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Generalized Discrete α, k-Fourier Transform

Research Article

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Abstract: In this paper, we obtain α, k-discrete Fourier transform and the closed form solutions of various functions by using inverse difference operators α and k. Appropriate examples are inserted to exemplify the main results.

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1. Introduction

The knowledge of Fourier transforms becomes an essential part of engineers and scientists. This provides easy and effective solutions of many problems arising in engineering. Let $R = (-\infty, \infty)$ and $L'(R)$ be the space of Lebesque integrable function define on R. Then the Fourier transform of $f \in L'(R)$ is defined by $F[f(t)] = \int_{-\infty}^{\infty} f(t)e^{ist}dt$. [10] This transform based on integral. In our research, we introduce discrete Fourier transform by using the difference operators $\underset{\alpha(\ell)}{\Delta}$ and $\underset{k(\ell)}{\Delta}$, where α is a parameter, k is a variable and $\ell > 0$. By varying value of ℓ and α we get better solutions. The α, k-discrete Fourier transform becomes discrete Fourier transform and Fourier transform when $\ell = 1$, $\ell \to 0$ and $\alpha \to 1$ respectively [1, 2]. The general theory on Δ_ℓ, Δ_α, $\Delta_{\alpha(\ell)}$ and $\Delta_{k(\ell)}$ one can refer [5]. In 2011, M.Maria Susai Manuel, et.al, [8], have extended the definition of Δ_α to $\Delta_{\alpha(\ell)}$ which is defined as $\Delta_{\alpha(\ell)}v(k) = v(k+\ell) - \alpha v(k)$ for the real valued function $v(k)$, $\ell \in (0, \infty)$. In [9], the authors have used the generalized α-difference equation;

$$v(k+\ell) - \alpha v(k) = u(k),\ k \in [0, \infty),\ \ell \in (0, \infty) \tag{1}$$

From α difference operator [9], if $\underset{\alpha(\ell)}{\Delta} v(k) = u(k)$ then we have

$$v(k) = \Delta^{-1}_{\alpha(\ell)}u(k) - \alpha^{\left[\frac{k}{\ell}\right]}\Delta^{-1}_{\alpha(\ell)}u(\hat{\ell}(k)) = \sum_{r=1}^{\left[\frac{k}{\ell}\right]} \alpha^{r-1}u(k - r\ell),\ \hat{\ell}(k) = k - [k/\ell]\ell \tag{2}$$

By replacing the parameter α by variable k, we define $k-$Difference operator with variable coefficient as

$$\underset{k(\ell)}{\Delta} v(k) = v(k+\ell) - kv(k) \tag{3}$$

In this paper, we establish discrete Fourier transform into α, k-discrete Fourier transform for certain functions, based on that we produce many theorems. Out of which one of the theorem demonstrate the relationship between discrete Laplace and discrete Fourier transforms.

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2. Preliminaries

In this section, we present basic concepts of the Generalized difference operators Δ_ℓ, $\underset{\alpha(\ell)}{\Delta}$, $\underset{k(\ell)}{\Delta}$ and inverse difference operators Δ_ℓ^{-1}, $\overset{-1}{\underset{\alpha(\ell)}{\Delta}}$, $\overset{-1}{\underset{k(\ell)}{\Delta}}$ for finding α, k-Discrete Fourier Transform. In [6, 7] the authors introduce $k_\ell^{(m)} = k(k-\ell)(k-2\ell)\cdots(k-(m-1)\ell)$, the operator Δ_ℓ as $\Delta_\ell u(k) = u(k+\ell) - u(k)$ and its inverse by

$$if\ \Delta_\ell v(k) = u(k),\ then\ v(k) \ = \ \Delta_\ell^{-1} u(k). \tag{4}$$

Let s_r^m and S_r^m are Stirling numbers of first and second kinds respectively, $\ell > 0$, m is non-negative integer and $k_\ell^{(m)} = k(k-\ell)(k-2\ell)\cdots(k-(m-1)\ell)$. From [3] we have the following identities:

$$(i)\ k_\ell^{(m)} = \sum_{r=1}^{m} s_r^m \ell^{m-r} k^r,\ (ii)\ k^m = \sum_{r=1}^{m} S_r^m \ell^{m-r} k_\ell^{(r)},\ (iii)\ \Delta_\ell k_\ell^{(m)} = (m\ell) k_\ell^{(m-1)}, \tag{5}$$

