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Partial Sums for Multivalent Harmonic Maps Defined by Generalized Hypergeometric Functions

Research Article

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Abstract: In this paper,we study partial sums of the series of certain multivalent harmonic functions involving the generalized hypergeometric functions which are in the class $P_H(m, A, B)$. We establish some new results giving the sharp bounds of the real parts of the ratios of harmonic multivalent functions to its sequences of partial sums.

MSC: 30C45, 30C55

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1. Introduction and Preliminaries

A continuous complex-valued function $f = u + iv$ defined in a simply connected domain $\mathbb{D}$ is said to be harmonic in $\mathbb{D}$ if both u and v are real-valued harmonic in $\mathbb{D}$. In any simply connected domain $\mathbb{D} \subset \mathbb{C}$, a harmonic function f can be written in the form: $f = h + \overline{g}$, where h and g are analytic in $\mathbb{D}$. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation preserving in $\mathbb{D}$ is that $|h'(z)| > |g'(z)|$ in $\mathbb{D}$ (see [6]). Let H denotes a class of functions $f = h + \overline{g}$ which are harmonic, univalent and orientation preserving in the open unit disc $\mathbb{U} = \{z : |z| < 1\}$ and are normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Harmonic functions are useful as they found their applications in the problems related to minimal surfaces [8].

Note that the family H reduces to the well known class S of normalized univalent functions if the co-analytic part of f is identically zero, that is if $g \equiv 0$.

The concept of multivalent harmonic complex valued functions by using argument principle, was given by Duren, Hengartner and Laugesen [7]. Using this concept, Ahuja and Jahangiri [1], [2] introduced a class $H(m)$ of m-valent harmonic and orientation preserving functions $f = h + \overline{g} \in H(m)$, where h and g are m-valent functions of the form

$$h(z) = z^m + \sum_{n=m+1}^{\infty} h_n z^n \ \text{ and } g(z) = \sum_{n=m}^{\infty} g_n z^n \ \ (|g_m| < 1, m \in \mathbb{N} = \{1, 2, 3....\}) \tag{1}$$

which are analytic in $\mathbb{U}$.

Motivated with the class conditions studied earlier in [2–4] and by observing various equivalent class conditions considered in [18], we define a unified classes $P_H(m, A, B)$ as follows:

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Definition 1.1. *A function $f = h + \overline{g} \in H(m)$ of the form (1), is said to be in the class $P_H(m, A, B)$ if it satisfies the condition*

$$\sum_{n=m+1}^{\infty}\left[\frac{(n-m)(1-B)}{m(A-B)}+1\right]|h_n|+\sum_{n=m}^{\infty}\left[\frac{(n+m)(1-B)}{m(A-B)}-1\right]|g_n|\leq 1, \tag{2}$$

where $-1 \leq B < A \leq 1$.

Denote by $TH(m)$ a subclass of functions $f = h + \overline{g} \in H(m)$ such that

$$h(z) = z^m - \sum_{n=m+1}^{\infty}|h_n|\,z^n \text{ and } g(z) = \sum_{n=m}^{\infty}|g_n|\,z^n. \tag{3}$$

Also, we denote $TP_H(m, A, B) = P_H(m, A, B) \cap TH(m)$. Let $p, q, \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $\alpha_i \in \mathbb{C}\,(i = 1, ..., p)$ and $\beta_i \in \mathbb{C}\,(\neq -n; i = 1, .., q, n \in \mathbb{N}_0)$, the generalized hypergeometric (gh) function

$${}_pF_q(\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; z) = {}_pF_q\left((\alpha_i);(\beta_i);z\right)$$

is defined by

$${}_pF_q\left((\alpha_i);(\beta_i);z\right) = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n}\frac{z^n}{n!} \quad (p \leq q+1; z \in \mathbb{U}) \tag{4}$$

which is analytic at $z = 1$ if (in case $p = q + 1$) $\Re\left(\sum_{i=1}^{q}\beta_i - \sum_{i=1}^{p}\alpha_i\right) > 0$, the symbol $(\lambda)_n$ is the Pochhammer symbol defined in terms of gamma function by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & n = 0, \lambda \neq 0 \\ \lambda(\lambda+1)\dots(\lambda+n-1), & n \in \mathbb{N} \end{cases}.$$

In terms of generalized hypergeometric functions ${}_pF_q\left((\alpha_i);(\beta_i);z\right)$ and ${}_rF_s\left((\gamma_i);(\delta_i);z\right)$, we consider a harmonic function $F(z) = H(z) + \overline{G(z)} \in H(m)$, where $H(z)$ and $G(z)$ are defined by

$$H(z) = z^m\,{}_pF_q\left((\alpha_i);(\beta_i);z\right) \text{ and } G(z) = z^{m-1}\left[{}_rF_s\left((\gamma_i);(\delta_i);z\right) - 1\right] \tag{5}$$

with $\prod_{i=1}^{r}|\gamma_i| < \prod_{i=1}^{s}|\delta_i|$. The series expression of $F(z)$ is given by

$$F(z) = z^m + \sum_{n=m+1}^{\infty}\frac{\prod_{i=1}^{p}(\alpha_i)_{n-m}}{\prod_{i=1}^{q}(\beta_i)_{n-m}}\frac{z^n}{(n-m)!} + \overline{\sum_{n=m}^{\infty}\frac{\prod_{i=1}^{r}(\gamma_i)_{n-m+1}}{\prod_{i=1}^{s}(\delta_i)_{n-m+1}}\frac{z^n}{(n-m+1)!}} \quad (z \in \mathbb{U}). \tag{6}$$

In 1985, Silvia [20] studied the partial sums of convex functions of order α. Later on, Silverman [19], Abubaker and darus [5], Dixit and Porwal [9], Frasin [10, 11], Murugusundaramoorthy et al. [12], Orahan et al. [21], Raina and Bansal [13] and Rosy et al. [15] studied and generalized the results on partial sums for various classes of analytic functions. Also Ravichandran [14] discussed the geometric properties of partial sums of univalent functions, Recently, Porwal [16], Porwal and Dixit [17] studied the partial sums of harmonic univalent functions. Further, Yasar and Yalcin [22] studied the partial sums of certain harmonic multivalent functions.

In this paper, we study partial sums of certain multivalent harmonic functions involving the generalized hypergeometric functions for the class $P_H(m, A, B)$. We establish some new results giving the sharp bounds of the real parts of ratios of harmonic multivalent functions involving the generalized hypergeometric functions to its sequences of partial sums.

We denote the following sequences of certain partial sums of $F(z)$ fo $n_1 \geq m+1,\ n_2 \geq m$,

$$\begin{aligned} F_{n_1}(z) &= z^m + \sum_{n=m+1}^{n_1} A_n z^n + \sum_{n=m}^{\infty} B_n \overline{z^n} \\ F_{n_2}(z) &= z^m + \sum_{n=m+1}^{\infty} A_n z^n + \sum_{n=m}^{n_2} B_n \overline{z^n} \\ F_{n_1,n_2}(z) &= z^m + \sum_{n=m+1}^{n_1} A_n z^n + \sum_{n=m}^{n_2} B_n \overline{z^n}, \end{aligned}$$

where

$$\lim_{n_1\to\infty} F_{n_1}(z) = F(z), \lim_{n_2\to\infty} F_{n_2}(z) = F(z) \text{ and } \lim_{n_1,n_2\to\infty} F_{n_1,n_2}(z) = F(z)$$

and

$$A_n = \frac{\prod_{i=1}^{p}(\alpha_i)_{n-m}}{\prod_{i=1}^{q}(\beta_i)_{n-m}} \frac{1}{(n-m)!},\ B_n = \frac{\prod_{i=1}^{r}(\gamma_i)_{n-m+1}}{\prod_{i=1}^{s}(\delta_i)_{n-m+1}} \frac{1}{(n-m+1)!}. \tag{7}$$

In the subsequent results we determine sharp upper and lower bounds for $\Re\left\{\frac{F(z)}{F_{n_1}(z)}\right\}$, $\Re\left\{\frac{F_{n_1}(z)}{F(z)}\right\}$, $\Re\left\{\frac{F'_{n_1}(z)}{F'(z)}\right\}$, $\Re\left\{\frac{F'(z)}{F'_{n_1}(z)}\right\}$, $\Re\left\{\frac{F(z)}{F_{n_2}(z)}\right\}$, $\Re\left\{\frac{F_{n_2}(z)}{F(z)}\right\}$, $\Re\left\{\frac{F'_{n_2}(z)}{F'(z)}\right\}$, $\Re\left\{\frac{F'(z)}{F'_{n_2}(z)}\right\}$, $\Re\left\{\frac{F(z)}{F_{n_1,n_2}(z)}\right\}$, $\Re\left\{\frac{F_{n_1,n_2}(z)}{F(z)}\right\}$, $\Re\left\{\frac{F'_{n_1,n_2}(z)}{F'(z)}\right\}$, $\Re\left\{\frac{F'(z)}{F'_{n_1,n_2}(z)}\right\}$ for $F \in P_H(m,A,B)$, where $F'(z) = \frac{\partial}{\partial\theta}F(z)\ \left(z = re^{i\theta}\right)$.

