k x\right) = A(x) \tag{16}$$

for all $x \in X$. Replacing (x, y, z) by $\left(n^k x, n^k y, n^k z\right)$ in (6) respectively. Then we get,

$$\left\|Df\left(n^k x, n^k y, n^k z\right)\right\| \le \phi\left(n^k x, n^k y, n^k z\right)$$

for all $x, y, z \in X$ with $x \perp y$, $y \perp z$ and $z \perp x$. Since $n^k x \perp n^k y$, $n^k y \perp n^k z$ and $n^k z \perp n^k x$, then,

$$\left\|\frac{1}{n^k} Df\left(n^k x, n^k y, n^k z\right)\right\| \le \frac{1}{|n|^k}\phi\left(n^k x, n^k y, n^k z\right)$$

for all $x, y, z \in X$ with $x \perp y$, $y \perp z$ and $z \perp x$. So

$$\begin{aligned}\|DA(x, y, z)\| &= \lim_{k\to\infty}\left\|\frac{1}{n^k} Df\left(n^k x, n^k y, n^k z\right)\right\| \\ &\le \lim_{k\to\infty}\frac{1}{|n|^k}\phi\left(n^k x, n^k y, n^k z\right) = 0\end{aligned} \tag{17}$$

for all $x, y, z \in X$ with $x \perp y$, $y \perp z$ and $z \perp x$. Thus

$$\|DA(x, y, z)\| = 0 \tag{18}$$

for all $x, y, z \in X$ with $x \perp y$, $y \perp z$ and $z \perp x$. Since $f(x)$ is an odd mapping, $A(x)$ is an odd mapping. So the mapping $A : X \to Y$ is an orthogonally additive mapping. Letting $k = 0$ and $m \to \infty$ in (8), we get the inequality (7). To prove the uniqueness of A, let $L : X \to Y$ be the another orthogonally mapping satisfying (7)

$$\begin{aligned}\|A(x) - L(x)\| &\le \left\|\frac{1}{n^k} A\left(n^k x\right) - \frac{1}{n^k} L\left(n^k x\right)\right\| \\ &\le max\left\{\left\|\frac{1}{n^k} A\left(n^k x\right) - \frac{1}{n^k} f\left(n^k x\right)\right\|, \left\|\frac{1}{n^k} f\left(n^k x\right) - \frac{1}{n^k} L\left(n^k x\right)\right\|\right\} \\ &\le \frac{1}{6|n|^{k+1}}\phi\left(n^k x, 0, 0\right)\end{aligned} \tag{19}$$

which tends to zero as $k \to \infty$. So $A : X \to Y$ is unique. Therefore, there exists a unique orthogonally additive mapping $A : X \to Y$ satisfying (6). This completes the proof of the theorem. □

Corollary 3.3. *Let $p > 1$. Let $f : X \to Y$ be an odd mapping satisfying*

$$\|Df(x, y, z)\| = \theta\left(\|x\|^p + \|y\|^p + \|z\|^p\right) \tag{20}$$

for all $x, y, z \in X$ with $x \perp y$, $y \perp z$, and $z \perp x$. Then there exists a unique orthogonally additive mapping $A : X \to Y$ such that,

$$\|f(x) - A(x)\| = \frac{1}{|6|}\left[\frac{\theta}{|n| + |n|^p}\|x\|^p\right] \tag{21}$$

for all $x \in X$.

Proof. The rest of proof that of the Theorem 3.2, we arrive (21). □

Theorem 3.4. *Let $\phi : X^3 \to [0,\infty)$ be a function such that,*

$$\phi(x,y,z) := \sum_{k=1}^{\infty} |n|^k \phi\left(\frac{x}{n^k}, \frac{y}{n^k}, \frac{z}{n^k}\right) < +\infty \tag{22}$$

for all $x,y,z \in X$ with $x \perp y \perp z$ and $z \perp x$. Let $f : X \to Y$ be an odd mapping satisfying (4). That there exists a unique orthogonally additive mapping $A : X \to X$ such that

$$\|f(x) - A(x)\| \le \frac{1}{6} \sum_{k=0}^{n} |n|^{k+1} \phi\left(\frac{x}{n^k}, 0, 0\right) \tag{23}$$

for all $x \in X$.

Proof. it follows from (5), that

$$\left\|f(x) - nf\left(\frac{x}{n}\right)\right\| \le \frac{1}{|6|} \phi\left(\frac{x}{n}, 0, 0\right) \tag{24}$$

for all $x \in X$. Since $x \perp 0$. The rest of the proof is similar to the proof of Theorem 3.2. □

Corollary 3.5. *Let $p < 1$. Let $f : X \to Y$ be an odd mapping satisfying (20) for all $x,y,z \in X$ with $x \perp y$, $y \perp z$ and $z \perp x$. Then there exists a unique orthogonally additive mapping $A : X \to Y$ such that,*

$$\|f(x) - A(x)\| \le \frac{\theta \|x\|^p |n|^{p+1}}{|6| \left[|n|^p - |n|\right]} \tag{25}$$

for all $x \in X$.

