International Journal of Mathematics And its Applications
Volume 5, Issue 2–A (2017), 57–62.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

On Certain Class of Meromorphically Starlike Functions with Alternating Coefficients

Research Article

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Abstract: In this paper, we introduce the subclass $\sigma(n, \alpha, \beta)$ of meromorphically starlike functions in the punctured unit disk. Also we investigate coefficient inequality, distortion properties, and closure properties. Further preserving integral operators are obtained.

MSC: 30C45.

Keywords: Regular, Meromorphic, starlike, coefficient inequality.

1. Introduction

Let $\sum$ denote the class of functions of the form

$$f(z) = \frac{a_{-1}}{z} + \sum_{k=1}^{\infty} a_k z^k, (a_{-1} \neq 0) \tag{1}$$

which are regular in the punctured unit disk $E = \{z : 0 < |z| < 1\}$. Define

$$D^0 f(z) = f(z) \tag{2}$$

$$Df(z) = D^1 f(z) = \frac{a_{-1}}{z} + \sum_{k=1}^{\infty} k a_k z^k = z f'(z) + \frac{2a_{-1}}{z} \tag{3}$$

$$D^2 f(z) = z\left(D^{n-1} f(z)\right)^1 + \frac{2a_{-1}}{z} \tag{4}$$

and in general

$$D^n f(z) = z\left(D^{n-1} f(z)\right)^1 + \frac{2a_{-1}}{z} \tag{5}$$

Let $B_n(\alpha)$ denote the class consisting of functions in $\sum$, which satisfies

$$Re\left\{\frac{D^{n+1} f(z)}{D^n f(z)} - 2\right\} < -\alpha, (z \in E, 0 \leq \alpha < 1, n \in N_0 = N \cup 0) \tag{6}$$

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Let σ be the subclass of $\sum$ which consists of functions of the form

$$f(z) = \frac{a_{-1}}{z} + \sum_{k-7}^{\infty} (-1)^{k-1} a_k z^k,\ (a_{-1} > 0, a_k \geq 0) \tag{7}$$

Further, let

$$\sigma(n, \alpha, \beta) = B(n, \alpha, \beta) \cap \sigma \tag{8}$$

In [6], Uralegaddi and Somanetha defined a class $B_n(\alpha)$ which consists of functions of the form

$$f(z) = \frac{a_{-1}}{z} + \sum_{k=1}^{\infty} a_k z^k\ (a_{-1} \neq 0)$$

which are analytic in E. Further, Aouf and Darwish [1] considered Meromorphic starlike univalent functions with alternating coefficients and obtained coefficient inequalities, distortion theorem and integral operators. In the present paper coefficient inequalities, distortion theorem and closure theorems for the class $\sigma(n, \alpha, \beta)$ are obtained. Techniques used are similar to those of Aouf and Darwish [1]. Finally, the class preserving integrals of the form

$$f_{c+1}(z) = (c+1) \int_0^1 u^{c+1} f(uz)\, du,\ (0 \leq u < 1,\ 0 < c < \infty) \tag{9}$$

is considered.

Definition 1.1. *Let the function $f(z)$ be defined by (7). The $f(z) \in \sigma(n, \alpha, \beta)$ if and only if*

$$\left| \frac{\frac{D^{n+1} f(z)}{D^n f(z)} - 1}{2\beta \left[\frac{D^{n+1} f(z)}{D^n f(z)} - 1\right] - \left[\frac{D^{n+1} f(z)}{D^n f(z)} + 1 - 2\alpha\right]} \right| < 1 \ \textit{for}\ \ |z| < 1,\ \ 0 \leq \alpha < 1,\ \ \frac{1}{2} < \beta \leq 1$$

2. Coefficient Inequalities

Theorem 2.1. *Let the function f(z) be defined by (1). If*

$$\sum_{k=1}^{\infty} k^n \left[(k-1)\beta - 1 + \alpha\right] |a_k| \leq (1-\alpha)|a_{-1}| \ \ \textit{then}\ \ f(z) \in B(n, \alpha, \beta) \tag{10}$$

Proof. To establish the theorem, it will be sufficient to show that

$$\left| \frac{\frac{D^{n+1} f(z)}{D^n f(z)} - 1}{2\beta \left[\frac{D^{n+1} f(z)}{D^n f(z)} - 1\right] - \left[\frac{D^{n+1} f(z)}{D^n f(z)} + 1 - 2\alpha\right]} \right| < 1 \ \text{for}\ \ |z| < 1, 0 \leq \alpha < 1, \frac{1}{2} < \beta \leq 1 \tag{11}$$