2. Main Results

Theorem 2.1. *Let $F(z)$ of the form (5) be in the class $P_H(m,A,B)$. Then*

$$\begin{aligned} \frac{(n_1+1)(1-B)+m(A-1)-m(A-B)}{(n_1+1)(1-B)+m(A-1)} &\leq \Re\left\{\frac{F(z)}{F_{n_1}(z)}\right\} \\ &\leq \frac{(n_1+1)(1-B)+m(A-1)+m(A-B)}{(n_1+1)(1-B)+m(A-1)} \quad (z \in \mathbb{U}). \end{aligned} \tag{8}$$

The result is sharp for the function

$$F(z) = z^m + \frac{m(A-B)}{(n_1+1)(1-B)+m(A-1)} z^{n_1+1} \quad (n_1 \geq m+1; z \in \mathbb{U}). \tag{9}$$

Proof. Since $F \in P_H(m,A,B)$, by Definition 1.1 we have

$$\sum_{n=m+1}^{\infty} \frac{n(1-B)+m(A-1)}{m(A-B)} |A_n| + \sum_{n=m}^{\infty} \frac{n(1-B)-m(A-1)}{m(A-B)} |B_n| \leq 1, \tag{10}$$

where A_n, B_n are given by (7). We observe that $\frac{n(1-B)+m(A-1)}{m(A-B)}$ and $\frac{n(1-B)-m(A-1)}{m(A-B)}$ are increasing functions of n, and

$$\frac{n(1-B)+m(A-1)}{m(A-B)} > 1\ (n \geq m+1),\ \frac{n(1-B)-m(A-1)}{m(A-B)} \geq 1\ (n \geq m). \tag{11}$$

Therefore, from (10), we obtain

$$\sum_{n=m+1}^{n_1} |A_n| + \frac{(n_1+1)(1-B)+m(A-1)}{m(A-B)} \sum_{n=n_1+1}^{\infty} |A_n| + \sum_{n=m}^{\infty} |B_n| \leq 1. \tag{12}$$

Let

$$p_1(z) = \frac{(n_1+1)(1-B)+m(A-1)}{m(A-B)} \left\{\frac{F(z)}{F_{n_1}(z)} - \frac{(n_1+1)(1-B)+m(A-1)-m(A-B)}{(n_1+1)(1-B)+m(A-1)}\right\} \tag{13}$$

which is analytic in $\mathbb{U}$ with $p_1(0)=1$. To obtain the lower bound of (8), we need to show that

$$p_1(z)=\frac{1+\omega_1(z)}{1-\omega_1(z)}, \tag{14}$$

where $\omega_1(z)$ is a Schwarz function with $\omega_1(0)=0$ and $|\omega_1(z)|<1$ in $\mathbb{U}$. Evidently, from (13) and (14), we get for $z\in\mathbb{U}$,

$$\begin{aligned}\omega_1(z) &= \frac{p_1(z)-1}{p_1(z)+1}\\ &= \frac{\frac{(n_1+1)(1-B)+m(A-1)}{m(A-B)}\sum\limits_{n=n_1+1}^{\infty}A_n z^{n-m}}{2+2\left(\sum\limits_{n=m+1}^{n_1}A_n z^{n-m}+\sum\limits_{n=m}^{\infty}B_n\overline{z}^n z^{-m}\right)+\frac{(n_1+1)(1-B)+m(A-1)}{m(A-B)}\sum\limits_{n=n_1+1}^{\infty}A_n z^{n-m}}\end{aligned}$$

and observe that

$$|\omega_1(z)|<\frac{\frac{(n_1+1)(1-B)+m(A-1)}{m(A-B)}\sum\limits_{n=n_1+1}^{\infty}|A_n|}{2-2\left(\sum\limits_{n=m+1}^{n_1}|A_n|+\sum\limits_{n=m}^{\infty}|B_n|\right)-\frac{(n_1+1)(1-B)+m(A-1)}{m(A-B)}\sum\limits_{n=n_1+1}^{\infty}|A_n|}\leq 1,$$

if (12) holds. Similarly, for upper bound of (8), let

$$p_2(z)=\frac{(n_1+1)(1-B)+m(A-1)}{m(A-B)}\left[\frac{(n_1+1)(1-B)+m(A-1)+m(A-B)}{(n_1+1)(1-B)+m(A-1)}-\frac{F(z)}{F_{n_1}(z)}\right] \tag{15}$$

which is analytic in $\mathbb{U}$ with $p_2(0)=1$. Now we need to show that

$$p_2(z)=\frac{1+\omega_2(z)}{1-\omega_2(z)} \tag{16}$$

with $\omega_2(0)=0$ and $|\omega_2(z)|<1$ in $\mathbb{U}$. From (15) and (16), we get that for $z\in\mathbb{U}$,

$$\begin{aligned}\omega_2(z) &= \frac{p_2(z)-1}{p_2(z)+1}\\ &= \frac{-\frac{(n_1+1)(1-B)+m(A-1)}{m(A-B)}\sum\limits_{n=n_1+1}^{\infty}A_n z^{n-m}}{2+2\left(\sum\limits_{n=m+1}^{n_1}A_n z^{n-m}+\sum\limits_{n=m}^{\infty}B_n\overline{z}^n z^{-m}\right)-\frac{(n_1+1)(1-B)+m(A-1)}{m(A-B)}\sum\limits_{n=n_1+1}^{\infty}A_n z^{n-m}}\end{aligned}$$

and

$$|\omega_2(z)|<\frac{\frac{(n_1+1)(1-B)+m(A-1)}{m(A-B)}\sum\limits_{n=n_1+1}^{\infty}|A_n|}{2-2\left(\sum\limits_{n=m+1}^{n_1}|A_n|+\sum\limits_{n=m}^{\infty}|B_n|\right)-\frac{(n_1+1)(1-B)+m(A-1)}{m(A-B)}\sum\limits_{n=n_1+1}^{\infty}|A_n|}\leq 1,$$

if (12) holds. For sharpness, we see that for the function $F(z)$ given by (9), if $z=re^{i\pi/(n_1+1-m)}$ $(n_1\geq m+1; r<1)$, we get equality at the left hand side of (8):

$$\begin{aligned}\frac{F(z)}{F_{n_1}(z)} &= 1+\frac{m(A-B)}{(n_1+1)(1-B)+m(A-1)}z^{n_1+1-m}\to 1-\frac{m(A-B)r^{n_1+1-m}}{(n_1+1)(1-B)+m(A-1)}\\ &= \frac{(n_1+1)(1-B)+m(A-1)-m(A-B)}{m(A-B)},\ \text{as } r\to 1^-\end{aligned}$$

and we get equality at the right hand side of (8), if $z=re^{2\pi i/(n_1+1-m)}$ $(n_1\geq m+1; r<1)$ as follows:

$$\begin{aligned}\frac{F(z)}{F_{n_1}(z)} &= 1+\frac{m(A-B)}{(n_1+1)(1-B)+m(A-1)}z^{n_1+1-m}\to 1+\frac{m(A-B)r^{n_1+1-m}}{(n_1+1)(1-B)+m(A-1)}\\ &= \frac{(n_1+1)(1-B)+m(A-1)+m(A-B)}{m(A-B)},\ \text{as } r\to 1^-.\end{aligned}$$

This completes the proof of Theorem 2.1. □

Theorem 2.2. *Let $F(z)$ of the form (5) be in the class $P_H(m, A, B)$, then*

$$\frac{(n_1+1)(1-B)+m(A-1)}{(n_1+1)(1-B)+m(A-1)+m(A-B)} \le \Re\left\{\frac{F_{n_1}(z)}{F(z)}\right\} \le \frac{(n_1+1)(1-B)+m(A-1)}{(n_1+1)(1-B)+m(A-1)-m(A-B)} \quad (z\in\mathbb{U}) \tag{17}$$

The result is sharp for the function given by (9).