4. Stability of the Orthogonally Additive - Quartic Functional Equation (6): Even Mapping Case and Mixed Case

In this section, we establish the Hyers - Ulam stability problem for the orthogonality additive - quadratic functional equation $Df(x,y,z) = 0$ given in the previous section: An even mapping case.

Definition 4.1. *An even mapping $f : X \to Y$ is called an orthogonally quartic mapping if*

$$\begin{aligned} f\left(nx + n^2y + n^3z\right) &+ f\left(nx - n^2y + n^3z\right) + f\left(nx + n^2y - n^3z\right) + f\left(-nx + n^2y + n^3z\right) \\ &= 2\left[f\left(nx + n^2y\right) + f\left(nx - n^2y\right)\right] + 2\left[f\left(n^2y + n^3z\right) + f\left(n^2y - n^3z\right)\right] + 2\left[f\left(nx + n^3z\right)\right. \\ &\left. + f\left(nx - n^3z\right)\right] - 2n^4\left[f(x) + f(-x)\right] - 2n^8\left[f(y) + f(-y)\right] - 2n^{12}\left[f(z) + f(-z)\right] \end{aligned}$$

for all x,y,z in with $x \perp y$, $y \perp z$ and $z \perp x$.

Theorem 4.2. *Let $\phi : X^3 \to [0,\infty)$ be a function such that,*

$$\phi(x,y,z) := \sum_{k=0}^{\infty} \frac{1}{-}\phi\left(n^k x, n^k y, n^k z\right) < +\infty \tag{26}$$

for all $x,y,z \in X$ with $x \perp y, y \perp z$ and $z \perp x$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (7). Then there exists a unique orthogonally quartic mapping $Q : X \to X$ such that

$$\|f(x) - Q(x)\| \le \sum_{k=1}^{n} \frac{\phi\left(n^k x, 0, 0\right)}{|4| \, |n|^{4(k)}} \tag{27}$$

for all $x \in X$.

Proof. Replacing (x, y, z) by $(x, 0, 0)$ in (7), we get,

$$\left\|4n^4 f(x) - 4f(nx)\right\| \leq \phi(x, 0, 0) \tag{28}$$

for all $x \in X$. Dividing by $4n^4$ in (27), we arrive

$$\left\|f(x) - \frac{f(nx)}{n^4}\right\| \leq \frac{1}{4n^4}\phi(x, 0, 0) \tag{29}$$

for all $x \in X$, since $x \perp 0$. The rest of the proof is similar to the proof of the Theorem 3.2. □

Corollary 4.3. *Let $p > 1$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (20). Then there exists a unique orthogonally quartic mapping $Q : X \to Y$ such that,*

$$\|f(x) - Q(x)\| = \frac{\theta}{|4|\left(|n|^4 - |n|^p\right)}\|x\|^p \tag{30}$$

for all $x \in X$.

Theorem 4.4. *Let $\phi : X^3 \to [0, \infty)$ be a function such that,*

$$\phi(x, y, z) := \sum_{k=0}^{\infty} \frac{|n|^{4(k+1)}}{4}\phi\left(\frac{x}{n^k}, \frac{y}{n^k}, \frac{z}{n^k}\right) < +\infty \tag{31}$$

for all $x, y, z \in X$ with $x \perp y$, $y \perp z$, and $z \perp x$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (6). Then there exists a unique orthogonally quartic mapping $Q : X \to X$ such that

$$\|f(x) - Q(x)\| \leq \frac{|n|^{4(k+1)}}{4}\phi\left(\frac{x}{n^k}, 0, 0\right) \tag{32}$$

for all $x \in X$.

Proof. It is follows from (28), that

$$\left\|f(x) - n^4 f\left(\frac{x}{n}\right)\right\| \leq \frac{1}{4}\phi\left(\frac{x}{n}, 0, 0\right) \tag{33}$$

for all $x \in X$. Since $x \perp 0$. The rest of the proof is similar to the proof of Theorem 3.2. □

Corollary 4.5. *Let $0 < p < 1$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (7). Then there exists a unique orthogonally quartic mapping $Q : X \to X$ such that*

$$\|f(x) - Q(x)\| = \frac{\phi}{4\left(|n|^p - |n|^4\right)}\|x\|^p \tag{34}$$

for all $x \in X$.