$$(iv)\ \Delta_\ell^{-1} k_\ell^{(m)} = \frac{k_\ell^{(m+1)}}{\ell(m+1)}\ (v)\ \Delta_\ell^{-1} k^m = \sum_{r=1}^{m} \frac{S_r^m \ell^{m-r} k_\ell^{(r)}}{(r+1)\ell}\ (vi)\ \Delta_\ell^{-1} e^{isk} = \frac{e^{isk}}{(e^{is\ell}-1)}, \tag{6}$$

$$(vii)\ \Delta_\ell^{-1} u(k)\Big|_a^b = \sum_{r=0}^{M-1} u(a+r\ell),\ M = \frac{b-a}{\ell}\ and\ (viii)\ \Delta_\ell^{-1} u(k)\Big|_0^\infty = \sum_{r=0}^{\infty} u(r\ell). \tag{7}$$

Lemma 2.1 ([3, 4]). *Let $\ell > 0$ and $u(k)$, $w(k)$ are real valued bounded functions. Then*

$$\Delta_\ell^{-1}(u(k)w(k)) = u(k)\Delta_\ell^{-1} w(k) - \Delta_\ell^{-1}(\Delta_\ell^{-1} w(k+\ell)\Delta_\ell u(k)) \tag{8}$$

$$and\ \overset{-1}{\underset{\alpha(\ell)}{\Delta}}(u(k)v(k)) = u(k) \overset{-1}{\underset{\alpha(\ell)}{\Delta}} v(k) - \overset{-1}{\underset{\alpha(\ell)}{\Delta}}(\overset{-1}{\underset{\alpha(\ell)}{\Delta}} v(k+\ell)\Delta_\ell u(k)). \tag{9}$$

Lemma 2.2 ([4]). *For $\ell > 0$,*

$$\Delta_{\alpha(\ell)}^{-1} u(k+\ell) - \alpha^{[\frac{k}{\ell}]+1} \Delta_{\alpha(\ell)}^{-1} u(\hat{\ell}(k)) = \sum_{r=0}^{[\frac{k}{\ell}]} \alpha^r u(k - r\ell) \tag{10}$$

and hence

$$\alpha^m \Delta_{\alpha(\ell)}^{-1} u(k+\ell-m\ell) - \alpha^{[\frac{k}{\ell}]+1} \Delta_{\alpha(\ell)}^{-1} u(\hat{\ell}(k)) = \sum_{r=m}^{[\frac{k}{\ell}]} \alpha^r u(k-r\ell)\ for\ \ m < \left[\frac{k}{\ell}\right]. \tag{11}$$

Theorem 2.3. *Let $k \in (-\infty, \infty)$ and $\ell > 0$. Then we have*

$$\overset{-1}{\underset{\alpha(\ell)}{\Delta}}(e^{-sk}\cos ak) = \frac{e^{-sk}(e^{-s\ell}\cos a(k-\ell)) - \alpha\cos ak}{e^{-2s\ell} - 2\alpha e^{-s\ell}\cos a\ell + \alpha^2}, \tag{12}$$

$$\overset{-1}{\underset{\alpha(\ell)}{\Delta}}(e^{-sk}\sin ak) = \frac{e^{-sk}(e^{-s\ell}\sin a(k-\ell) - \alpha\sin ak)}{e^{-2s\ell} - 2\alpha e^{-s\ell}\cos a\ell + \alpha^2}. \tag{13}$$

Proof. The proof follows by the definition of $\overset{-1}{\underset{\alpha(\ell)}{\Delta}}$ and solving the following relations:

$$\underset{\alpha(\ell)}{\Delta}(e^{-sk}\cos ak) = e^{-sk}\cos ak(e^{-s\ell}\cos a\ell - \alpha) - e^{-sk}e^{-s\ell}\sin ak\sin a\ell,$$

$$\underset{\alpha(\ell)}{\Delta}(e^{-sk}\sin ak) = e^{-sk}\sin ak(e^{-s\ell}\cos a\ell - \alpha) + e^{-sk}e^{-s\ell}\cos ak\sin a\ell.$$

∎

Lemma 2.4 ([5] (1(k)-series of u(k))). *The first order generalized $k-$difference equation $v(k+\ell)-kv(k)=u(k)$, $k\in[\ell,\ \infty)$, $\ell>0$, has a summation solution of the form*

$$\sum_{r=0}^{\left[\frac{k}{\ell}\right]} k_\ell^{(r)}u(k-r\ell)=\mathop{\Delta}_{(k+\ell)(\ell)}^{-1}u(k+\ell)-k_\ell^{([\frac{k}{\ell}]+1)}\mathop{\Delta}_{\hat{\ell}(k)(\ell)}^{-1}u(\hat{\ell}(k)). \tag{14}$$