Proof. Similar to the proof of Theorem 2.1. Let

$$\frac{1+\omega_3(z)}{1-\omega_3(z)} = \frac{(n_1+1)(1-B)+m(A-1)+m(A-B)}{m(A-B)}\left[\frac{F_{n_1}(z)}{F(z)} - \frac{(n_1+1)(1-B)+m(A-1)}{(n_1+1)(1-B)+m(A-1)+m(A-B)}\right]$$
$$= \frac{1+\sum\limits_{n=m+1}^{n_1} A_n z^{n-m} + \sum\limits_{n=m}^{\infty} B_n \overline{z}^n z^{-m} - \frac{(n_1+1)(1-B)+m(A-1)}{m(A-B)} \sum\limits_{n=n_1+1}^{\infty} A_n z^{n-m}}{1+\sum\limits_{n=m+1}^{n_1} A_n z^{n-m} + \sum\limits_{n=m}^{\infty} B_n \overline{z}^n z^{-m}}.$$

We see that

$$|\omega_3(z)| < \frac{\frac{(n_1+1)(1-B)+m(A-1)+m(A-B)}{m(A-B)} \sum\limits_{n=n_1+1}^{\infty} |A_n|}{2-2\left(\sum\limits_{n=m+1}^{n_1} |A_n| + \sum\limits_{n=m}^{\infty} |B_n|\right) - \frac{(n_1+1)(1-B)+m(A-1)-m(A-B)}{m(A-B)} \sum\limits_{n=n_1+1}^{\infty} |A_n|} \le 1,$$

if (12) holds. This proves the left-hand side of (17). Further, let

$$\frac{1+\omega_4(z)}{1-\omega_4(z)} = \frac{(n_1+1)(1-B)+m(A-1)-m(A-B)}{m(A-B)}\left[\frac{(n_1+1)(1-B)+m(A-1)}{(n_1+1)(1-B)+m(A-1)-m(A-B)} - \frac{F_{n_1}(z)}{F(z)}\right]$$
$$= \frac{1+\sum\limits_{n=m+1}^{n_1} A_n z^{n-m} + \sum\limits_{n=m}^{\infty} B_n \overline{z}^n z^{-m} + \frac{(n_1+1)(1-B)+m(A-1)}{m(A-B)} \sum\limits_{n=n_1+1}^{\infty} A_n z^{n-m}}{1+\sum\limits_{n=m+1}^{n_1} A_n z^{n-m} + \sum\limits_{n=m}^{\infty} B_n \overline{z}^n z^{-m}}.$$

Now to prove right-hand side of inequality (17), we see that

$$|\omega_4(z)| < \frac{\frac{(n_1+1)(1-B)+m(A-1)-m(A-B)}{m(A-B)} \sum\limits_{n=n_1+1}^{\infty} |A_n|}{2-2\left(\sum\limits_{n=m+1}^{n_1} |A_n| + \sum\limits_{n=m}^{\infty} |B_n|\right) - \frac{(n_1+1)(1-B)+m(A-1)+m(A-B)}{m(A-B)} \sum\limits_{n=n_1+1}^{\infty} |A_n|} \le 1,$$

if (12) holds. This completes the proof of Theorem 2.2. Sharpness can be verified for the function (9) similar to the proof of Theorem 2.1. □

Theorem 2.3. *Let $F(z)$ of the form (5) be in the class $P_H(m, A, B)$, then*

$$\frac{(n_1+1)(1-B)+m(A-1)}{(n_1+1)(1-B)+m(A-1)+(n_1+1)m(A-B)} \le \Re\left\{\frac{F'_{n_1}(z)}{F'(z)}\right\} \le \frac{(n_1+1)(1-B)+m(A-1)}{(n_1+1)(1-B)+m(A-1)-(n_1+1)m(A-B)} \quad (z\in\mathbb{U}). \tag{18}$$

The result is sharp for the function given by (9).

Proof. Since $F \in P_H(m, A, B)$ by Definition 1.1 we have (10). Again since $\frac{n(1-B)+m(A-1)}{m(A-B)}$ and $\frac{n(1-B)-m(A-1)}{m(A-B)}$ are increasing functions of n, so we observe that

$$\frac{n(1-B)+m(A-1)}{m(A-B)} \ge \frac{n}{m} \quad (n\ge m+1), \quad \frac{n(1-B)-m(A-1)}{m(A-B)} \ge \frac{n}{m} \quad (n \ge m)$$

and

$$\frac{n\,(1-B)+m\,(A-1)}{m\,(A-B)} \geq \frac{(n_1+1)\,(1-B)+m\,(A-1)}{(n_1+1)\,m\,(A-B)}\,\frac{n}{m}\quad (n \geq n_1+1)\,.$$

So, from (10) we obtain

$$\sum_{n=m+1}^{n_1} \frac{n}{m}\,|A_n| + \sum_{n=m}^{\infty} \frac{n}{m}\,|B_n| + \frac{(n_1+1)\,(1-B)+m\,(A-1)}{(n_1+1)\,m\,(A-B)} \sum_{n=n_1+1}^{\infty} \frac{n}{m}\,|A_n| \leq 1, \tag{19}$$

where A_n, B_n are given by (7). Let

$$\begin{aligned} p_5(z) = {} & \frac{(n_1+1)\,(1-B)+m\,(A-1)+(n_1+1)\,m\,(A-B)}{(n_1+1)m\,(A-B)} \\ & \left[\frac{F'_{n_1}(z)}{F'(z)} - \frac{(n_1+1)\,(1-B)+m\,(A-1)}{(n_1+1)\,(1-B)+m\,(A-1)+(n_1+1)m\,(A-B)}\right] \end{aligned} \tag{20}$$

which is analytic in $\mathbb{U}$ with $p_5(0)=1$. To obtain the lower bound of (18), we need to show that

$$p_5(z) = \frac{1+\omega_5(z)}{1-\omega_5(z)}, \tag{21}$$

where $\omega_5(z)$ is a Schwarz function with $\omega_5(0)=0$ and $|\omega_5(z)|<1$. From (20) and (21), we get for $z \in \mathbb{U}$ that

$$\begin{aligned} \omega_5(z) &= \frac{p_5(z)-1}{p_5(z)+1} \\ &= \frac{-\frac{(n_1+1)(1-B)+m(A-1)+m(n_1+1)(A-B)}{(n_1+1)m(A-B)} \sum\limits_{n=n_1+1}^{\infty} \frac{n}{m} A_n z^{n-m}}{2+2\left(\sum\limits_{n=m+1}^{n_1} \frac{n}{m} A_n z^{n-m} - \sum\limits_{n=m}^{\infty} \frac{n}{m} B_n \overline{z}^n z^{-m}\right) - \frac{(n_1+1)(1-B)+m(A-1)-(n_1+1)m(A-B)}{(n_1+1)m(A-B)} \sum\limits_{n=n_1+1}^{\infty} \frac{n}{m} A_n z^{n-m}} \end{aligned}$$