Theorem 4.6. *Let $\phi : X^3 \to [0, \infty)$ be a function such that, (7) and (25) for all $x, y, z \in X$. Let $f : X \to Y$ be an even mapping satisfying inequality*

$$\|Df(x, y, z)\| \leq \phi(x, y, z) \tag{35}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to X$ and unique quadratic mapping $Q : X \to X$ such that

$$\begin{aligned} \|f(x) - A(x) - Q(x)\| \leq \frac{1}{2}&\left[\frac{1}{|n|\,|6|}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\phi\left(n^k x, 0, 0\right)}{|n|^{kj}} + \frac{\phi\left(-n^k x, 0, 0\right)}{|n|^{kj}}\right)\right] \\ &+ \frac{1}{2}\left[\frac{1}{|4|}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\phi\left(n^k x, 0, 0\right)}{|n|^{4kj}} + \frac{\phi\left(-n^k x, 0, 0\right)}{|n|^{4kj}}\right)\right] \end{aligned} \tag{36}$$

for all $x \in X$. The mapping $A(x)$ and $Q(x)$ are defined by in (8) and (26) respectively, for all $x \in X$.

Proof. Let us assume that

$$f_0(x) = \frac{f_0(x) - f_0(-x)}{2}$$

for all $x \in X$. Then $f_0(x) = 0$ and $f_0(-x) = -f_0(x)$ for all $x \in X$. Hence

$$\|Df_0(x,y,z)\| = \frac{\phi(x,y,z)}{2} + \frac{\phi(-x,-y,-z)}{2} \tag{37}$$

for all $x, y, z \in X$. By a Theorem 3.2, we have

$$\|f_0(x) - A(x)\| \leq \frac{1}{|2|\,|n|} \sum_{i=\frac{1-j}{2}}^{\infty} \left(\frac{\phi(n^{kj}x,0,0)}{|n|^{kj}} + \frac{\phi(-n^{kj}x,0,0)}{|n|^{kj}} \right) \tag{38}$$

for all $x \in X$. Also let

$$f_e(x) = \frac{f_q(x) + f_q(-x)}{2}$$

for all $x \in X$. Therefore $f_e(0) = 0$ and $f_e(-x) = f_e(x)$ for all $x \in X$. Hence

$$\|Df_e(x,y,z)\| = \frac{\phi(x,y,z)}{2} + \frac{\phi(-x,-y,-z)}{2} \tag{39}$$

for all $x, y, z \in X$. By a Theorem 4.2, we have

$$\|f(x) - Q(x)\| \leq \frac{1}{|4|\,|n|^4} \sum_{i=\frac{1-j}{2}}^{\infty} \left(\frac{\phi(n^k x,0,0)}{|n|^{4kj}} + \frac{\phi(-n^k x,0,0)}{|n|^{4kj}} \right) \tag{40}$$

for all $x \in X$. Define $f(x) = f_e(x) + f_0(x)$ for all $x \in X$. From the Theorem 3.2 and Theorem 4.2, we arrive

$$\begin{aligned} \|f(x) - A(x) - Q(x)\| &= \|f_e(x) + f_0(x) - A(x) - Q(x)\| \\ &\leq \|f_0(x) - A(x)\| + \|f_e(x) - Q(x)\| \\ &\leq \frac{1}{|6|\,|n|} \sum_{i=\frac{1-j}{2}}^{\infty} \left(\frac{\phi(n^k x,0,0)}{|n|^{kj}} + \frac{\phi(-n^k x,0,0)}{|n|^{kj}} \right) \\ &+ \frac{1}{|4|\,|n|^4} \sum_{i=\frac{1-j}{2}}^{\infty} \left(\frac{\phi(n^k x,0,0)}{|n|^{4kj}} + \frac{\phi(-n^k x,0,0)}{|n|^{4kj}} \right) \end{aligned}$$

for all $x \in X$. Hence the Theorem is proved. □

Corollary 4.7. *Let $p > 1$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$. Then there exists an orthogonally additive mapping $A : X \to Y$ and a orthogonally quartic mapping $Q : X \to Y$ such that,*

$$\|f(x) - A(x) - Q(x)\| \leq \left(\frac{1}{|6|\left(|n| - |n|^p\right)} + \frac{1}{|4|\left(|n|^4 - |n|^p\right)} \right) \theta \|x\|^p \tag{41}$$

for all $x \in X$.

Corollary 4.8. *Let $0 < p < 1$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$. Then there exists an orthogonally additive mapping $A : X \to Y$ and a orthogonally quartic mapping $Q : X \to Y$ such that,*

$$\|f(x) - A(x) - Q(x)\| \leq \left(\frac{1}{|6|\left(|n|^p - |n|\right)} + \frac{1}{|4|\left(|n|^p - |n|^4\right)} \right) \theta \|x\|^p \tag{42}$$

for all $x \in X$.

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