Lemma 2.5. *Let $\ell>0$ and $u(k)$, $v(k)$ are real valued bounded functions. Then*

$$\mathop{\Delta}_{k(\ell)}^{-1}(u(k)v(k))=u(k)\mathop{\Delta}_{k(\ell)}^{-1}v(k)-\mathop{\Delta}_{k(\ell)}^{-1}(\mathop{\Delta}_{k(\ell)}^{-1}v(k+\ell)\Delta_\ell u(k)). \tag{15}$$

Proof. From (3), we get

$$\mathop{\Delta}_{k(\ell)}(u(k)w(k))=u(k)\mathop{\Delta}_{k(\ell)}w(k)+w(k+\ell)\Delta_\ell u(k). \tag{16}$$

By taking $\mathop{\Delta}_{k(\ell)}w(k)=v(k)$ and $w(k)=\mathop{\Delta}_{k(\ell)}^{-1}v(k)$ in equation (16), we obtain (15) □

3. Main Results

In this section we derive closed form solution of the generalized difference equation $\mathop{\Delta}_{\alpha(\ell)}w(k)=v(k)$ and $\mathop{\Delta}_{k(\ell)}w(k)=v(k)$, where $w(k)$ takes polynomials, polynomial factorial with exponential and geometric with trigonometric functions. Also we find that the α, k-discrete Fourier transform of several functions.

Theorem 3.1. *Let $k\in(-\infty,\infty)$ and $\ell>0$. Then we have*

$$\mathop{\Delta}_{\alpha(\ell)}^{-1}(k_\ell^{(p)}e^{isk})=\Big(\sum_{r=0}^{p}(-\ell)^r(p)_1^{(r)}k_\ell^{(p-r)}(1-\alpha e^{-is\ell})^{-(r+1)}\Big)e^{is(k-\ell)}. \tag{17}$$

Proof. Taking $u(k)=k_\ell^{(1)}$, $w(k)=e^{isk}$ in (8) and using (5), we get

$$\mathop{\Delta}_{\alpha(\ell)}^{-1}(k_\ell^{(1)}e^{isk})=\Big(k_\ell^{(1)}(1-\alpha e^{-is\ell})^{-1}-\ell(1-\alpha e^{-is\ell})^{-2}\Big)e^{is(k-\ell)}.$$

Taking $u(k)=k_\ell^{(2)}$, $w(k)=e^{isk}$ in (8), and using (5), we get

$$\mathop{\Delta}_{\alpha(\ell)}^{-1}(k_\ell^{(2)}e^{isk})=\Big(k_\ell^{(2)}(1-\alpha e^{-is\ell})^{-1}-2\ell k_\ell^{(1)}(1-\alpha e^{-is\ell})^{-2}+2\ell^2\ell(1-\alpha e^{-is\ell})^{-3}\Big)e^{is(k-\ell)}.$$

Continuing the above process, we get the proof of (17). □

Theorem 3.2. *Let $k\in(-\infty,\infty)$and $\ell>0$. Then we have*

$$\mathop{\Delta}_{\alpha(\ell)}^{-1}(k^qe^{\pm isk})=\sum_{p=1}^{q}S_q^p\ell^{q-p}\Big(\sum_{r=0}^{p}(-\ell)^r(p)_1^{(r)}k_\ell^{(p-r)}(1-\alpha e^{-is\ell})^{-(r+1)}\Big)e^{is(k-\ell)}. \tag{18}$$

Proof. The proof follows from second term of (5) and (17) □

Theorem 3.3. *Let $k\in(-\infty,\infty)$ and $\ell>0$, then we have*

$$\mathop{\Delta}_{\alpha(\ell)}^{-1}(a^ke^{isk})=\frac{a^ke^{is(k-\ell)}}{(a^\ell-\alpha e^{-is\ell})}. \tag{19}$$

Proof. Since $\underset{\alpha(\ell)}{\Delta} a^k e^{isk} = a^{k+\ell} e^{is(k+\ell)} - \alpha a^k e^{isk}$, the proof follows from (4). □