and

$$|\omega_5(z)| < \frac{\frac{(n_1+1)(1-B)+m(A-1)+(n_1+1)m(A-B)}{(n_1+1)m(A-B)} \sum\limits_{n=n_1+1}^{\infty} \frac{n}{m}\,|A_n|}{2-2\left(\sum\limits_{n=m+1}^{n_1} \frac{n}{m}\,|A_n| + \sum\limits_{n=m}^{\infty} \frac{n}{m}\,|B_n|\right) - \frac{(n_1+1)(1-B)+m(A-1)-(n_1+1)m(A-B)}{(n_1+1)m(A-B)} \sum\limits_{n=n_1+1}^{\infty} \frac{n}{m}\,|A_n|} \leq 1$$

if (19) holds. Similarly to prove upper bound of (18), we consider

$$\begin{aligned} p_6(z) = {} & \frac{(n_1+1)\,(1-B)+m\,(A-1)-(n_1+1)m\,(A-B)}{(n_1+1)m\,(A-B)} \\ & \left[\frac{(n_1+1)\,(1-B)+m\,(A-1)}{(n_1+1)\,(1-B)+m\,(A-1)-(n_1+1)m\,(A-B)} - \frac{F'_{n_1}(z)}{F'(z)}\right] \end{aligned} \tag{22}$$

$$= \frac{1+\omega_6(z)}{1-\omega_6(z)} \tag{23}$$

with $|\omega_6(z)|<1$ in $\mathbb{U}$. From (22) and (23), we have for $z \in \mathbb{U}$ that

$$\begin{aligned} \omega_6(z) &= \frac{p_6(z)-1}{p_6(z)+1} \\ &= \frac{\frac{(n_1+1)(1-B)+m(A-1)-(n_1+1)m(A-B)}{(n_1+1)m(A-B)} \sum\limits_{n=n_1+1}^{\infty} \frac{n}{m} z^{n-m}}{2+2\left(\sum\limits_{n=m+1}^{n_1} \frac{n}{m} A_n z^{n-m} - \sum\limits_{n=m}^{\infty} \frac{n}{m} B_n \overline{z}^n z^{-m}\right) + \frac{(n_1+1)(1-B)+m(A-1)+(n_1+1)m(A-B)}{(n_1+1)m(A-B)} \sum\limits_{n=n_1+1}^{\infty} \frac{n}{m} A_n z^{n-m}}, \end{aligned}$$

and

$$|\omega_6(z)| < \frac{\frac{(n_1+1)(1-B)+m(A-1)-(n_1+1)m(A-B)}{(n_1+1)m(A-B)} \sum\limits_{n=n_1+1}^{\infty} \frac{n}{m}\,|A_n|}{2-2\left(\sum\limits_{n=m+1}^{n_1} \frac{n}{m}\,|A_n| + \sum\limits_{n=m}^{\infty} \frac{n}{m}\,|B_n|\right) - \frac{(n_1+1)(1-B)+m(A-1)+(n_1+1)(A-B)}{(n_1+1)m(A-B)} \sum\limits_{n=n_1+1}^{\infty} \frac{n}{m}\,|A_n|} \leq 1,$$

if (19) holds. This completes the proof of Theorem 2.3. Sharpness can be verified for the function (9) similar to Theorem 2.1. □

Theorem 2.4. *Let $F(z)$ of the form (5) be in the class $P_H(m,A,B)$, then*

$$\frac{(n_1+1)(1-B)+m(A-1)-(n_1+1)m(A-B)}{(n_1+1)(1-B)+m(A-1)} \le \Re\left\{\frac{F'(z)}{F'_{n_1}(z)}\right\} \le \frac{(n_1+1)(1-B)+m(A-1)+(n_1+1)m(A-B)}{(n_1+1)(1-B)+m(A-1)} \quad (z\in\mathbb{U}).$$

The result is sharp for the function given by (9).

Proof. The proof of the above theorem is based on the proof of Theorem 2.3, so we omit the details involved. □

We next determine bounds for $\frac{F(z)}{F_{n_2}(z)}$ and $\frac{F_{n_2}(z)}{F(z)}$.

Theorem 2.5. *Let $F(z)$ of the form (5) be in the class $P_H(m,A,B)$, then*

$$\frac{(n_2+1)(1-B)-m(A-1)-m(A-B)}{(n_2+1)(1-B)-m(A-1)} \le \Re\left\{\frac{F(z)}{F_{n_2}(z)}\right\} \le \frac{(n_2+1)(1-B)-m(A-1)+m(A-B)}{(n_2+1)(1-B)-m(A-1)} \quad (z\in\mathbb{U}). \tag{24}$$

The result is sharp for the function

$$F(z)=z^m+\frac{m(A-B)}{(n_2+1)(1-B)-m(A-1)}\overline{z}^{n_2+1}\,(n_2\ge m). \tag{25}$$

Proof. Since $F\in P_H(m,A,B)$, by Definition 1.1 we have the coefficient inequality (10). Also we have

$$\frac{n(1-B)+m(A-1)}{m(A-B)}>1\ \ (n\ge m+1)\ \text{ and }\ \frac{n(1-B)-m(A-1)}{m(A-B)}\ge 1\ \ (n\ge m).$$

Therefore, from coefficient inequality (10), we obtain

$$\sum_{n=m+1}^{\infty}|A_n|+\sum_{n=m}^{n_2}|B_n|+\frac{(n_2+1)(1-B)-m(A-1)}{m(A-B)}\sum_{n=n_2+1}^{\infty}|B_n|\le 1, \tag{26}$$

where A_n, B_n are given by (7). Let

$$p_7(z)=\frac{(n_2+1)(1-B)-m(A-1)}{m(A-B)}\left[\frac{F(z)}{F_{n_2}(z)}-\frac{(n_2+1)(1-B)-m(A-1)-m(A-B)}{(n_2+1)(1-B)-m(A-1)}\right] \tag{27}$$

which is analytic in $\mathbb{U}$ with $p_7(0)=1$. To obtain the lower bound of (24), we need to show that

$$p_7(z)=\frac{1+\omega_7(z)}{1-\omega_7(z)}, \tag{28}$$

where $\omega_7(z)$ is a Schwarz function with $\omega_7(0)=0$ and $|\omega_7(z)|<1$ in $\mathbb{U}$. Evidently, from (27) and (28), we get

$$\omega_7(z)=\frac{p_7(z)-1}{p_7(z)+1}\quad (z\in\mathbb{U})$$
$$=\frac{\frac{(n_2+1)(1-B)-m(A-1)}{m(A-B)}\sum\limits_{n=n_2+1}^{\infty}B_n\overline{z}^nz^{-m}}{2+2\left(\sum\limits_{n=m+1}^{\infty}A_nz^{n-m}+\sum\limits_{n=m}^{n_2}B_n\overline{z}^nz^{-m}\right)+\frac{(n_2+1)(1-B)-m(A-1)}{m(A-B)}\sum\limits_{n=n_2+1}^{\infty}B_n\overline{z}^nz^{-m}}\quad (n_2\ge m; z\in\mathbb{U}).$$

We observe that

$$|\omega_7(z)| < \frac{\frac{(n_2+1)(1-B)-m(A-1)}{m(A-B)} \sum\limits_{n=n_2+1}^{\infty} B_n}{2-2\left(\sum\limits_{n=m+1}^{\infty} A_n + \sum\limits_{n=m}^{n_2} B_n\right) - \frac{(n_2+1)(1-B)-m(A-1)}{m(A-B)} \sum\limits_{n=n_2+1}^{\infty} B_n} \leq 1$$

if (26) holds. Similarly for upper bound of (24), let

$$p_8(z) = \frac{(n_2+1)(1-B)-m(A-1)}{m(A-B)} \left[\frac{(n_2+1)(1-B)-m(A-1)+m(A-B)}{(n_2+1)(1-B)-m(A-1)} - \frac{F(z)}{F_{n_2}(z)}\right] \tag{29}$$

which is analytic in $\mathbb{U}$ with $p_8(0) = 1$. Now we need to show that

$$p_8(z) = \frac{1+\omega_8(z)}{1-\omega_8(z)} \tag{30}$$

with $\omega_8(0) = 0$ and $|\omega_2(z)| < 1$ in $\mathbb{U}$. From (29) and (30), we get for $z \in \mathbb{U}$,

$$\begin{aligned}\omega_8(z) &= \frac{p_8(z)-1}{p_8(z)+1} \quad (z \in \mathbb{U}) \\ &= \frac{-\frac{(n_2+1)(1-B)-m(A-1)}{m(A-B)} \sum\limits_{n=n_2+1}^{\infty} B_n \overline{z}^n z^{-m}}{2+2\left(\sum\limits_{n=m+1}^{\infty} A_n z^{n-m} + \sum\limits_{n=m}^{n_2} B_n \overline{z}^n z^{-m}\right) - \frac{(n_2+1)(1-B)-m(A-1)}{m(A-B)} \sum\limits_{n=n_2+1}^{\infty} B_n \overline{z}^n z^{-m}} \\ &\quad (n_2 \geq m).\end{aligned}$$