We have

$$\left| \frac{\frac{D^{n+1} f(z)}{D^n f(z)} - 1}{2\beta \left[\frac{D^{n+1} f(z)}{D^n f(z)} - 1\right] - \left[\frac{D^{n+1} f(z)}{D^n f(z)} + 1 - 2\alpha\right]} \right| = \left| \frac{\sum\limits_{k=1}^{\infty} k^n (k-1) a_k z^k}{(2-2\alpha)\frac{a_{-1}}{z} - \sum\limits_{k=1}^{\infty} k^n \left[2\beta(k-1) - (k+1-2\alpha)\right] a_k z^k} \right|$$

$$\leq \frac{\sum\limits_{k=1}^{\infty} k^n (k-1) |a_k|}{2(1-\alpha) a_{-1} - \sum\limits_{k=1}^{\infty} k^n \left[2\beta(k-1) - (k+1-2\alpha)\right] |a_k|}$$

The last expression is bounded by 1 if

$$\sum_{k=1}^{\infty} k^n (k-1) |a_k| \leq 2(1-\alpha)|a_{-1}| - \sum_{k-1}^{\infty} k^n \left[2\beta(k-1) - (k+1-2\alpha)\right] |a_k|$$

which reduces to

$$\sum_{k=1}^{\infty} k^n \left[(k-1)\beta - 1 + \alpha\right] |a_k| \leq (1-\alpha) |a_{-1}| \tag{12}$$

But (12) is true by hypothesis. Hence the result follows. □

Theorem 2.2. *Let the function f(z) be defined by (7) then $f(z) \in \sigma(n, \alpha, \beta)$ if and only if*

$$\sum_{k=1}^{\infty} k^n \left[(k-1)\beta - 1 + \alpha\right] a_k \leq (1-\alpha) a_{-1} \tag{13}$$

Proof. In view of Theorem 2.1, it is sufficient to prove the only if part. Let us assume that $f(z)$ defined by (7) is in $\sigma(n, \alpha, \beta)$ then

$$\left| \frac{\frac{D^{n+1}f(z)}{D^n f(z)} - 1}{2\beta \left[\frac{D^{n+1}f(z)}{D^n f(z)} - 1\right] \left[\frac{D^{n+1}f(z)}{D^n f(z)} + 1 - 2\alpha\right]} \right| < 1$$

reduces to

$$\left| \frac{\sum_{k=1}^{\infty} (-1)^{k-1} k^n (k-1) a_k z^{k+1}}{2(1-\alpha) a_{-1} \sum_{k=1}^{\infty} (-1)^{k-1} k^n \left[2\beta(k-1)(k+1-2\alpha)\right] a_k z^{k+1}} \right| < 1 \tag{14}$$

Hence $\sum\limits_{k=1}^{\infty} k^n \left[(k-1)\beta - 1 + \alpha\right] a_k \leq (1-\alpha) a_{-1}$. Thus the result follows. □

Corollary 2.3. *Let the function $f(z)$ defined by (7), be in the class $\sigma(n, \alpha, \beta)$ then*

$$a_k \leq \frac{(1-\alpha) a_{-1}}{k^n \left[(k-1)\beta - 1 + \alpha\right]} \tag{15}$$

The result is sharp for the function

$$f(z) = \frac{a_{-1}}{z} + \frac{(1-\alpha) a_{-1}}{k^n \left[(k-1)\beta - 1 + \alpha\right]} z^k, \ (k \geq 1) \tag{16}$$

3. Distortion Theorem

Theorem 3.1. *Let the function f(z), defined by (7) be in the class $\sigma(n, \alpha, \beta)$ then for $0 < |z| = r < 1$ we have*

$$\frac{a_{-1}}{r} - a_{-1} r \leq |f(z)| \leq \frac{a_{-1}}{r} + a_{-1} r \tag{17}$$

where equality holds for the function

$$f(z) = \frac{a_{-1}}{z} + \frac{(1-\alpha) a_{-1}}{k^n \left[(k-1)\beta - 1 + \alpha\right]} z, z = ir, r \tag{18}$$

Proof. In view of Theorem 2.2 we have

$$\sum_{k=1}^{\infty} a_k \leq \frac{(1-\alpha) a_{-1}}{k^n \left[(k-1)\beta - 1 + \alpha\right]} \tag{19}$$

Thus

$$|f(z)| \leq \frac{a_{-1}}{r} + r \sum_{k_{-1}}^{\infty} a_k, 0 < |z| = r < 1$$

$$\sum \frac{a_{-1}}{r} + a_{-1} r \tag{20}$$

and

$$|f(z)| \geq \frac{a_{-1}}{r} - r \sum_{k_{-1}}^{\infty} a_k \geq \frac{a_{-1}}{r} - a_{-1} r \tag{21}$$

