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General Solution and Stability of Quadratic Functional Equation

Research Article

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Abstract: In this paper, the authors present the general solution and generalized Ulam-Hyers stability of quadratic functional equation of the form $f\left(nx+n^2y+n^3z+n^4t\right) + f\left(nx-n^2y+n^3z+n^4t\right) + f\left(nx+n^2y-n^3z+n^4t\right) + f\left(nx+n^2y+n^3z-n^4t\right) + f\left(-nx+n^2y+n^3z+n^4t\right) = f\left(nx+n^2y\right) + f\left(nx+n^3z\right) + f\left(nx+n^4t\right) + +f\left(-nx+n^2y+n^3z+n^4t\right) = f\left(nx+n^2y\right) + f\left(nx+n^3z\right) + f\left(nx+n^4t\right) + f\left(n^2y+n^3z\right) + f\left(n^2y+n^4t\right) + f\left(n^3z+n^4t\right) + 2n^2f(x) + 2n^4f(y) + 2n^6f(z) + 2n^8f(t)$ where n is positive integer with $n \neq 0$ in Banach algebra using direct and fixed point methods.

MSC: 39B52, 32B72, 32B82.

Keywords: Quadratic functional equation, Banach algebra, generalized Ulam-Hyers stability, fixed point theory.

1. Introduction

The study of stability problems for functional equations is related to a question of Ulam [33] concerning the stability of group hommomorphisms and Affirmatively answered for Banach spaces by Hyers [18] the result of Hyers was Generalized by Aoki [2] for approximate additive mappings and Rassias [26,27] for approximate linear mapping by allowing the Cauchy difference operator $CDf(x,y) = f(x+y) - [f(x)+f(y)]$to be controlled by $\in (\|x\|^p + \|y\|^p)$. In 1994, a further generalization was obtained by Gavruta [14], who replaced $\in (\|x\|^p + \|y\|^p)$ by a general control function $\phi(x,y)$.J.M Rassias [29,30] treated the Ulam-Gavruta- Rassias stability on linear and non-linear mappings and generalized Hyers result. The reader is referred to the following books and research articles which provide an existence account of progress made on Ulam's problem during the last Seventy years (see[18-27]). The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1}$$

is related to symmetric bi-additive function [15]. It is a natural that such question is called a quadratic functional equation. In particular, every solution of the quadratic equation (**??**) is said to be quadratic function. It is well known thata function f between real vector spaces in quadratic if and only if there exists a unique symmetric biadditive function B such that $f(x) = B(x,x)$ for all$x$ [see 15]. The bi-additive function B is given by $B(x,x) = \frac{1}{4}\left[f(x+y)+f(x-y)\right]$ in[12], Czerwik,

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proved the Hyers-Ulam Stability of the quadratic functional equation (**??**). A Hyers-Ulam stability Problem for the quadratic functional equation (**??**) was proved by Skof For function $f : E_1 \to E_2$, where E_1 is a normed space and E_2 a BanachSpace [see [6]). S.Czerwik [12] noticed that the theorem Skof is still True if the relevant domain E_1 is replaced by an abelian group. M.Egordjn has generalized Apollonius type quadratic functional equation (see[18]). The quadratic functional equation are useful to characteristic inner product space [8,24,28,29,34]. In this paper we introduce and investigate that the general solution and generalized Ulam-Hyers stability of a generalized quadratic functional equation of the form

$$\begin{aligned} & f\left(nx+n^2y+n^3z+n^4t\right)+f\left(nx-n^2y+n^3z+n^4t\right)+f\left(nx+n^2y-n^3z+n^4t\right)+f\left(nx+n^2y+n^3z-n^4t\right) \\ & +f\left(-nx+n^2y+n^3z+n^4t\right)=f\left(nx+n^2y\right)+f\left(nx+n^3z\right)+f\left(nx+n^4t\right)+f\left(-nx+n^2y+n^3z+n^4t\right) \\ & =f\left(nx+n^2y\right)+f\left(nx+n^3z\right)+f\left(nx+n^4t\right)+f\left(n^2y+n^3z\right)+f\left(n^2y+n^4t\right) \\ & +f\left(n^3z+n^4t\right)+2n^2f(x)+2n^4f(y)+2n^6f(z)+2n^8f(t) \end{aligned} \tag{2}$$

where n is positive integer with $n \neq 0$ in Banach algebra using direct and fixed point methods.

2. General Solution of the Functional Equation (4)

In this section, the general solution of the functional eauation(A) for even case is given Throughout this section, Let us consider X and Y to be real vector spaces.

Theorem 2.1. *If an even mapping $f : X \to Y$ satisfies the functional equation*

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{3}$$

for all $x, y \in X$ if and only if $f : X \to Y$ satisfies the functional equation

$$\begin{aligned} & f\left(nx+n^2y+n^3z+n^4t\right)+f\left(nx-n^2y+n^3z+n^4t\right)+f\left(nx+n^2y-n^3z+n^4t\right)+f\left(nx+n^2y+n^3z-n^4t\right) \\ & +f\left(-nx+n^2y+n^3z+n^4t\right)=f\left(nx+n^2y\right)+f\left(nx+n^3z\right)+f\left(nx+n^4t\right)+f\left(-nx+n^2y+n^3z+n^4t\right) \\ & =f\left(nx+n^2y\right)+f\left(nx+n^3z\right)+f\left(nx+n^4t\right)+f\left(n^2y+n^3z\right)+f\left(n^2y+n^4t\right) \\ & +f\left(n^3z+n^4t\right)+2n^2f(x)+2n^4f(y)+2n^6f(z)+2n^8f(t) \end{aligned} \tag{4}$$

for all $x, y, z, t \in X$.