Now

$$|\omega_8(z)| < \frac{\frac{(n_2+1)(1-B)-m(A-1)}{m(A-B)} \sum\limits_{n=n_2+1}^{\infty} |B_n|}{2-2\left(\sum\limits_{n=m+1}^{\infty} |A_n| + \sum\limits_{n=m}^{n_2} |B_n|\right) - \frac{(n_2+1)(1-B)-m(A-1)}{m(A-B)} \sum\limits_{n=n_2+1}^{\infty} |B_n|} \leq 1$$

if (26) holds. For sharpness, we see that for the function given by (25), if $z = re^{\pi i/(n_2+1-m)}$ $(r < 1)$ we get equality at the left hand side of (24) as follows:

$$\begin{aligned}\frac{F(z)}{F_{n_2}(z)} &= 1 + \frac{m(A-B)}{(n_2+1)(1-B)-m(A-1)} r^{n_2+1-m} e^{-i\pi} \quad (n_2 \geq m) \\ &\to 1 - \frac{m(A-B)}{(n_2+1)(1-B)-m(A-1)} r^{n_2+1-m} \\ &= \frac{(n_2+1)(1-B)-m(A-1)-m(A-B)}{(n_2+1)(1-B)-m(A-1)} \text{ as } r \to 1^-\end{aligned}$$

and we get equality at the right hand side of (24), if $z = re^{2\pi i/(n_2+1-m)}$ $(r < 1)$ as follows:

$$\begin{aligned}\frac{F(z)}{F_{n_2}(z)} &= 1 + \frac{m(A-B)}{(n_2+1)(1-B)-m(A-1)} r^{n_2+1-m} e^{2\pi i} \quad (n_2 \geq m) \\ &\to 1 + \frac{m(A-B)}{(n_2+1)(1-B)-m(A-1)} r^{n_2+1-m} \\ &= \frac{(n_2+1)(1-B)-m(A-1)+m(A-B)}{(n_2+1)(1-B)-m(A-1)} \text{ as } r \to 1^-.\end{aligned}$$

This completes the proof of Theorem 2.5. □

Theorem 2.6. *Let $F(z)$ of the form (5) be in the class $P_H(m, A, B)$, then*

$$\begin{aligned}\frac{(n_2+1)(1-B)-m(A-1)}{(n_2+1)(1-B)-m(A-1)+m(A-B)} &\leq \Re\left\{\frac{F_{n_2}(z)}{F(z)}\right\} \\ &\leq \frac{(n_2+1)(1-B)-m(A-1)}{(n_2+1)(1-B)-m(A-1)-m(A-B)} \quad (z \in \mathbb{U}).\end{aligned} \tag{31}$$

The result is sharp for the function given by (25).

Proof. Similar to the proof of Theorem 2.5, let

$$\frac{1+\omega_9(z)}{1-\omega_9(z)}=\frac{(n_2+1)(1-B)-m(A-1)+m(A-B)}{m(A-B)}\left[\frac{F_{n_2}(z)}{F(z)}-\frac{(n_2+1)(1-B)-m(A-1)}{(n_2+1)(1-B)-m(A-1)+m(A-B)}\right]$$
$$=\frac{1+\sum_{n=m+1}^{\infty}A_nz^{n-m}+\sum_{n=m}^{n_2}B_n\overline{z}^nz^{-m}-\frac{(n_2+1)(1-B)-m(A-1)}{m(A-B)}\sum_{n=n_2+1}^{\infty}B_n\overline{z}^nz^{-m}}{1+\sum_{n=m+1}^{\infty}A_nz^{n-m}+\sum_{n=m}^{\infty}B_n\overline{z}^nz^{-m}}\quad(n_2\geq m; z\in\mathbb{U}).$$

We see that

$$|\omega_9(z)|<\frac{\frac{(n_2+1)(1-B)-m(A-1)-m(A-B)}{m(A-B)}\sum_{n=n_2+1}^{\infty}|B_n|}{2-2\left(\sum_{n=m+1}^{\infty}|A_n|+\sum_{n=m}^{n_2}|B_n|\right)-\frac{(n_2+1)(1-B)-m(A-1)+m(A-B)}{m(A-B)}\sum_{n=n_2+1}^{\infty}|B_n|}\leq 1,$$

if (26) holds. This proves the left-hand side of (31). Further, let

$$\frac{1+\omega_{10}(z)}{1-\omega_{10}(z)}=\frac{(n_2+1)(1-B)-m(A-1)-m(A-B)}{m(A-B)}\left[\frac{(n_2+1)(1-B)-m(A-1)}{(n_2+1)(1-B)-m(A-1)-m(A-B)}-\frac{F_{n_2}(z)}{F(z)}\right]$$
$$=\frac{1+\sum_{n=m+1}^{\infty}A_nz^{n-m}+\sum_{n=m}^{n_2}B_n\overline{z}^nz^{-m}+\frac{(n_2+1)(1-B)-m(A-1)}{m(A-B)}\sum_{n=n_2+1}^{\infty}B_n\overline{z}^nz^{-m}}{1+\sum_{n=m+1}^{\infty}A_nz^{n-m}+\sum_{n=m}^{\infty}B_n\overline{z}^nB_n\overline{z}^nz^{-m}}\quad(n_2\geq m; z\in\mathbb{U}).$$

Now to prove right-hand side of inequality (31), we see that

$$|\omega_{10}(z)|<\frac{\frac{(n_2+1)(1-B)-m(A-1)+m(A-B)}{m(A-B)}\sum_{n=n_2+1}^{\infty}|B_n|}{2-2\left(\sum_{n=m+1}^{\infty}|A_n|+\sum_{n=m}^{n_2}|B_n|\right)-\frac{(n_2+1)(1-B)-m(A-1)-m(A-B)}{m(A-B)}\sum_{n=n_2+1}^{\infty}|B_n|}\leq 1,$$

if (26) holds. This complete the proof of Theorem 2.5. Sharpness can be verified for the function (25) similar to Theorem 2.5. □

We next give bounds for $\frac{F'_{n_2}(z)}{F'(z)}$ and $\frac{F'(z)}{F'_{n_2}(z)}$, in the form of Theorems 2.7 and 2.8 without proof as the proofs are similar to that of Theorem 2.5.

Theorem 2.7. *Let $F(z)$ of the form (5) be in the class $P_H(m,A,B)$, then*

$$\frac{(n_2+1)(1-B)-m(A-1)}{(n_2+1)(1-B)-m(A-1)+(n_2+1)m(A-B)}\leq\Re\left\{\frac{F'_{n_2}(z)}{F'(z)}\right\}$$
$$\leq\frac{(n_2+1)(1-B)-m(A-1)}{(n_2+1)(1-B)-m(A-1)-(n_2+1)m(A-B)}\quad(z\in\mathbb{U}).$$

The result is sharp for the function given by (25).

Theorem 2.8. *Let $F(z)$ of the form (5) be in the class $P_H(m,A,B)$, then*

$$\frac{(n_2+1)(1-B)-m(A-1)-(n_2+1)m(A-B)}{(n_2+1)m(1-B)-m(A-1)}\leq\Re\left\{\frac{F'(z)}{F'_{n_2}(z)}\right\}$$
$$\leq\frac{(n_2+1)(1-B)-m(A-1)+(n_2+1)m(A-B)}{(n_2+1)m(1-B)-m(A-1)}\quad(z\in\mathbb{U}).$$

The result is sharp for the function given by (25).