Theorem 3.4. *Let $k \in (-\infty, \infty)$ and $\ell > 0$, then we have*

$$\underset{k(\ell)}{\overset{-1}{\Delta}} \left\{ \sum_{r=0}^{m} \ell^r + \sum_{r=1}^{m-1} \sum_{p=0}^{r-1} \binom{m-p}{r-p} \ell^{r-p} k^{m-r} - k^{m+1} \right\} = \sum_{r=0}^{m} k^r. \tag{20}$$

Proof. From (3), we get $\underset{k(\ell)}{\Delta} k^0 = 1 - k$, $\underset{k(\ell)}{\Delta} k = k + \ell - k^2$, $\underset{k(\ell)}{\Delta} k^2 = (k+\ell)^2 - k^3$ and proceeding like this we get $\underset{k(\ell)}{\Delta} k^m = (k+\ell)^m - k^{m+1}$. Adding all and then taking $\underset{k(\ell)}{\overset{-1}{\Delta}}$ on both sides we get the proof of (20). □

Example 3.5. *Taking $m = 5$ in (20), we get*

$$\underset{k(\ell)}{\overset{-1}{\Delta}} \left\{ 5\ell k^4 + (10\ell^2 + 4\ell)k^3 + (10\ell^3 + 6\ell^2 + 3\ell)k^2 + (5\ell^4 + 4\ell^3 + 3\ell^2 + 2\ell)k + (1 + \ell + \ell^2 + \ell^3 + \ell^4 + \ell^5) - k^6 \right\} = 1 + k + k^2 + k^3 + k^4 + k^5.$$

Corollary 3.6. *Let $k \in (-\infty, \infty)$ and $\ell > 0$, then we have*

$$\sum_{r=0}^{m} \underset{k(\ell)}{\overset{-1}{\Delta}} ((k+\ell)^r - k^{r+1}) = \sum_{r=0}^{m} k^r. \tag{21}$$

Proof. The similar proof of the Theorem 3.4 proves (21). □

Theorem 3.7. *Let $k \in (-\infty, \infty)$ and $\ell > 0$, then we have*

$$\underset{k(\ell)}{\overset{-1}{\Delta}} \left\{ (a^\ell e^{is\ell} - k) a^k e^{isk} \right\} = a^k e^{isk}. \tag{22}$$

Proof. From (3), we get $\underset{k(\ell)}{\Delta} a^k e^{isk} = a^{k+\ell} e^{is(k+\ell)} - k a^k e^{isk}$. Now taking $\underset{k(\ell)}{\overset{-1}{\Delta}}$ on both sides we get the proof of (22). □

Example 3.8. *Taking $a = 2$, $\ell = 3$, $k = 6$ and $s = 2$ in (22), we get*

$$\underset{k(\ell)}{\overset{-1}{\Delta}} (2^6(2^6(\cos 6 + i \sin 6) - 6))(\cos 12 + i \sin 12) = 64(\cos 12 + i \sin 12).$$

Theorem 3.9. *Let $k \in (-\infty, \infty)$ and $\ell > 0$, then we have*

$$\underset{k(\ell)}{\overset{-1}{\Delta}} \left\{ (a^\ell e^{is\ell}(k+\ell) - k^2) a^k e^{isk} \right\} = k a^k e^{isk}. \tag{23}$$

Proof. Taking $v(k) = k a^k e^{isk}$ in (3) and then applying $\underset{k(\ell)}{\overset{-1}{\Delta}}$, we get the proof of (23). □

Theorem 3.10. *Let $k \in (-\infty, \infty)$ and $\ell > 0$, then we have*

$$\underset{k(\ell)}{\overset{-1}{\Delta}} \big((\cos a\ell + \sin a\ell - k) \cos ak + (\cos a\ell - \sin a\ell - k) \sin ak \big) = \cos ak + \sin ak. \tag{24}$$

Proof. From (3), we have $\underset{k(\ell)}{\Delta} \cos ak = \cos a(k+\ell) - k \cos ak$ and $\underset{k(\ell)}{\Delta} \sin ak = \sin a(k+\ell) - k \sin ak$. Which completes the proof of (24). □