Proof. Let $f : X \to Y$ satisfies the functional equation (3) setting (x, y) by $(0, 0)$ in (3). We get $f(0) = 0$. Replacing y by x in (3), we obtain

$$f(2x) = 4f(x) \tag{5}$$

for all $x \in X$. Replacing y by $2x$ in (3), we get

$$f(3x) = 9f(x) \tag{6}$$

for all $x \in X$. In general for any positive integer b Such that

$$f(bx) = b^2 f(x) \tag{7}$$

for all $x \in X$. It is easy to verify from (7) that

$$f(b^2x) = b^2 f(x) \text{ and } f(b^3x) = b^6 f(x) \tag{8}$$

for all $x \in X$, putting $x = nx + n^4t$, $y = n^2y + n^3z$ in (3), we have

$$f\left(nx + n^4t + n^2y + n^3z\right) + f\left(nx + n^4t - n^2y - n^3z\right) = 2f\left(nx + n^4t\right) + 2f\left(n^2y + n^3z\right) \tag{9}$$

for all $x, y, z, t \in X$. Substitute $x = nx + n^3z$, $y = n^2y + n^4t$ in (3), we arrive

$$f\left(nx + n^3z + n^2y + n^4t\right) + f\left(nx + n^3z - n^2y - n^4t\right) = 2f\left(nx + n^3z\right) + 2f\left(n^2y + n^4t\right) \tag{10}$$

for all $x, y, z, t \in X$. Setting $x = nx + n^2y$, $y = n^3z + n^4t$ in (3), we obtain

$$f\left(nx + n^2y + n^3z + n^4t\right) + f\left(nx + n^2y - n^3z - n^4t\right) = 2f\left(nx + n^2y\right) + 2f\left(n^3z + n^4t\right) \tag{11}$$

for all $x, y, z, t \in X$. Replacing (x, y) by $(nx - n^2y,\ n^3z - n^4t)$ in (3), we lead to

$$f\left(nx - n^2y + n^3z - n^4t\right) + f\left(nx - n^2y - n^3z + n^4t\right) = 2f\left(nx - n^2y\right) + 2f\left(n^3z - n^4t\right) \tag{12}$$

for all $x, y, z, t \in X$. Forming (x, y) by $(n^3z + n^4t,\ nx - n^2y)$ in (3), we obtain

$$f\left(n^3z + n^4t + nx - n^2y\right) + f\left(n^3z - n^4t - nx + n^2y\right) = 2f\left(n^3z + n^4t\right) + 2f\left(nx - n^2y\right) \tag{13}$$

for all $x, y, z, t \in X$. Adding (9) and (10) and remodifying, we get

$$\begin{aligned} &f\left(nx + n^2y + n^3z + n^4t\right) + f\left(nx - n^2y + n^3z + n^4t\right) + f\left(nx + n^2y + n^3z + n^4t\right) + f\left(nx - n^2y + n^3z - n^4t\right) \\ &\qquad = 2f\left(nx + n^4t\right) + 2f\left(n^2y + n^3z\right) + 2f\left(nx + n^3z\right) + 2f\left(n^2y + n^4t\right) \end{aligned} \tag{14}$$

for all $x, y, z, t \in X$. From the Equation (14), which implies that

$$\begin{aligned} &f\left(nx + n^2y + n^3z + n^4t\right) + f\left(nx - n^2y - n^3z + n^4t\right) + f\left(nx - n^2y + n^3z - n^4t\right) = 2f\left(nx + n^4t\right) \\ &\qquad + 2f\left(n^2y + n^3z\right) + 2f\left(nx + n^3z\right) + 2f\left(n^2y + n^4t\right) \ \to (14) \end{aligned} \tag{15}$$

for all $x, y, z, t \in X$. Using the equations (12) in (15), we lead to

$$\begin{aligned} &\Rightarrow 2f\left(nx + n^2y + n^3z + n^4t\right) + 2f\left(nx - n^2y\right) + 2f\left(n^3z - n^4t\right) \\ &\qquad = 2f\left(nx + n^4t\right) + 2f\left(n^2y + n^3z\right) + 2f\left(nx + n^3z\right) + 2f\left(n^2y + n^4t\right) \end{aligned} \tag{16}$$

for all $x, y, z, t \in X$. Dividing by 2 in (16), we assumed that

$$f\left(nx + n^2y + n^3z + n^4t\right) + f\left(nx - n^2y\right) + f\left(n^3z - n^4t\right) = f\left(nx + n^4t\right) + f\left(n^2y + n^3z\right) + f\left(nx + n^3z\right) + f\left(n^2y + n^4t\right) \tag{17}$$

for all $x, y, z, t \in X$. Adding $f(nx - n^2y)$ and $f(n^3z \to -n^4t)$ on both sides in (17), we get

$$\begin{aligned} &f\left(nx + n^2y + n^3z + n^4t\right) + f\left(nx - n^2y\right) + f\left(n^3z - n^4t\right) + f\left(nx - n^2y\right) + f\left(n^3z - n^4t\right) \\ &= f\left(nx + n^4t\right) + f\left(n^2y + n^3z\right) + f\left(nx + n^3z\right) + f\left(n^2y + n^4t\right) + f\left(nx - n^2y\right) + f\left(n^3z - n^4t\right) \end{aligned} \tag{18}$$

for all $x, y, z, t \in X$ using equation (18) and remodifying, we arrive

$$f\left(nx + n^2y + n^3z + n^4t\right) + 2f\left(nx - n^2y\right) + 2f\left(n^3z - n^4t\right) = f\left(nx + n^4t\right) + f\left(n^2y + n^3z\right) + f\left(nx + n^3z\right) + f\left(n^2y + n^4t\right) + f\left(nx - n^2y\right) + f\left(n^3z - n^4t\right) \tag{19}$$