We next determine bounds for $\frac{F(z)}{F_{n_1,n_2}(z)}$ and $\frac{F_{n_1,n_2}(z)}{F(z)}$.

Theorem 2.9. *Let $F(z)$ of the form (5) be in the class $P_H(m,A,B)$, then*

(i)

$$\frac{(n_1+1)(1-B)+m(A-1)-m(A-B)}{(n_1+1)(1-B)+m(A-1)} \le \Re\left\{\frac{F(z)}{F_{n_1,n_2}(z)}\right\} \le \frac{(n_1+1)(1-B)+m(A-1)+m(A-B)}{(n_1+1)(1-B)+m(A-1)} \quad (n_2 \ge n_1 \ge m+1; z \in \mathbb{U}). \tag{32}$$

(ii)

$$\frac{(n_2+1)(1-B)-m(A-1)-m(A-B)}{(n_2+1)(1-B)-m(A-1)} \le \Re\left\{\frac{F(z)}{F_{n_1,n_2}(z)}\right\} \le \frac{(n_2+1)(1-B)-m(A-1)+m(A-B)}{(n_2+1)(1-B)-m(A-1)} \quad (n_1 \ge \max(n_2, m+1); z \in \mathbb{U}). \tag{33}$$

The above results (i) and (ii) are sharp for the functions given, respectively, by (9) and (25).

Proof. (i) Since $F \in P_H(m,A,B)$, by Definition 1.1 we have inequality (10), and the observation 11. Further, observe that

$$\frac{n(1-B)+m(A-1)}{m(A-B)} \ge \frac{(n_1+1)(1-B)+m(A-1)}{m(A-B)} \quad (n \ge n_1+1, n \ge n_2+1).$$

Therefore, from coefficient inequality (10), and since $n_1 \le n_2$ we obtain

$$\sum_{n=m+1}^{n_1}|A_n| + \sum_{n=m}^{n_2}|B_n| + \frac{(n_1+1)(1-B)+m(A-1)}{m(A-B)}\left[\sum_{n=n_1+1}^{\infty}|A_n| + \sum_{n=n_2+1}^{\infty}|B_n|\right] \le 1, \tag{34}$$

where A_n, B_n are given by (7). Let

$$p_{11}(z) = \frac{(n_1+1)(1-B)+m(A-1)}{m(A-B)}\left[\frac{F(z)}{F_{n_1,n_2}(z)} - \frac{(n_1+1)(1-B)+m(A-1)-m(A-B)}{(n_1+1)(1-B)+m(A-1)}\right] \tag{35}$$

which is analytic in $\mathbb{U}$ with $p_{11}(0)=1$. To obtain lower bound of (32), we need to show that

$$p_{11}(z) = \frac{1+\omega_{11}(z)}{1-\omega_{11}(z)}, \tag{36}$$

where $\omega_{11}(z)$ is a Schwarz function with $\omega_{11}(0)=0$ and $|\omega_{11}(z)|<1$ in $\mathbb{U}$. Evidently, from (35) and (28), we get

$$\omega_{11}(z) = \frac{p_{11}(z)-1}{p_{11}(z)+1} = \frac{\frac{(n_1+1)(1-B)+m(A-1)}{m(A-B)}\left[\sum\limits_{n=n_1+1}^{\infty}A_n z^{n-m} + \sum\limits_{n=n_2+1}^{\infty}B_n\overline{z}^n z^{-m}\right]}{2+2\left(\sum\limits_{n=m+1}^{n_1}A_n z^{n-m} + \sum\limits_{n=m}^{n_2}B_n\overline{z}^n z^{-m}\right) + \frac{(n_1+1)(1-B)+m(A-1)}{mA-B}\left[\sum\limits_{n=n_1+1}^{\infty}A_n z^{n-m} + \sum\limits_{n=n_2+1}^{\infty}B_n\overline{z}^n z^{-m}\right]}$$

$(n_2 \ge m, n_1 \ge m+1; z \in \mathbb{U})$ and hence,

$$|\omega_{11}(z)| < \frac{\frac{(n_1+1)(1-B)+m(A-1)}{m(A-B)}\left[\sum\limits_{n=n_1+1}^{\infty}|A_n| + \sum\limits_{n=n_2+1}^{\infty}|B_n|\right]}{2-2\left(\sum\limits_{n=m+1}^{n_1}|A_n| + \sum\limits_{n=m}^{n_2}|B_n|\right) - \frac{(n_1+1)(1-B)+m(A-1)}{m(A-B)}\left[\sum\limits_{n=n_1+1}^{\infty}|A_n| + \sum\limits_{n=n_2+1}^{\infty}|B_n|\right]} \le 1$$

if (34) holds. Similarly for upper bound of (32), let

$$p_{12}(z) = \frac{(n_1+1)(1-B)+m(A-1)}{m(A-B)}\left[\frac{(n_1+1)(1-B)+m(A-1)+m(A-B)}{(n_1+1)(1-B)+m(A-1)} - \frac{F(z)}{F_{n_1,n_2}(z)}\right] \tag{37}$$

which is analytic in $\mathbb{U}$ with $p_{12}(0) = 1$. Now we need to show that

$$p_{12}(z) = \frac{1+\omega_{12}(z)}{1-\omega_{12}(z)} \tag{38}$$

with $\omega_{12}(0) = 0$ and $|\omega_{12}(z)| < 1$ in $\mathbb{U}$. From (37) and (38), we get

$$\omega_{12}(z) = \frac{p_{12}(z)-1}{p_{12}(z)+1}$$

$$= \frac{-\frac{(n_1+1)(1-B)+m(A-1)}{m(A-B)}\left[\sum\limits_{n=n_1+1}^{\infty} A_n z^{n-m} + \sum\limits_{n=n_2+1}^{\infty} B_n \overline{z}^n z^{-m}\right]}{2+2\left(\sum\limits_{n=m+1}^{n_1} A_n z^{n-m} + \sum\limits_{n=m}^{n_2} B_n \overline{z}^n z^{-m}\right) - \frac{(n_1+1)(1-B)+m(A-1)}{m(A-B)}\left[\sum\limits_{n=n_1+1}^{\infty} A_n z^{n-m} + \sum\limits_{n=n_2+1}^{\infty} B_n \overline{z}^n z^{-m}\right]}$$

$(n_2 \geq m, n_1 \geq m+1; z \in \mathbb{U})$. Hence

$$|\omega_{12}(z)| < \frac{\frac{(n_1+1)(1-B)+m(A-1)}{m(A-B)}\left[\sum\limits_{n=n_1+1}^{\infty} |A_n| + \sum\limits_{n=n_2+1}^{\infty} |B_n|\right]}{2-2\left(\sum\limits_{n=m+1}^{n_1} |A_n| + \sum\limits_{n=m}^{n_2} |B_n|\right) - \frac{(n_1+1)(1-B)+m(A-1)}{m(A-B)}\left[\sum\limits_{n=n_1+1}^{\infty} A_n + \sum\limits_{n=n_2+1}^{\infty} B_n\right]} \leq 1,$$

if (34) holds.