Thus (17) follows. □

4. Closure Theorems

Let the function $f_j(z)$, be defined for $j \in \{1, 2, 3, \ldots, m\}$, by

$$f_j(z) = \frac{a_{-1},j}{z} + \sum_{k=1}^{\infty} (-1)^{k-1} a_{k,j} z^k, \left(a_{-1,j} > 0, a_{k,j} \geq 0\right), z \in E \tag{22}$$

We shall prove the following closure theorems for the class $\sigma(n, \alpha, \beta)$

Theorem 4.1. *Let the functions $f_j(z)$ defined by (22) be in the class $\sigma(n, \alpha, \beta)$ for every $j = 1, 2, \ldots, m$. Then the function $f(z)$ defined by*

$$f(z) = \frac{b_{-1}}{z} + \sum_{k=1}^{\infty} (-1)^{k-1} b_k z^k, b_{-1} > 0,\ b_k \geq 0 \tag{23}$$

is a member of the class $\sigma(n, \alpha, \beta)$ where

$$b_{-1} = \frac{1}{m} \sum_{j=1}^{m} a_{-1,\,j} \text{ and } b_k = \frac{1}{m} a_{k,\,j}, (k = 1,\ 2, \ldots.) \tag{24}$$

Proof. Since $f_j(z) \in \sigma(n, \alpha, \beta)$ it follows from Theorem 2.2, that

$$\sum_{k=1}^{\infty} k^n \left[(k-1)\beta - 1 + \alpha\right]\ a_{k,\,j} \leq (1-\alpha)\ a_{-1,\,j} \tag{25}$$

for every $j = 1, 2, 3, \ldots, m$. Hence

$$\begin{aligned} \sum_{k=1}^{\infty} k^n \left[(k-1)\beta - 1 + \alpha\right]\ b_k &= \frac{1}{m} \sum_{j=1}^{\infty} \left[\sum_{k=1}^{\infty} \left(k^n (k-1)\beta - 1 + \alpha\right) a_{k,\,j}\right] \\ &\leq (1-\alpha)\left(\frac{1}{m} \sum_{j=1}^{m} a_{-1,j}\right) \\ &= (1-\alpha)\, b_{-1} \end{aligned}$$

which (in view of Theorem 2) implies that $f(z) \in \sigma(n, \alpha, \beta)$. □

Theorem 4.2. *The class $\sigma(n, \alpha, \beta)$ is closed under convex linear combination.*

Proof. Let the function $f_j(z)$, (j=1, 2) defined by (22) be in the class $\sigma(n, \alpha, \beta)$. It is sufficient to prove that the function

$$H(z) = \lambda_1\ f_1(z)\ +\ (1-\lambda)\ f_2(z),\ (0 \leq \lambda \leq 1) \tag{26}$$

is also in the class $\sigma(n, \alpha, \beta)$. Since for $0 \leq \lambda \leq 1$,

$$H(z) = \frac{\lambda a_{-1,1} + (1-\lambda)\ a_{-1,\,2}}{z} - \sum_{k=1}^{\infty} \left\{\lambda a_{k,1} + (1-\lambda)\ a_{k,2}\right\} z^k \tag{27}$$

we observe by virtue of Theorem 2.2 that

$$\begin{aligned} \sum_{k=1}^{\infty} k^n \left[(k-1)\beta - 1 + \alpha\right] \left\{\lambda a_{k,1} + (1-\lambda) a_{k,2}\right\} &= \lambda \sum_{k=1}^{\infty} \left[k^n (k-1)\beta - 1 + \alpha\right] a_{k,1} + (1-\lambda) \sum_{k=1}^{\infty} \left[k^n (k-1)\beta - 1 + \alpha\right]\ a_{k,2} \\ &\leq (1-\alpha)\left\{\lambda a_{-1,1} + (1-\lambda) a_{-1,2}\right\} \end{aligned} \tag{28}$$

Hence, $H(z) \in \sigma(n, \alpha, \beta)$. □

Theorem 4.3. *Let*

$$f_0(z) = \frac{a_{-1}}{z} \tag{29}$$

and

$$f_k(z) = \frac{a_{-1}}{z} + (-1)^{k-1}\frac{(1-\alpha)\,a_{-1}}{k^n\,[(k-1)\,\beta - 1 + \alpha]}z^k, (k \geq 1) \tag{30}$$

Then, $f(z) \in \sigma(n, \alpha, \beta)$ and only if it can be expressed in the form

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z) \tag{31}$$

where $\lambda_k \geq 0\,(k \geq 1)$ and $\sum\limits_{k=1}^{\infty} \lambda_k = 1$.