for all $x, y, z, t \in X$. Adding (11), (19) and resultant of the equation, we get,

$$\begin{aligned} f\left(nx + n^2y + n^3z + n^4t\right) + 2f\left(nx - n^2y\right) + 2f\left(n^3z - n^4t\right) + f\left(nx + n^2y + n^3z + n^4t\right) + f\left(nx + n^2y - n^3z - n^4t\right) \\ = 2f\left(nx + n^2y\right) + 2f\left(n^3z + n^4t\right) + f\left(nx + n^3z\right) + f\left(n^2y + n^4t\right) + f\left(nx + n^4t\right) \\ + f\left(n^2y + n^3z\right) + f\left(nx - n^2y\right) + f\left(n^3z - n^4t\right) \end{aligned} \tag{20}$$

for all $x, y, z, t \in X$. Using (11) in (20), we obtain that the equation

$$\begin{aligned} f\left(nx + n^2y + n^3z + n^4t\right) + 2f\left(nx - n^2y\right) + 2f\left(n^3z - n^4t\right) + 2f\left(nx + n^2y\right) + 2f\left(n^3z + n^4t\right) \\ = f\left(nx + n^2y\right) + f\left(n^3z + n^4t\right) + f\left(nx + n^3z\right) + f\left(nx + n^4t\right) + f\left(n^2y + n^4t\right) + f\left(n^2y + n^3z\right) \\ + f\left(nx - n^2y\right) + f\left(nx + n^2y\right) + f\left(n^3z - n^4t\right) + f\left(n^3z + n^4t\right) \end{aligned} \tag{21}$$

for all $x, y, z, t \in X$. Using (13) in (21), we obtain that

$$\begin{aligned} f\left(nx + n^2y + n^3z + n^4t\right) + f\left(nx - n^2y + n^3z + n^4t\right) + f\left(-nx + n^2y + n^3z + n^4t\right) + 2f\left(nx + n^2y\right) + 2f\left(n^3z - n^4t\right) \\ = f\left(nx + n^4y\right) + f\left(n^2y + n^3z\right) + f\left(nx + n^4t\right) + f\left(nx - n^2y\right) + f\left(n^3z + n^4t\right) + f\left(n^3z - n^4t\right) \end{aligned} \tag{22}$$

for all $x, y, z, t \in X$. Setting (x, y) by $\left(nx, n^2y\right)$ in (1), we obtain

$$f\left(nx + n^2y\right) + f\left(nx - n^2y\right) = 2f\left(nx\right) + 2f\left(n^2y\right) \tag{23}$$

for all $x, y, z, t \in X$. Replacing (x, y) by $f\left(n^3z + n^4t\right)$ in (**??**), we lead to

$$f\left(n^3z + n^4t\right) + f\left(n^3z - n^4t\right) = 2f\left(n^3z\right) 2f\left(n^4t\right) \tag{24}$$

for all $x, y, z, t \in X$. Setting (x, y) by $\left(nx + n^2y,\ n^3z - n^4t\right)$ in (1). We have

$$f\left(nx + n^2y + n^3z - n^4t\right) + f\left(nx + n^2y - n^3z + n^4t\right) = 2f\left(nx + n^4t\right) + 2f\left(n^3z - n^4t\right) \tag{25}$$

for all $x, y, z, t \in X$. Substitute (23), (24), (25), in (22) we arrive at

$$\begin{aligned} f\left(nx + n^2y + n^3z + n^4t\right) + f\left(nx - n^2y + n^3z + n^4t\right) + f\left(-nx + n^2y + n^3z + n^4t\right) + f\left(nx + n^2y + n^3z - n^4t\right) \\ + f\left(nx + n^2y - n^3z + n^4t\right) = f\left(nx + n^2y\right) + f\left(n^2y + n^3z\right) + f\left(nx + n^4t\right) + f\left(n^2y - n^3z\right) \\ + f\left(n^2y + n^4t\right) + f\left(n^3z + n^4t\right) + 2f\left(nx\right) + 2f\left(n^2y\right) + 2f\left(n^{3z}\right) + 2f\left(n^4t\right) \end{aligned} \tag{26}$$

for all $x, y, z, t \in X$. Using (26) and rearranging ,we obtain that

$$\begin{aligned} f\left(nx + n^2y + n^3z + n^4t\right) + f\left(nx - n^2y + n^3z + n^4t\right) + f\left(-nx + n^2y + n^3z + n^4t\right) + f\left(nx + n^2y + n^3z - n^4t\right) \\ + f\left(nx + n^2y - n^3z + n^4t\right) = f\left(nx + n^2y\right) + f\left(nx + n^3z\right) + f\left(nx + n^4t\right) + f\left(n^2y + n^3z\right) \\ + f\left(n^2y + n^4t\right) + f\left(n^3z + n^4t\right) + 2n^2f\left(x\right) + 2n^4f\left(y\right) + 2n^6f\left(z\right) + 2n^8f\left(t\right) \end{aligned} \tag{27}$$

Conversely, $f : X \to Y$ satisfies the functional equation (3) and Setting $n^3z = 0$ and $n^4t = 0$ in (27), we obtain that

$$\Rightarrow 3f\left(nx+n^2y\right)+f\left(-nx+n^2y\right)+f\left(nx-n^2y\right)+\left(f\left(nx-n^2y\right)\right)=4n^2f\left(x\right)+4n^4f\left(y\right) \tag{28}$$

for all $x, y, z, t \in X$ Using evenness of f in (28), we arrive at

$$2f\left(nx+n^2y\right)+2f\left(nx-n^2y\right)\ =4f\left(nx\right)+4f\left(n^2y\right) \tag{29}$$

for all $x, y \in X$. In (29) Divided by 2, we obtain

$$f\left(nx+n^2y\right)+f\left(nx-n^2y\right)=2f\left(nx\right)+2f\left(n^2y\right) \tag{30}$$

for all $x, y \in X$. Put $nx = x$, $n^2y = y$ in (30), we arrive at (3) □

3. Stability Results for (A): Even Case-Direct Method

In this section ,we investigate the generalized Ulam-Hyers stability Of the functional equation (A) for even case.