Theorem 3.11. *Let $k \in (-\infty, \infty)$ and $\ell > 0$, then we have*

$$\overset{-1}{\underset{k(\ell)}{\Delta}} \big\{(e^{s\ell}(\cos a\ell - \sin a\ell) - k)\sin ak + (e^{s\ell}(\cos a\ell + \sin a\ell) - k)\cos ak\big\}e^{sk} = (\sin ak + \cos ak)e^{sk}, \tag{25}$$

$$\overset{-1}{\underset{k(\ell)}{\Delta}} \big\{(e^{s\ell}(\cos a\ell + \sin a\ell) - k)\sin ak + (e^{s\ell}(\sin a\ell - \cos a\ell) + k)\cos ak\big\}e^{sk} = (\sin ak - \cos ak)e^{sk}, \tag{26}$$

$$\overset{-1}{\underset{k(\ell)}{\Delta}} \big\{(e^{s\ell}\cos ak - k\cos a(k-\ell)\big\}e^{sk} = e^{sk}\cos a(k-\ell), \tag{27}$$

$$\overset{-1}{\underset{k(\ell)}{\Delta}} \big\{(e^{s\ell}\sin ak - k\sin a(k-\ell)\big\}e^{sk} = e^{sk}\sin a(k+\ell). \tag{28}$$

Proof. From (3), we have

$$\underset{k(\ell)}{\Delta} e^{sk}\sin ak = (e^{s\ell}\cos a\ell\sin ak + e^{s\ell}\sin a\ell\cos ak - k\sin ak)e^{sk} \tag{29}$$

$$\text{and} \quad \underset{k(\ell)}{\Delta} e^{sk}\cos ak = (e^{s\ell}\cos a\ell\cos ak - e^{s\ell}\sin a\ell\sin ak - k\cos ak)e^{sk} \tag{30}$$

By solving (29) and (30), we get the proof of (25), (26), (27) and (28). □

Definition 3.12. *The α, k-Fourier transform of $u(k)$ is defined as*

$$\underset{\alpha(\ell)}{F} w(k) = \underset{\alpha(\ell)}{W}(s) = \ell \overset{-1}{\underset{\alpha(\ell)}{\Delta}} w(k)e^{isk}\Big|_{-\infty}^{\infty} \text{ and } \underset{k(\ell)}{F} w(k) = \underset{k(\ell)}{W}(s) = \ell \overset{-1}{\underset{k(\ell)}{\Delta}} w(k)e^{isk}\Big|_{-\infty}^{\infty}. \tag{31}$$

Similarly α, k-Fourier sine and cosine transforms of $u(k)$ are defined as

$$\underset{\alpha_s(\ell)}{F} w(k) = \underset{\alpha_s(\ell)}{W}(s) = \ell \overset{-1}{\underset{\alpha(\ell)}{\Delta}} w(k)\sin sk\Big|_0^{\infty} \text{ and } \underset{k_s(\ell)}{F} w(k) = \underset{k_s(\ell)}{W}(s) = \ell \overset{-1}{\underset{k(\ell)}{\Delta}} w(k)\sin sk\Big|_0^{\infty} \tag{32}$$

$$\underset{\alpha_c(\ell)}{F} w(k) = \underset{\alpha_c(\ell)}{W}(s) = \ell \overset{-1}{\underset{\alpha(\ell)}{\Delta}} w(k)\cos sk\Big|_0^{\infty} \text{ and } \underset{k_c(\ell)}{F} w(k) = \underset{k_c(\ell)}{W}(s) = \ell \overset{-1}{\underset{k(\ell)}{\Delta}} w(k)\cos sk\Big|_0^{\infty} \tag{33}$$

Theorem 3.13. *Let $\ell > 0$, $k \in [0, \infty)$, then*

$$\underset{\alpha(\ell)}{F}(\Delta_\ell^n u(k)) = (e^{-is\ell} - \alpha)^{n-1} \underset{\alpha(\ell)}{F}(\Delta_\ell u(k)). \tag{34}$$

In particular, when $\alpha \to 1$ we get

$$F(\Delta_\ell^n u(k)) = (e^{-is\ell} - 1)^{n-1} F(\Delta_\ell u(k)). \tag{35}$$

Proof. Taking $w(k) = \Delta_\ell u(k)$ in (15), we get $F(\Delta_\ell u(k)) = \dfrac{-e^{is\ell}}{e^{is\ell} - 1} F(\Delta_\ell^2 u(k))$. From this we can get $F(\Delta_\ell^2 u(k)) = (e^{-is\ell} - 1)F(\Delta_\ell u(k))$. Now taking $w(k) = \Delta_\ell^2 u(k)$, we get $F(\Delta_\ell^3 u(k)) = (e^{-is\ell} - 1)^2 F(\Delta_\ell u(k))$. Repeating this process n times we get the proof of (34). □