(ii) Since $F \in P_H(m, A, B)$, by Definition 1.1 we have the coefficient inequality (10) and the observation (11). Further we observe that

$$\frac{n(1-B)+m(A-1)}{m(A-B)} \geq \frac{(n_1+1)(1-B)+m(A-1)}{m(A-B)} \quad (n \geq n_1+1)$$

and

$$\frac{n(1-B)-m(A-1)}{m(A-B)} \geq \frac{(n_2+1)(1-B)-m(A-1)}{m(A-B)} \quad (n \geq n_2+1).$$

Therefore, from (10) by using the fact that $n_1 \geq n_2$, we obtain

$$\sum_{n=m+1}^{n_1} |A_n| + \sum_{n=m}^{n_2} |B_n| + \frac{(n_2+1)(1-B)-m(A-1)}{m(A-B)}\left[\sum_{n=n_1+1}^{\infty} |A_n| + \sum_{n=n_2+1}^{\infty} |B_n|\right] \leq 1, \tag{39}$$

where A_n, B_n are given by (7). Let

$$p_{13}(z) = \frac{(n_2+1)(1-B)-m(A-1)}{m(A-B)}\left[\frac{F(z)}{F_{n_1,n_2}(z)} - \frac{(n_2+1)(1-B)-m(A-1)-m(A-B)}{(n_2+1)(1-B)-m(A-1)}\right] \tag{40}$$

which is analytic in $\mathbb{U}$ with $p_{13}(0) = 1$. To obtain lower bound of (33), we need to show that

$$p_{13}(z) = \frac{1+\omega_{13}(z)}{1-\omega_{13}(z)}, \tag{41}$$

where $\omega_{13}(z)$ is a Schwarz function with $\omega_{13}(0) = 0$ and $|\omega_{13}(z)| < 1$ in $\mathbb{U}$. Evidently, from (40) and (41), we get

$$\omega_{13}(z) = \frac{p_{13}(z)-1}{p_{13}(z)+1}$$

$$= \frac{\frac{(n_2+1)(1-B)-m(A-1)}{m(A-B)}\left[\sum\limits_{n=n_1+1}^{\infty} A_n z^{n-m} + \sum\limits_{n=n_2+1}^{\infty} B_n \overline{z}^n z^{-m}\right]}{2+2\left(\sum\limits_{n=m+1}^{n_1} A_n z^{n-m} + \sum\limits_{n=m}^{n_2} B_n \overline{z}^n z^{-m}\right) + \frac{(n_2+1)(1-B)-m(A-1)}{m(A-B)}\left[\sum\limits_{n=n_1+1}^{\infty} A_n z^{n-m} + \sum\limits_{n=n_2+1}^{\infty} B_n \overline{z}^n z^{-m}\right]}$$

$(n_2 \geq m, n_1 \geq m+1; z \in \mathbb{U})$. We observe that

$$|\omega_{13}(z)| < \frac{\frac{(n_2+1)(1-B)-m(A-1)}{m(A-B)}\left[\sum\limits_{n=n_2+1}^{\infty}|A_n| + \sum\limits_{n=n_2+1}^{\infty}|B_n|\right]}{2-2\left(\sum\limits_{n=m+1}^{n_1}|A_n| + \sum\limits_{n=m}^{n_2}B_n\right) - \frac{(n_2+1)(1-B)-m(A-1)}{m(A-B)}\left[\sum\limits_{n=n_2+1}^{\infty}|A_n| + \sum\limits_{n=n_2+1}^{\infty}|B_n|\right]} \leq 1,$$

if (39) holds. Similarly for upper bound of (33), let

$$p_{14}(z) = \frac{(n_2+1)(1-B)-m(A-1)}{m(A-B)}\left[\frac{(n_2+1)(1-B)-m(A-1)+m(A-B)}{(n_2+1)(1-B)-m(A-1)} - \frac{F(z)}{F_{n_1,n_2}(z)}\right] \tag{42}$$

which is analytic in $\mathbb{U}$ with $p_{14}(0) = 1$. Now we need to show that

$$p_{14}(z) = \frac{1+\omega_{14}(z)}{1-\omega_{14}(z)} \tag{43}$$

with $\omega_{14}(0) = 0$ and $|\omega_{14}(z)| < 1$ in $\mathbb{U}$. From (42) and (43), we get

$$\begin{aligned}\omega_{14}(z) &= \frac{p_{14}(z)-1}{p_{14}(z)+1} \\ &= \frac{-\frac{(n_2+1)(1-B)-m(A-1)}{m(A-B)}\left[\sum\limits_{n=n_1+1}^{\infty}A_n z^{n-m} + \sum\limits_{n=n_2+1}^{\infty}B_n\overline{z}^n z^{-m}\right]}{2+2\left(\sum\limits_{n=m+1}^{n_1}A_n z^{n-m} + \sum\limits_{n=m}^{n_2}B_n\overline{z}^n z^{-m}\right) - \frac{(n_2+1)(1-B)-m(A-1)}{m(A-B)}\left[\sum\limits_{n=n_1+1}^{\infty}A_n z^{n-m} + \sum\limits_{n=n_2+1}^{\infty}B_n\overline{z}^n z^{-m}\right]}\end{aligned}$$

$(n_2 \geq m, n_1 \geq m+1; z \in \mathbb{U})$. Now

$$|\omega_{14}(z)| < \frac{\frac{(n_2+1)(1-B)-m(A-1)}{m(A-B)}\left[\sum\limits_{n=n_1+1}^{\infty}|A_n| + \sum\limits_{n=n_2+1}^{\infty}|B_n|\right]}{2-2\left(\sum\limits_{n=m+1}^{n_1}|A_n| + \sum\limits_{n=m}^{n_2}|B_n|\right) - \frac{(n_2+1)(1-B)-m(A-1)}{m(A-B)}\left[\sum\limits_{n=n_1+1}^{\infty}|A_n| + \sum\limits_{n=n_2+1}^{\infty}|B_n|\right]} \leq 1$$

if (39) holds. □

Theorem 2.10. *Let $F(z)$ of the form (5) be in the class $P_H(m, A, B)$, then*

(i)

$$\begin{aligned}\frac{(n_1+1)(1-B)+m(A-1)}{(n_1+1)(1-B)+m(A-1)+m(A-B)} &\leq \Re\left\{\frac{F_{n_1,n_2}(z)}{F(z)}\right\} \\ &\leq \frac{(n_1+1)(1-B)+m(A-1)}{(n_1+1)(1-B)+m(A-1)-m(A-B)} \quad (z \in \mathbb{U}).\end{aligned} \tag{44}$$

(ii)

$$\begin{aligned}\frac{(n_2+1)(1-B)-m(A-1)}{(n_2+1)(1-B)-m(A-1)+m(A-B)} &\leq \Re\left\{\frac{F_{n_1,n_2}(z)}{F(z)}\right\} \\ &\leq \frac{(n_2+1)(1-B)-m(A-1)}{(n_2+1)(1-B)-m(A-1)-m(A-B)} \quad (z \in \mathbb{U}).\end{aligned} \tag{45}$$

The above results (i) and (ii) are sharp for the functions given, respectively, by (9) and (25).

Proof. (i) Similar to proof of Theorem 2.9. Let

$$\frac{1+\omega_{15}(z)}{1-\omega_{15}(z)}=\frac{(n_1+1)(1-B)+m(A-1)+m(A-B)}{m(A-B)}\left[\frac{F_{n_1,n_2}(z)}{F(z)}-\frac{(n_1+1)(1-B)+m(A-1)}{(n_1+1)(1-B)+m(A-1)+m(A-B)}\right]$$
$$=\frac{1+\sum\limits_{n=m+1}^{n_1}A_nz^{n-m}+\sum\limits_{n=m}^{n_2}B_n\overline{z}^nz^{-m}}{1+\sum\limits_{n=m+1}^{\infty}A_nz^{n-m}+\sum\limits_{n=m}^{\infty}B_n\overline{z}^nz^{-m}}-\frac{\frac{(n_1+1)(1-B)+m(A-1)}{m(A-B)}\left[\sum\limits_{n=n_1+1}^{\infty}A_nz^{n-m}+\sum\limits_{n=n_2+1}^{\infty}B_n\overline{z}^nz^{-m}\right]}{1+\sum\limits_{n=m+1}^{\infty}A_nz^{n-m}+\sum\limits_{n=m}^{\infty}B_n\overline{z}^nz^{-m}}$$