Theorem 3.1. *Let $j \in \{-1, 1\}$ and $\alpha : X^4 \to [0, \infty)$ be a function such that*

$$\sum_{k=0}^{\infty}\frac{\alpha\left(n^{kj}x, n^{kj}y, n^{kj}z, n^{kj}t\right)}{n^{2kj}}$$

converges to R and

$$\lim_{k\to\infty}\frac{\alpha\left(n^{kj}x, n^{kj}y, n^{kj}z, n^{kj}t\right)}{n^{2kj}}=0 \tag{31}$$

for all $x, y, z, t \in X$, then there exists unique quadratic mapping $Q : X \to Y$ which satisfies the functional equation (A) and

$$\|f_q(x)-Q(x)\|\le\frac{1}{4n^2}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\alpha\left(n^{kj}x,0,0,0\right)}{n^{2kj}} \tag{32}$$

for all $x, y, z, t \in X$. Let $f_q : X \to Y$ be an even function satisfying the Inequality.

$$\|Df_q(x,y,z,t)\|\le\alpha(x,y,z,t) \tag{33}$$

for all $x, y, z, t \in X$. Then there exists unique quadratic mapping $Q : X \to Y$ which satisfies the functional equation (4) and

$$\|f_q(x)-Q(x)\|\le\frac{1}{4n^2}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\alpha(n^{kj}x,0,0,0)}{n^{2kj}} \tag{34}$$

for all $x \in X$. The mapping $Q(x)$ is defined by

$$Q(x)=\lim_{k\to\infty}\frac{f_q(n^{kj}x)}{n^{2kj}} \tag{35}$$

for all$x \in X$.

Proof. Assume that $j = 1$, replacing(x, y, z, t) by $(x, 0, 0, 0)$ in (33), we get

$$\left\| 2f_q(nx) - 2n^2 f_q(x) \right\| \leq \alpha(x, 0, 0, 0) \tag{36}$$

for all $x \in X$. It follows from (36), that

$$\left\| \frac{f_q(nx)}{n^2} - f_q(x) \right\| \leq \frac{\alpha}{2n^2}(x, 0, 0, 0) \tag{37}$$

for all $x \in X$. Replacing x by nx in (37), we lead to

$$\left\| \frac{f_q(n^2x)}{n^2} - f_q(nx) \right\| \leq \frac{\alpha}{2n^2}(nx, 0, 0, 0) \tag{38}$$

for all $x \in X$. Dividing by n^2 in (38), we get

$$\left\| \frac{f_q(n^2x)}{n^4} - \frac{f_q(nx)}{n^2} \right\| \leq \frac{\alpha}{2n^4}(nx, 0, 0, 0) \tag{39}$$

for all $x \in X$. It follows from (38) and (39), we get

$$\left\| \frac{f_q(n^2x)}{n^4} - f_q(nx) \right\| \leq \frac{1}{2n^2}\left[\alpha(nx, 0, 0, 0) + \frac{\alpha}{n^2}(nx, 0, 0, 0) \right] \tag{40}$$

for all $x \in X$. In general for any positive integer n, such that

$$\begin{aligned} \left\| f_q(x) - \frac{f_q(n^kx)}{n^{2k}} \right\| &\leq \frac{1}{2n^2} \sum_{k=0}^{n-1} \frac{\alpha(n^kx, 0, 0, 0)}{n^{2k}} \\ &\leq \frac{1}{2n^2} \sum_{k=0}^{\infty} \frac{\alpha(n^kx, 0, 0, 0)}{n^{2k}} \end{aligned} \tag{41}$$

for all $x \in X$. In order to prove the convergence of the sequence $\left\{ \frac{f_q(n^kx)}{n^{2k}} \right\}$. Replacing x by n^lx and dividing n^l in (41) for any $k, l > 0$ to deduce

$$\begin{aligned} \left\| \frac{f_q(n^lx)}{n^{2l}} - \frac{f_q(n^{R+k}x)}{n^{2(k+l)}} \right\| &= \frac{1}{n^{2l}} \left\| f_q(n^lx) - \frac{f_q(n^k.n^lx)}{n^{2k}} \right\| \\ &\leq \frac{1}{2n^{2l}} \sum_{k=0}^{n-1} \frac{\alpha(n^{k+l}x, 0, 0, 0)}{n^{2(k+l)}} \\ &\leq \frac{1}{2n^2} \sum_{k=0}^{\infty} \frac{\alpha(n^{k+l}x, 0, 0, 0)}{n^{2(k+l)}} \end{aligned} \tag{42}$$

for all $x \in X$. Hence the sequence $\left\{ \frac{f_q(n^kx)}{n^{2k}} \right\}$ is a Cauchy sequence . Since Y is complete, there exists a mapping $Q : X \to Y$ such that

$$Q(x) = \lim_{k \to \infty} \frac{f_q(n^kx)}{n^{2k}} \ \forall x \in X \tag{43}$$