Example 3.14. *Taking $u(k) = e^k$ in (34) and using (31), we get*

$$\underset{\alpha(\ell)}{F}(\Delta_\ell^n e^k) = (e^{-is\ell} - \alpha)^{n-1}\ell(e^\ell - 1)\Big(\frac{e^{(is+1)k}}{e^{(is+1)\ell} - \alpha}\Big)\Big|_{-\infty}^{\infty}.$$

In particular, when $n = 108$ and $-4 < k < 4$, we have

$$\underset{\alpha(\ell)}{F}(\Delta_\ell^{108} e^k) = \frac{\ell(e^{-is\ell} - \alpha)^{107}(e^\ell - 1)}{e^{(is+1)\ell} - \alpha}(e^{4(is+1)} - e^{-4(is+1)}).$$

Theorem 3.15. *Let $\ell > 0$, $k \in [0,\infty)$, $\underset{\alpha(\ell)}{F} w(k) = \underset{\alpha(\ell)}{W}(s)$ and $\underset{\alpha(\ell)}{F} z(k) = \underset{\alpha(\ell)}{Z}(s)$, then*

$$\underset{\alpha(\ell)}{F}(aw(k)+bz(k)) = a\underset{\alpha(\ell)}{W}(s) + b\underset{\alpha(\ell)}{Z}(s) \text{ and } \underset{\alpha(\ell)}{F}(w(ak)) = \frac{1}{a}\underset{\alpha(\ell)}{W}\Big(\frac{s}{a}\Big). \tag{36}$$

Proof. The proof follow from (31) and the linearity of $\underset{\alpha(\ell)}{\Delta}$. □

Example 3.16. *Taking $w(k) = k$ and $-2 < k < 2$ in (36), then the α-Fourier transform of ak as*

$$\underset{\alpha(\ell)}{F}(ak) = \frac{\ell}{a}\Big\{\frac{4\cos 2(s/a)}{(e^{i(s/a)\ell}-\alpha)} - \frac{2ie^{i(s/a)\ell}\sin 2(s/a)}{(e^{i(s/a)\ell}-\alpha)^2}\Big\}.$$

Theorem 3.17. *Let $\ell > 0$, $k \in [0,\infty)$, $\underset{\alpha_s(\ell)}{F} w(k) = \underset{\alpha_s(\ell)}{W}(s)$ and $\underset{\alpha_c(\ell)}{F} w(k) = \underset{\alpha_c(\ell)}{W}(s)$, then*

$$\underset{\alpha_s(\ell)}{F} w(ak) = \frac{1}{a}\underset{\alpha_s(\ell)}{W}\Big(\frac{s}{a}\Big) \text{ and } \underset{\alpha_c(\ell)}{F} w(ak) = \frac{1}{a}\underset{\alpha_c(\ell)}{W}\Big(\frac{s}{a}\Big) \tag{37}$$

Proof. The proof follow from (32) and (33). □

Theorem 3.18. *Let $\ell > 0$, $k \in [0,\infty)$, $\underset{\alpha(\ell)}{F} g(k) = \underset{\alpha(\ell)}{G}(s)$, then $\underset{\alpha(\ell)}{F}(g(k-a)) = e^{isa}\underset{\alpha(\ell)}{G}(s)$.*

Proof. The proof follow from (31). □

Example 3.19. *Taking $g(k) = k_\ell^{(2)}$ in the Theorem 3.18, using (17) and (31), we get the α-Fourier transform of $(k-a)_\ell^{(2)}$ as*

$$\underset{\alpha(\ell)}{F}((k-a)_\ell^{(2)}) = \ell e^{isa}\Big\{\frac{k_\ell^{(2)}e^{isk}}{(e^{is\ell}-\alpha)} - \frac{2\ell k_\ell^{(1)}e^{is(k+\ell)}}{(e^{is\ell}-\alpha)^2} + \frac{2\ell^2 e^{is(k+2\ell)}}{(e^{is\ell}-\alpha)^3}\Big\}$$

In particular, when $\alpha = 2$, $\ell = 2$ and $0 < k < 3$, we have

$$\underset{\alpha(\ell)}{F}((k-a)_\ell^{(2)}) = \frac{6e^{is(3+a)}}{(e^{i2s}-2)} - \frac{24e^{is(5+a)}}{(e^{i2s}-2)^2} + \frac{8e^{is(7+a)}}{(e^{i2s}-2)^3} - \frac{8e^{is(4+a)}}{(e^{i2s}-2)^3}.$$