Proof. Let $f(z) = \sum\limits_{k=1}^{\infty} \lambda_k f_k(z)$ where $\lambda_k \geq 0\,(k \geq 1)$ and $\sum\limits_{k=1}^{\infty} \lambda_k = 1$. Then

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z) = \lambda_0 f_0(z) + \sum_{k=1}^{\infty} \lambda_k f_k(z) \tag{32}$$

$$= \frac{a_{-1}}{z} + \sum_{k=1}^{\infty} (-1)^{k-1}\frac{(1-\alpha)\,a_{-1}\lambda_k}{k^n\,[(k-1)\,\beta - 1 + \alpha]}z^k \tag{33}$$

since, by Theorem 2, we have

$$= \sum_{k=1}^{\infty} k^n\,[(k-1)\,\beta - 1 + \alpha]\,\frac{(1-\alpha)\,a_{-1}\lambda_k}{k^n\,[(k-1)\,\beta - 1 + \alpha]} \tag{34}$$

$$= (1-\alpha)\,a_{-1}\sum_{k=1}^{\infty}\lambda_k \tag{35}$$

$$= (1-\alpha)\,a_{-1}\,(1-\lambda_0) \leq (1-\alpha)\,a_{-1} \tag{36}$$

Hence, $f(z) \in \sigma(n, \alpha, \beta)$.

Conversely, we suppose that $f(z)$, defined by (7), is in the class $\sigma(n, \alpha, \beta)$. Then by using (15) we get,

$$a_k \leq \frac{(1-\alpha)\,a_{-1}}{k^n\,[(k-1)\,\beta - 1 + \alpha]}, (k \geq 1) \tag{37}$$

Setting

$$\lambda_k = \frac{k^n\,[(k-1)\,\beta - 1 + \alpha]}{(1-\alpha)\,a_{-1}}, (k \geq 1) \tag{38}$$

and $\lambda_0 = 1 - \sum\limits_{k=1}^{\infty} \lambda_k$. □

5. Integral Operators

In this section we consider integral transforms of functions in the class $\sigma(n, \alpha, \beta)$.

Theorem 5.1. *Let the function f(z), defined by (7) be in the class $\sigma(n, \alpha, \beta)$ then the integral transforms.*

$$f_{c+1}(z) = (c+1)\int_0^1 u^{c+1} f(uz)\,du, (0 \leq u \leq 1,\ 0 < c < \infty) \tag{39}$$

are in $\sigma(n, \delta)$ where

$$\delta(\alpha, \beta, c) = \frac{[(k-1) + \alpha - 1]\,(k + c + 2) - [(k-1)\,\beta - 1]\,(c+1)\,(1-\alpha)}{[(k-1)\,\beta + \alpha - 1]\,(k + c + 2) + (c+1)\,(1-\alpha)} \tag{40}$$

The result is sharp for the function

$$f(z) = \frac{a_{-1}}{z} + (-1)^{k-1}\frac{(1-\alpha)\,a_{-1}}{k^n\,[(k-1)\,\beta + \alpha - 1]}zs. \tag{41}$$

Proof. Let

$$f_{c+1}(z) = (c+1)\int_0^1 u^{c+1} f(uz)\,du \tag{42}$$

$$= \frac{a_{-1}}{z} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{c+1}{k+c+2} a_k z^k \tag{43}$$

In view of Theorem 2, it is sufficient to show that

$$\sum_{k=1}^{\infty} \frac{k^n\left[(k-1)\beta+\delta-1\right]}{(1-\delta)\,a_{-1}} \left(\frac{c+1}{(k+c+2)}\right) a_k \leq 1 \tag{44}$$

since $f(z) \in \sigma(n,\alpha,\beta)$, we have

$$\sum_{k=1}^{\infty} \frac{k^n\left[(k-1)\beta-1+\alpha\right]}{(1-\alpha)\,a_{-1}} a_k \leq 1 \tag{45}$$

Thus (43) will be satisfied if $\frac{[(k-1)\beta+\delta-1](c+1)}{(1-\delta)(k+c+2)} \leq \frac{(k-1)\beta+\alpha-1}{1-\alpha}$, for each k, solving for δ, we obtain

$$\delta \leq \frac{\left[(k-1)\beta+\alpha-1\right](k+c+2) - \left[(k-1)\beta-1\right](c+1)(1-\alpha)}{(k+c+2)\left((k-1)\beta+\alpha-1\right) + (c+1)(1-\alpha)} \tag{46}$$

for each α, β and c fixed, let

$$\psi(k) = \frac{\left[(k-1)\beta+\alpha-1\right](k+c+2) - \left[(k-1)\beta-1\right](c+1)(1-\alpha)}{\left((k-1)\beta+\alpha-1\right)(k+c+2) + (c+1)(1-\alpha)}$$

Then $\psi(k+1) - \psi(k) > 0$ for each k. Hence $\psi(k)$ is an increasing function of k. Since $\psi(1) = \frac{(c+3)(\alpha-1)+(1-\alpha)(c+1)}{(c+3)(\alpha-1)+(c+1)(1-\alpha)}$ The result follows. □

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