Letting $k \to \infty$ in (41), we see that (34) holds for $x \in X$. To prove that a satisfies (A). Replacing (x, y, z, t) by $\left(n^{2k}x, n^{2k}y, n^{2k}z, n^{2k}t\right)$ dividing by n^{2k} in (33), we obtain

$$\frac{1}{n^{2k}} \left\| Df_q\left(n^{2k}x, n^{2k}y, n^{2k}z, n^{2k}t\right) \right\| \leq \frac{1}{n^{2k}} \alpha\left(n^{2k}x, n^{2k}y, n^{2k}z, n^{2k}t\right)$$

for all $x, y, z, t \in X$. Letting $k \to \infty$ in the above inequality and using the definition of $Q(x)$ we see that, $DQ(x, y, z, t) = 0$. Hence Q satisfies (A) for all $x, y, z, t \in X$. To show that Q is unique let $B(x)$ be another quadratic mapping Satisfying (32) and (A), then

$$\begin{aligned} \|Q(x) - B(x)\| &= \frac{1}{n^{2k}} \left\| Q(n^l x) - B(n^l x) \right\| \\ &\leq \frac{1}{n^{2k}} \left\{ \left\| Q(n^l x) - f_q(n^l x) \right\| + \left\| f_q(n^l x) - B(n^l x) \right\| \right\} \quad \to 0 \; as \; n \to \infty \end{aligned}$$

for all $x \in X$ hence Q is unique. Now, replacing x by $\frac{x}{n}$ in (36), we get

$$\left\| 2f_q(x) - 2n^2 f_q\left(\frac{x}{n}\right) \right\| \leq \alpha\left(\frac{x}{n}, 0, 0, 0\right) \tag{44}$$

for all x belongs to X. It follows from (44) that

$$\left\| f_q(x) - n^2 f_q\left(\frac{x}{n}\right) \right\| \leq \alpha\left(\frac{x}{n}, 0, 0, 0\right) \tag{45}$$

For all $x \in X$ the rest of the proof is similar to that of $j = 1$. Hence for $j = -1$ also the theorem is true. This completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 1.1 concerning the stability of (4).

Corollary 3.2. *Let λ and s be a non-negative real numbers. Let an even function $f_q : X \to Y$ satisfying the inequality*

$$\|Df_q(x, y, z, t)\| \leq \begin{cases} \lambda; \\ \lambda\left(\|x\|^s + \|y\|^s + \|z\|^s + \|t\|^s\right); \\ \lambda\left[\left(\|x\|^s + \|y\|^s + \|z\|^s + \|t\|^s\right) + \left\{\left(\|x\|^{4s} + \|y\|^{4s} + \|z\|^{4s} + \|t\|^{4s}\right)\right\}\right]; \end{cases} \tag{46}$$

for all $x, y, z, t \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f_q(x) - Q(x)\| \leq \begin{cases} \frac{\lambda}{4\left|n^2 - 1\right|}; \\ \frac{\lambda\|x\|^s}{4\left|n^2 - n^s\right|} \\ \frac{\lambda\|x\|^{4s}}{4\left|n^2 - n^{4s}\right|} \end{cases} \tag{47}$$

for all$x \in X$.

4. Fixed Point Stability Results of (4)

The following theorems are useful to prove our fixed point Stability results.

Theorem 4.1 (Banach Contraction Principle). *Let (X, d) be a complete metric spaces and consider a mapping $T : X \to X$ which is strictly contractive mapping that is*

(A_1) *$d(T_x, T_y) \leq d(x, y)$, for some (Lipchitz constant) $L < 1$, then*

(1). the mapping T has only one fixed point $x^* = T(x^*)$

(2). the fixed point for each given element x^* is globally contractive, that is

(A_2) $\lim\limits_{n\to\infty} T^n x \leq x^*$. *For any starting point* $x \in X$

(3). one has the following estimation inequalities

(A_3) $d(T^n x, x^*) \leq \frac{1}{1-L} d\left(T^n x, T^{n+1}x\right)$, $\forall n \geq 0 \forall x \in X$.

(A_4) $d(x, x^*) = \frac{1}{1-L} d(x, x^*)$, $\forall x \in X$.

Theorem 4.2 (The Alternative Fixed Point)**.** *Suppose that for a complete generalized metric space* (X, d) *and a strictly contractive mapping* $T : X \to X$ *with lipschitz constant* L*, then for each given element* $x \in X$ *either*

(B_1) $d\left(T^n x, T^{n+1}x\right) = \infty \ \forall\, n \geq 0$.

(B_2) *There exists a natural numbers* n_0 *such that*

(1). $d\left(T^n x . T^{n+1}x\right) < \infty \ \forall n \geq 0$

(2). The sequence $\{T^n x\}$ *is convergent to a fixed point* y^* *of* T

(3). y^* *is the unique fixed point of* T *in the set* $y = \{y \in Y; d(T^{n_0}x, y) < \infty\}$.

(4). $d(y^*, y) \leq \frac{1}{1-L} d(y, T_y)$ *for all* $y \in Y$.