Theorem 3.20 (α-Modulation Theorem)**.** *Let $\ell > 0$, $k \in [0,\infty)$, we have the following*

$$(i)\ If\ \underset{\alpha(\ell)}{F} w(k) = \underset{\alpha(\ell)}{W}(s)\ \ then\ \ \underset{\alpha(\ell)}{F}(w(k)\cos ak) = \frac{1}{2}\{\underset{\alpha(\ell)}{W}(s+a) + \underset{\alpha(\ell)}{W}(s-a)\} \tag{38}$$

$$(ii)\ If\ \underset{\alpha_s(\ell)}{F} w(k)) = \underset{\alpha_s(\ell)}{W}(s)\ \ then\ \ \underset{\alpha_s(\ell)}{F}(w(k)\cos ak) = \frac{1}{2}\{\underset{\alpha_s(\ell)}{W}(s+a) + \underset{\alpha_s(\ell)}{W}(s-a)\} \tag{39}$$

$$(iii)\ If\ \underset{\alpha_s(\ell)}{F} w(k)) = \underset{\alpha_s(\ell)}{W}(s)\ \ then\ \ \underset{\alpha_c(\ell)}{F}(w(k)\sin ak) = \frac{1}{2}\{\underset{\alpha_s(\ell)}{W}(s+a) - \underset{\alpha_s(\ell)}{W}(s-a)\} \tag{40}$$

$$(iv)\ If\ \underset{\alpha_c(\ell)}{F} w(k)) = \underset{\alpha_c(\ell)}{W}(s)\ \ then\ \ \underset{\alpha_s(\ell)}{F}(w(k)\sin ak) = \frac{1}{2}\{\underset{\alpha_c(\ell)}{W}(s-a) - \underset{\alpha_c(\ell)}{W}(s+a)\} \tag{41}$$

Example 3.21. *Taking $w(k)$ in (41) and using (13), we get the α-sine Fourier transform of $e^{-mk\sin ak}$ as*

$$\underset{\alpha_s(\ell)}{F}(e^{-mk}\sin ak) = \frac{\ell}{2}\Big\{\frac{e^{-mk}(e^{-m\ell}\cos(s-a)(k-\ell) - \alpha\cos(s-a)k)}{e^{-2m\ell} - 2\alpha e^{-m\ell}\cos(s-a)\ell + \alpha^2} - \frac{e^{-mk}(e^{-m\ell}\cos(s+a)(k-\ell) - \alpha\cos(s+a)k)}{e^{-2m\ell} - 2\alpha e^{-m\ell}\cos(s+a)\ell + \alpha^2}\Big\}\Big|_0^\infty.$$

In particular, when $m = 5$, $\ell = 3$ and $0 < k < 9$, then we have

$$\underset{\alpha_s(3)}{F}(e^{-5k}\sin ak) = \frac{3}{2}\Big\{\frac{e^{-45}(e^{-15}\cos 6(s-a) - \alpha\cos 9(s-a))}{e^{-30} - 2\alpha e^{-15}\cos 3(s-a) + \alpha^2} - \frac{e^{-45}(e^{-15}\cos 6(s+a) - \alpha\cos 9(s+a))}{e^{-30} - 2\alpha e^{-15}\cos 3(s+a) + \alpha^2}$$
$$- \frac{e^{-15}\cos 3(s-a) - \alpha}{e^{-30} - 2\alpha e^{-15}\cos 3(s-a) + \alpha^2} + \frac{e^{-15}\cos 3(s+a) - \alpha}{e^{-30} - 2\alpha e^{-15}\cos 3(s+a) + \alpha^2}\Big\}.$$

Theorem 3.22. *Let $\ell > 0$, $k \in [0, \infty)$, $\underset{k(\ell)}{F} h(k) = \underset{k(\ell)}{H}(s)$ and $\underset{k(\ell)}{F} g(k) = \underset{k(\ell)}{G}(s)$, then*

$$\underset{k(\ell)}{F}(ah(k) + bg(k)) = a \underset{k(\ell)}{H}(s) + b \underset{k(\ell)}{G}(s) \text{ and } \underset{k(\ell)}{F}(h(ak)) = \frac{1}{a} \underset{k(\ell)}{H}\left(\frac{s}{a}\right). \tag{42}$$

Proof. The proof follow from (31) and the linearity of $\underset{k(\ell)}{\Delta}$. □

Example 3.23. *Taking $h(k) = a^k(a^\ell e^{is\ell} - k)$ in (42), using (31) and (22) we get k-Fourier transform of $a^k(a^\ell e^{is\ell} - k)$ for $-5 < k < 5$ is*

$$\underset{k(\ell)}{F}(a^k(a^\ell e^{is\ell} - k)) = \ell(a^5 e^{i5s} - a^{-5} e^{-i5s}).$$