$(n_2\geq m, n_1\geq m+1; z\in\mathbb{U})$. Now, we see that

$$|\omega_{15}(z)|<\frac{\frac{(n_1+1)(1-B)+m(A-1)+m(A-B)}{m(A-B)}\left[\sum\limits_{n=n_1+1}^{\infty}|A_n|+\sum\limits_{n=n_2+1}^{\infty}|B_n|\right]}{2-2\left(\sum\limits_{n=m+1}^{n_1}|A_n|+\sum\limits_{n=m}^{n_2}|B_n|\right)-\frac{(n_1+1)(1-B)+m(A-1)-m(A-B)}{m(A-B)}\left[\sum\limits_{n=n_1+1}^{\infty}|A_n|+\sum\limits_{n=n_2+1}^{\infty}|B_n|\right]}\leq 1,$$

if (34) holds. This proves the left-hand side of (44). Further, to prove right hand side of inequality (44), let

$$\frac{1+\omega_{16}(z)}{1-\omega_{16}(z)}=\frac{(n_1+1)(1-B)+m(A-1)-m\delta(A-B)}{m(A-B)}\left[\frac{(n_1+1)(1-B)+m(A-1)}{(n_1+1)(1-B)+m(A-1)-m(A-B)}-\frac{F_{n_1,n_2}(z)}{F(z)}\right]$$
$$=\frac{1+\sum\limits_{n=m+1}^{n_1}A_nz^{n-m}+\sum\limits_{n=m}^{n_2}B_n\overline{z}^nz^{-m}}{1+\sum\limits_{n=m+1}^{n_1}A_nz^{n-m}+\sum\limits_{n=m}^{n_2}B_n\overline{z}^nz^{-m}}+\frac{\frac{(n_1+1)(1-B)+m(A-1)}{m(A-B)}\left[\sum\limits_{n=n_1+1}^{\infty}A_nz^{n-m}+\sum\limits_{n=n_2+1}^{\infty}B_n\overline{z}^nz^{-m}\right]}{1+\sum\limits_{n=m+1}^{n_1}A_nz^{n-m}+\sum\limits_{n=m}^{n_2}B_n\overline{z}^nz^{-m}}$$

$(n_2\geq m, n_1\geq m+1; z\in\mathbb{U})$, from which we see that

$$|\omega_{16}(z)|<\frac{\frac{(n_1+1)(1-B)+m(A-1)-m(A-B)}{m(A-B)}\left[\sum\limits_{n=n_1+1}^{\infty}|A_n|+\sum\limits_{n=n_2+1}^{\infty}|B_n|\right]}{2-2\left(\sum\limits_{n=m+1}^{n_1}|A_n|+\sum\limits_{n=m}^{n_2}|B_n|\right)-\frac{(n_1+1)(1-B)+m(A-1)+m(A-B)}{m(A-B)}\left[\sum\limits_{n=n_1+1}^{\infty}|A_n|+\sum\limits_{n=n_2+1}^{\infty}|B_n|\right]}\leq 1,$$

if (34) holds.

(ii) Similar to proof of Theorem 2.9. Let

$$\frac{1+\omega_{17}(z)}{1-\omega_{17}(z)}=\frac{(n_2+1)(1-B)-m(A-1)+m(A-B)}{m(A-B)}\left[\frac{F_{n_1,n_2}(z)}{F(z)}-\frac{(n_2+1)(1-B)-m(A-1)}{(n_2+1)(1-B)-m(A-1)+m(A-B)}\right]$$
$$=\frac{1+\sum\limits_{n=m+1}^{n_1}A_nz^{n-m}+\sum\limits_{n=m}^{n_2}B_n\overline{z}^nz^{-m}}{1+\sum\limits_{n=m+1}^{\infty}A_nz^{n-m}+\sum\limits_{n=m}^{\infty}B_n\overline{z}^nz^{-m}}-\frac{\frac{(n_2+1)(1-B)-m(A-1)}{m(A-B)}\left[\sum\limits_{n=n_1+1}^{\infty}A_nz^{n-m}+\sum\limits_{n=n_2+1}^{\infty}B_n\overline{z}^nz^{-m}\right]}{1+\sum\limits_{n=m+1}^{\infty}A_nz^{n-m}+\sum\limits_{n=m}^{\infty}B_n\overline{z}^nz^{-m}}$$

$(n_2\geq m, n_1\geq m+1; z\in\mathbb{U})$. We see that

$$|\omega_{17}(z)|<\frac{\frac{(n_2+1)(1-B)-m(A-1)+m(A-B)}{m(A-B)}\left[\sum\limits_{n=n_2+1}^{\infty}|A_n|+\sum\limits_{n=n_2+1}^{\infty}|B_n|\right]}{2-2\left(\sum\limits_{n=m+1}^{n_1}|A_n|+\sum\limits_{n=m}^{n_2}|B_n|\right)-\frac{(n_2+1)(1-B)-m(A-1)-m(A-B)}{m(A-B)}\left[\sum\limits_{n=n_1+1}^{\infty}|A_n|+\sum\limits_{n=n_2+1}^{\infty}|B_n|\right]}\leq 1$$

if (39) holds. This proves the left-hand side of (45). Further, to prove right hand side of inequality (45), let

$$\frac{1+\omega_{18}(z)}{1-\omega_{18}(z)}=\frac{(n_2+1)(1-B)-m(A-1)-m(A-B)}{m(A-B)}\left[\frac{(n_2+1)(1-B)-m(A-1)}{(n_2+1)(1-B)-m(A-1)-m(A-B)}-\frac{F_{n_1,n_2}(z)}{F(z)}\right]$$
$$=\frac{1+\sum\limits_{n=m+1}^{n_1}A_nz^{n-m}+\sum\limits_{n=m}^{n_2}B_n\overline{z}^nz^{-m}}{1+\sum\limits_{n=m+1}^{\infty}A_nz^{n-m}+\sum\limits_{n=m}^{\infty}B_n\overline{z}^nz^{-m}}+\frac{\frac{(n_2+1)(1-B)-m(A-1)}{m(A-B)}\left[\sum\limits_{n=n_1+1}^{\infty}A_nz^{n-m}+\sum\limits_{n=m}^{\infty}B_n\overline{z}^nz^{-m}\right]}{1+\sum\limits_{n=m+1}^{\infty}A_nz^{n-m}+\sum\limits_{n=m}^{\infty}B_n\overline{z}^nz^{-m}}$$

$(n_2 \geq m, n_1 \geq m+1; z \in \mathbb{U})$ and we see that

$$|\omega_{18}(z)| < \frac{\frac{(n_2+1)(1-B)-m(A-1)-m(A-B)}{m(A-B)}\left[\sum\limits_{n=n_1+1}^{\infty}|A_n| + \sum\limits_{n=n_2+1}^{\infty}|B_n|\right]}{2-2\left(\sum\limits_{n=m+1}^{n_1}|A_n| + \sum\limits_{n=m}^{n_2}|B_n|\right) - \frac{(n_2+1)(1-B)-m(A-1)+m(A-B)}{m(A-B)}\left[\sum\limits_{n=n_1+1}^{\infty}|A_n| + \sum\limits_{n=n_2+1}^{\infty}|B_n|\right]} \leq 1,$$

if (39) holds. □

Theorem 2.11. *Let $F(z)$ of the form (5) be in the class $P_H(m, A, B)$, then*

$$\frac{(n_1+1)(1-B)+m(A-1)}{(n_1+1)(1-B)+m(A-1)+(n_1+1)(A-B)} \leq \Re\left\{\frac{F'_{n_1,n_2}(z)}{F'(z)}\right\} \leq \frac{(n_1+1)(1-B)+m(A-1)}{(n_1+1)(1-B)+m(A-1)-(n_1+1)(A-B)}$$

$(n_2 \geq m, n_1 \geq m+1; z \in \mathbb{U})$. *The result is sharp for the function given by (9).*

Proof. The proof of the above theorem is based on the similar lines of the proof of Theorem 2.10 so we omit the details involved. □

Theorem 2.12. *Let $F(z)$ of the form (5) be in the class $P_H(m, A, B)$, then*

$$\frac{(n_1+1)(1-B)+m(A-1)-(n_1+1)(A-B)}{(n_1+1)(1-B)+m(A-1)} \leq \Re\left\{\frac{F'(z)}{F'_{n_1,n_2}(z)}\right\} \leq \frac{(n_1+1)(1-B)+m(A-1)+(n_1+1)(A-B)}{(n_1+1)(1-B)+m(A-1)}$$

$(n_2 \geq m, n_1 \geq m+1; z \in \mathbb{U})$. *The result is sharp for the function given by (9).*

Proof. The proof of the above theorem is based on the similar lines of the proof of Theorem 2.10 so we omit the details involved. □

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