Fixed point stability of (4.1): even case-fixed point method:

In this section, we give the generalized Ulam-hyers stability of the Functional equation (4) for even case

Theorem 4.3. *Let* $f_q : W \to B$ *be an even mapping for which there exists a Function* $\alpha : W^* \to [0, \infty)$ *with the condition*

$$\lim_{k\to\infty} \frac{\alpha\left(ni^k x, ni^k y, ni^k z, ni^k t\right)}{ni^{2k}} = 0 \tag{48}$$

Where

$$\eta_i = \begin{cases} n, & i = 0; \\ \frac{1}{n}, & i = 1; \end{cases}$$

Such that the functional inequality

$$\|Df_q(x, y, z, t)\| \leq \alpha(x, y, z, t) \tag{49}$$

for all $x, y, z, t \in W$*. If there exists* $L = L(i) < 1$ *such that the function*

$$x \to \beta(x) = \frac{1}{2}\alpha\left(\frac{x}{n}, 0, 0, 0\right)$$

Has the property.

$$\frac{1}{\eta_i^2}\beta(\eta, x) = L\beta(x) \tag{50}$$

For all $x \in W$*. Then there exists a unique quadratic function* $Q : W \to B$ *satisfying the functional equation* (4) *and*

$$\|f_q(x) - Q(x)\| \leq \frac{L^{1-i}}{1-L}\beta(x) \tag{51}$$

holds for all $x \in W$*.*

Proof. Consider the set $X = \{P/P : W \to B, P(0) = 0\}$ and introduce the generalized Metric on X.

$$d(p,q) = \inf\{k \in (o,\infty) : \|p(x) - q(x)\| \le k\beta(x), x \in W\}$$

It is easy to see that (x,d) is complete. Define $T : x \to x$ by $T_p(x) = \frac{1}{\eta_i^2}P(\eta_i x)$ for all $x \in W$. Now $p.q \in X$.

$$\begin{aligned}
d(p,q) \le k &\Rightarrow \|p(x) - q(x)\| \le k\beta(x)\,;\ x \in W \\
&\Rightarrow \left\|\frac{1}{\eta_i^2}P(\eta_i x) - \frac{1}{\eta_i^2}q(\eta_i x)\right\| \le \frac{1}{\eta_i^2}k\beta(\eta_i x);\ x \in W \\
&\Rightarrow \left\|\frac{1}{\eta_i^2}P(\eta_i x) - \frac{1}{\eta_i^2}q(\eta_i x)\right\| \le Lk\beta(x);\ x \in W \\
&\Rightarrow \|T_p(\eta_i x) - T_q(\eta_i x)\| \le Lk\beta(x);\ x \in W \\
&\Rightarrow d(T_p, T_q) \le Lk
\end{aligned}$$

This implies $d(T_p, T_q) \le Ld(p,q)$ for all $p, q \in X$. (ie) T is strictly contractive mapping on X with Lipschtiz constant L.Replacing (x,y,z,t) by $(x,0,0,0)$ in (49) and using evenness of f we get

$$\left\|2f(nx) - 2n^2 f(x)\right\| \le \alpha(x,0,0,0) \tag{52}$$

for all $x \in X$. Using (52) and remodifying, that

$$\left\|f(x) - \frac{f(nx)}{n^2}\right\| \le \frac{1}{2n^2}\alpha(x,0,0,0) \tag{53}$$

for all $x \in W$. Using (50) for the case $i = 0$, it reduces to

$$\left\|f_q(x) - \frac{1}{n^2}f_q(nx)\right\| \le \frac{1}{n^2}\beta(x) \tag{54}$$

for all $x \in W$. ie., $d(f_q, Tf_q) \le \frac{1}{n^2} = L = L^1 < \infty$. Again replacing $x = \frac{x}{n}$ in (52), we get

$$\left\|f_q(x) - n^2 f_q\left(\frac{x}{n}\right)\right\| \le \frac{1}{2}\alpha\left(\frac{x}{n}, 0, 0, 0\right) \tag{55}$$

for all $x \in W$. Using (50) for the case $i = 1$, it reduces to

$$\left\|f_q(x) - n^2 f_q\left(\frac{x}{n}\right)\right\| \le \beta(x) \tag{56}$$

for all $x \in W$. ie., $d(f_q, Tf_q) \le 1 \Rightarrow L^o < \infty$. In above cases, we arrive $d(f_q, Tf_q) \le L^{1-i}$. Therefore $(B_2(i))$ holds by $(B_2(ii))$, it follows that there exists a fixed Point Q of T in X. Such that

$$Q(x) = \lim_{k\to\infty} \frac{f_q(n_i^k x)}{n_i^{2k}}\ \forall x \in W \tag{57}$$

In order to prove $Q : W \to B$ is quadratic replacing (x,y,z,t) by $(\eta_i^k x, \eta_i^k y, \eta_i^k z, \eta_i^k t)$ in (49) and dividing by η_i^{2k}, it follows from (47) and (55), we see that Q satisfies (A) for all $x, y, z, t \in W$ hence Q satisfies the functional equation (A). By $(B_2(iii))$. Q is the unique fixed point of T in the set

$$Y = \{f_q \in X,\ d(Tf_q, A) < \infty\}$$

Using the fixed point alternative result . Q is the unique function such that $\|f_q(x) - Q(x)\| \leq k\beta(x)$ for all $x \in W$ and $k > 0$ finally by $(B_2(iv))$. We obtain $d(f_q, Q) \leq \frac{1}{1-L} d(f_q, Tf_q)$

$$d(f_q, Q) \leq \frac{L^{1-i}}{1-L}$$

Hence, we conclude that

$$d(f_q(x), Q(x)) \leq \frac{L^{1-i}}{1-L}\beta(x)$$

for all $x \in W$. This completes the proof of the Theorem. □

Corollary 4.4. *Let $f_q : W \to B$ be an even mapping and there exists a real numbers λ and s such that*

$$\|Df_q(x,y,z,t)\| \leq \begin{cases} \lambda, \\ \lambda\left\{\|x\|^s + \|y\|^s + \|z\|^s + \|t\|^s\right\} & s \neq 2 \\ \lambda\left\{\|x\|^s + \|y\|^s + \|z\|^s + \|t\|^s + \left\{\|x\|^{4s} + \|y\|^{4s} + \|z\|^{4s} + \|t\|^{4s}\right\}\right\} & s \neq \frac{1}{4} \end{cases} \tag{58}$$

for all $x, y, z, t \in W$. There exists a unique quadratic mapping $Q : W \to B$ Such that

$$\|f_q(x) - Q(x)\| \leq \begin{cases} \frac{\lambda}{4\left|n^2-1\right|} \\ \frac{\lambda\|x\|^s}{4\left|n^2-n^s\right|}, \\ \frac{\lambda\|x\|^{4s}}{4\left|n^2-n^{4s}\right|} \end{cases} \tag{59}$$

for all $x \in W$.