From this we can get the k-Fourier transform of $a^{ak}(a^\ell e^{is\ell} - ak)$ is

$$\underset{k(\ell)}{F}(a^{ak}(a^\ell e^{is\ell} - ak)) = \frac{\ell}{a^6}\{\cos 5(s/a)(a^{10} - 1) + i \sin 5(s/a)(a^{10} + 1)\}.$$

Theorem 3.24. *Let $\ell > 0$, $k \in [0, \infty)$, $\underset{k_s(\ell)}{F} h(k) = \underset{k_s(\ell)}{H}(s)$ and $\underset{k_c(\ell)}{F} h(k) = \underset{k_c(\ell)}{H}(s)$, then*

$$\underset{k_s(\ell)}{F} h(ak) = \frac{1}{a} \underset{k_s(\ell)}{H}\left(\frac{s}{a}\right) \text{ and } \underset{k_c(\ell)}{F} h(ak) = \frac{1}{a} \underset{k_c(\ell)}{H}\left(\frac{s}{a}\right) \tag{43}$$

Proof. The proof follow from (32) and (33). □

Theorem 3.25. *Let $\ell > 0$, $k \in [0, \infty)$, $\underset{k(\ell)}{F} g(k) = \underset{k(\ell)}{G}(s)$, then $\underset{k(\ell)}{F}(g(k - a)) = e^{isa} \underset{k(\ell)}{G}(s)$.*

Proof. The proof follow from (31). □

Theorem 3.26 (*k*-Modulation Theorem)**.** *Let $\ell > 0$, $k \in [0, \infty)$, we have the following*

$$(i)\ If \quad \underset{k(\ell)}{F} h(k) = \underset{k(\ell)}{H}(s) \quad then \quad \underset{k(\ell)}{F}(h(k)\cos ak) = \frac{1}{2}\{\underset{k(\ell)}{H}(s + a) + \underset{k(\ell)}{H}(s - a)\} \tag{44}$$

$$(ii)\ If \quad \underset{k_s(\ell)}{F} h(k) = \underset{k_s(\ell)}{H}(s) \quad then \quad \underset{k_s(\ell)}{F}(h(k)\cos ak) = \frac{1}{2}\{\underset{k_s(\ell)}{H}(s + a) + \underset{k_s(\ell)}{H}(s - a)\} \tag{45}$$

$$(iii)\ If \quad \underset{k_s(\ell)}{F} h(k) = \underset{k_s(\ell)}{H}(s) \quad then \quad \underset{k_c(\ell)}{F}(h(k)\sin ak) = \frac{1}{2}\{\underset{k_s(\ell)}{H}(s + a) - \underset{k_s(\ell)}{H}(s - a)\} \tag{46}$$

$$(iv)\ If \quad \underset{k_c(\ell)}{F} h(k) = \underset{k_c(\ell)}{H}(s) \quad then \quad \underset{k_s(\ell)}{F}(h(k)\sin ak) = \frac{1}{2}\{\underset{k_c(\ell)}{H}(s - a) - \underset{k_c(\ell)}{H}(s + a)\} \tag{47}$$

Theorem 3.27 (Relationship between Laplace and Fourier Transform)**.** *Let $\ell > 0$, $w(k) = e^{-pk} g(k)$ for $k > 0$ and $w(k) = 0$ for $k < 0$, then $\underset{k(\ell)}{F} w(k) = \underset{\alpha(\ell)}{L} g(k)$. In particular, when $\alpha \to 1$ we get $F(w(k)) = L(g(k))$.*

Proof. From (31), we get

$$\underset{\alpha(\ell)}{F} w(k) = \ell \overset{-1}{\underset{\alpha(\ell)}{\Delta}} w(k) e^{isk}\Big|_{-\infty}^{0} + \ell \overset{-1}{\underset{\alpha(\ell)}{\Delta}} w(k) e^{isk}\Big|_{0}^{\infty}.$$

Which gives

$$\underset{\alpha(\ell)}{F} w(k) = \ell \overset{-1}{\underset{\alpha(\ell)}{\Delta}} e^{-pk} e^{isk}\Big|_{0}^{\infty}.$$

Which completes the proof of the theorem. □

4. Conclusion

By using Fourier transform we get only one output for input. But by using the α and k-discrete Fourier transform we can get different outputs by varying the value of α for given inputs.

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