5. Stability Results for (4): Even Case-Direct Method

In this section we discuss the generalized Ulam-Hyers stability of the functional equation (4), when f is even.

Definition 5.1. *Let X be abanach Algebra A mapping $Q : X \to X$ is said to be Quadratic derivation if the quadratic function Q satisfies,*

$$Q(ab) = Q(a)b^2 + a^2Q(b) \tag{60}$$

for all $a, b \in X$. Also the quadratic derivation for four variables Satisfies that

$$Q(abcd) = Q(a)b^2c^2d^2 + a^2Q(b)c^2d^2 + a^2b^2Q(c)d^2 + a^2b^2c^2Q(d) \tag{61}$$

for all $a, b, c, d \in X$.

Theorem 5.2. *Let $j = \pm 1$. Let $f_q : X \to Y$ be an even mapping for which there exists function $\alpha, \beta : X^4 \to [0, \infty)$ with the conditions $\sum\limits_{k=0}^{\infty} \frac{\alpha(n^{kj}x, n^{kj}y, n^{kj}z, n^{kj}t)}{n^{2kj}}$ converges in$R$ and*

$$\lim_{k\to\infty} \frac{\alpha(n^{kj}x, n^{kj}y, n^{kj}z, n^{kj}t)}{n^{2kj}} = 0 \tag{62}$$

$\sum\limits_{k=0}^{\infty} \frac{\beta(n^{kj}x, n^{kj}y, n^{kj}z, n^{kj}t)}{n^{8kj}}$ *converges in R and*

$$\lim_{k\to\infty} \frac{\beta(n^{kj}x, n^{kj}y, n^{kj}z, n^{kj}t)}{n^{8kj}} = 0 \tag{63}$$

Such that the functional inequalities

$$\|Df_q(x,y,z,t)\| \leq \alpha(x,y,z,t). \tag{64}$$

And

$$\left\|f_q(x,y,z,t) - y^2z^2t^2 f_q(x) - x^2 f_q(y) z^2t^2 - x^2y^2 f_q(z) - x^2y^2z^2 f_q(t)\right\| \leq \beta(x,y,z,t) \tag{65}$$

for all $x,y,z,t \in X$. There exists a unique quadratic derivation $Q : X \to X$ satisfying the functional equation (4) and

$$\|f_q(x) - Q(x)\| \leq \frac{1}{2n^2} \sum_{k=0}^{\infty} \frac{\alpha(n^{kj}x,0,0,0)}{n^{2kj}} \tag{66}$$

for all $x \in X$. The mapping $Q(x)$ is defined by

$$Q(x) = \lim_{k\to\infty} \frac{f_q(n^{kj}x)}{n^{2kj}} \tag{67}$$

for all$x \in X$.

Proof. It follows from Theorem 3.1 that Q is a unique quadratic Mapping and satisfies (4) for all $x,y,z,t \in X$. It follows form (65) that

$$\begin{aligned}
&\left\|Q(xyzt) - Q(x)y^2z^2t^2 - x^2Q(y)z^2t^2 - x^2y^2Q(z)t^2 - x^2y^2z^2Q(t)\right\| \\
&\leq \frac{1}{n^{8k}} \left\| \begin{array}{l} f_q\left(n^k(xyzt)\right) - f_q n^k(x) n^{2k}y\, n^{2k}z\, n^{2k}t \\ -n^{2k}x\, f_q(n^1 y) n^{2k}z\; n^{2k}t - n^{2k}x\; n^{2k}y\; f_q(n^k z) n^{2k}t - n^{2k}x\; n^{2k}y\; n^{2k}z\; f_q(t) \end{array} \right\| \\
&\leq \frac{1}{n^{8k}} \beta(n^kx, n^ky, n^kz, n^kt) \;\to 0 \; as \; k \to \infty
\end{aligned}$$

For all $x,y,z,t \in X$. Hence the mapping $Q : X \to Y$ is a unique quadratic derivation satisfying (66). □

Corollary 5.3. *Let $f_q : X \to Y$ be an even mapping and there exists a real no's λ and S such that*

$$\|Df_q(x,y,z)\| \leq \begin{cases} \lambda, & \\ \lambda\left\{\|x\|^s + \|y\|^s + \|z\|^s + \|t\|^s\right\} & s \neq 2 \\ \lambda\left\{\|x\|^s + \|y\|^s + \|z\|^s + \|t\|^s + \left\{\|x\|^{4s} + \|y\|^{4s} + \|z\|^{4s} + \|t\|^{4s}\right\}\right\} & s \neq \frac{1}{2} \end{cases} \tag{68}$$

$$\begin{aligned}
&\left\|f_q(xyzt) - f_q(x)y^2z^2t^2 - \; f_q(y)x^2y^2z^2 - x^2y^2f_q(z)t^2 - x^2y^2z^2 f_q(t)\right\| \\
&\leq \begin{cases} \lambda, \\ \lambda\left\{\|x\|^s + \|y\|^s + \|z\|^s + \|t\|^s\right\}; \\ \lambda\left[\|x\|^s + \|y\|^s + \|z\|^s + \|t\|^s\right] + \left\{\|x\|^{4s} + \|y\|^{4s} + \|z\|^{4s} + \|t\|^{4s}\right\}; \end{cases}
\end{aligned} \tag{69}$$

for all $x \in X$. Then there exists a unique quadratic derivation $Q : X \to Y$ such that

$$\|f_q(x) - Q(x)\| \leq \begin{cases} \frac{\lambda}{4\left|n^2-1\right|}, \\ \frac{\lambda\|x\|^s}{4\left|n^2-n^s\right|} \\ \frac{\lambda\|x\|^s}{4\left|n^2-n^{4s}\right|} \end{cases} \tag{70}$$

6. Stability Results for (4): Even Case - Fixed Point Method

In this section, we discuss the generalized Ulam-Hyers stability of the functional equation (A) when f is even case.

Theorem 6.1. *Let $j = \pm 1$, let $f_q : X \to Y$ be an even mapping for which there exists functions $\alpha, \beta : X^4 \to [0, \infty)$ with the conditions $\sum_{k=0}^{\infty} \frac{\alpha(n^{kj}x, n^{kj}y, n^{kj}z, n^{kj}t)}{n^{2kj}}$ converges in R and*

$$\lim_{k\to\infty} \frac{\alpha(n^{kj}x, n^{kj}y, n^{kj}z, n^{kj}t)}{n^{2kj}} = 0 \tag{71}$$

$\sum_{k=0}^{\infty} \frac{\beta(n^{kj}x, n^{kj}y, n^{kj}z, n^{kj}t)}{n^{8kj}}$ converges in R and

$$\lim_{k\to\infty} \frac{\beta(n^{kj}x, n^{kj}y, n^{kj}z, n^{kj}t)}{n^{8kj}} = 0 \tag{72}$$

Where n_i is defined in (48) satisfying the functional inequalities

$$\|Df_q(x, y, z, t)\| \le \alpha(x, y, z, t) \tag{73}$$

and

$$\left\| f_q(x, y, z, t) - f_q(x)y^2z^2t^2 - x^2 f_q(y)z^2t^2 - x^2y^2 f_q(z) - x^2y^2z^2 f_q(t) \right\| \le \beta(x, y, z, t) \tag{74}$$

for all $x, y, z, t \in X$ then there exists $L = L(i) < 1$ such that the function

$$x \to \beta(x) = \frac{1}{2}\left[\frac{x}{n}, 0, 0, 0\right]$$

As the property

$$\frac{1}{\eta_i^2}\beta(\eta_i x) = L\beta(x) \tag{75}$$

for all $x \in X$ then there exists a unique quadratic derivation mapping $Q : X \to Y$ satisfying the functional equation (4) and

$$\|f_q(x) - Q(x)\| \le \frac{L^{1-i}}{1-L}\beta(x) \tag{76}$$

for all$x \in X$.

Proof. It follows from Theorem 4.1 that is a unique quadratic mapping and Satisfying (A) forall $x \in X$. It follows from (71), (72) and (74) that

$$\begin{aligned}
&\left\| Q(xyzt) - Q(x)y^2z^2t^2 - x^2Q(y)z^2t^2 - x^2y^2Q(z)t^2 - x^2y^2z^2Q(t) \right\| \\
&\le \frac{1}{n^{8k}} \left\| \begin{array}{l} f_q\left(n^k(xyzt)\right) - f_q n^k(x) n^{2k}y\, n^{2k}z\, n^{2k}t - \\ n^{2k}x\, f_q n^k(y) n^{2k}z\; n^{2k}t - n^{2k}x\; n^{2k}y\; f_q(n^k z) n^{2k}t - n^{2k}x\; n^{2k}y\; n^{2k}z\; f_q n^k(t) \end{array} \right\| \\
&\le \frac{1}{n^{8k}}\beta(n^kx, n^ky, n^kz, n^kt) \to 0 \text{ as } k \to \infty.
\end{aligned}$$

Thus the mapping $Q : X \to Y$ is a unique quadratic derivation mapping Satisfying (4). □

Corollary 6.2. *Let $f_q : X \to Y$ be an even mapping and there exists a real number λ and s such that*

$$\|Df_q(x,y,z,t)\| \leq \begin{cases} \lambda, & \\ \lambda\left\{\|x\|^s+\|y\|^s+\|z\|^s+\|t\|^s\right\} & s\neq 2 \\ \lambda\left\{\|x\|^s+\|y\|^s+\|z\|^s+\|t\|^s+\left\{\|x\|^{4s}+\|y\|^{4s}+\|z\|^{4s}+\|t\|^{4s}\right\}\right\} & s\neq \frac{1}{2} \end{cases} \tag{77}$$

$$\left\|f_q(xyzt)-f_q(x)y^2z^2t^2-\ x^2f_q(y)y^2z^2-x^2y^2f_q(z)t^2-x^2y^2z^2f_q(t)\right\|$$

$$\leq \begin{cases} \lambda, & \\ \lambda\left\{\|x\|^s+\|y\|^s+\|z\|^s+\|t\|^s\right\}; & s\neq 1 \\ \lambda\left[\|x\|^s+\|y\|^s+\|z\|^s+\|t\|^s\right]+\left\{\|x\|^{4s}+\|y\|^{4s}+\|z\|^{4s}+\|t\|^{4s}\right\}; & s\neq \frac{1}{4} \end{cases} \tag{78}$$

for all $x \in X$. Then there exists a quadratic derivation $Q : X \to Y$ such that

$$\|f_q(x)-Q(x)\| \leq \begin{cases} \frac{\lambda}{4\left|n^2-1\right|}, \\ \frac{\lambda\|x\|^s}{4\left|n^2-n^s\right|} \\ \frac{\lambda\|x\|^s}{4\left|n^2-n^{4s}\right|} \end{cases} \tag{79}$$

for all $x \in X$.

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