



Superior Julia Sets and Superior Mandelbrot Sets in SP Orbit

Research Article

Mandeep Kumari^{1*}, Sudesh Kumari¹ and Renu Chugh¹

¹ Department of Mathematics, Maharshi Dayanand University, Rohtak, Haryana, India.

Abstract: The aim of this paper is to generate new Superior Julia and Superior Mandelbrot sets for complex-valued polynomials such as quadratic, cubic and higher degree polynomials using SP orbit, which is an example of four-step iterative procedure. In this paper, we visualize the graphical images of Superior Julia sets and Superior Mandelbrot sets and analyze their patterns. It is surprising to see that a few Superior Mandelbrot sets take the shape of decorated coupled Urns (Kalash in Hindi language), a decorated coupled Lightening lamp (Diya in Hindi language) and beautiful scared patterns (Rangolies in Hindi language). We believe that our results generalize and extend the existing results in the literature of Fractal theory.

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Keywords: Superior Julia set, Superior Mandelbrot set, Superior escape criterion, SP orbit, complex polynomials.

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1. Introduction

Fractal is a rough or fragmented geometric shape that can be subdivided into congruent parts, each of which is a reduced size copy of the whole. Mathematics has used fractal geometry to present some interesting complex objects to computer graphics. The term “fractal” was first used by Gaston Julia, when he was studying Cayley’s problem in complex plane which is related to the behavior of Newton’s method. In 1918, Gaston Julia obtained a Julia set using the iteration process of complex function [7]. Further in 1975, the idea of Gaston Julia was extended by Benoit Mandelbrot. He introduced the Mandelbrot set; a set of all complex values c , for which the filled Julia set is connected. The fractal geometry of Mandelbrot and Julia sets have been studied for quadratic [1, 7, 17, 19], cubic [2, 4, 5, 16, 17, 18] and higher degree polynomials [30] using Picard orbit, which is an example of one-step feedback process. In 2000, Rochon [6] studied a more generalized form of a Mandelbrot set in a bi-complex plane. Later on, Wang et al. [24-27] carried further analysis of generalized Julia and Mandelbrot sets. In 2006, the fractal structure and discontinuity evolution law of the generalized Julia sets generated from the extended complex mapping $z^n - c (n \in \mathbb{R})$ was obtained [23].

In 2004, Rani and Kumar [8, 9] introduced Superior Julia sets and Superior Mandelbrot sets using Mann iteration scheme which is a two-step feedback process. After this, Rani et al. obtained Superior Julia sets and Superior Mandelbrot sets for quadratic [10,12,15], cubic [13] and n^{th} degree [11] complex polynomials. In 2010, Chauhan et al. [28, 29] obtained new Julia and Mandelbrot sets via Ishikawa iterate which is a three-step feedback process. In 2014, Ashish et al. [3] generated

* E-mail: kumarimandeep28@gmail.com

Julia and Mandelbrot sets using Noor iteration - a four – step feedback process. Recently, M. Kumari et al.[14] obtained further generalizations of Julia and Mandelbrot sets for a new iterative process which is an example of four step feedback process. They generated new Julia and Mandelbrot sets for quadratic, cubic and higher degree polynomials.

In 2011, W. Phuengrattana and S. Suantai [22] proposed the SP-iteration for approximating a fixed point of continuous functions on an arbitrary interval. They gave a necessary and sufficient condition for the convergence of the SP-iteration of continuous functions on an arbitrary interval. They also compared the convergence speed of Mann, Ishikawa, Noor and SP-iterations using some numerical examples and proved that the SP-iteration is equivalent to and converges faster than the other iterations. The purpose of this paper is to generate new Superior Julia sets and Superior Mandelbrot sets via SP iterative procedure, which is also an example of a four-step feedback process. In Section 2, we give some definitions, which are the basis of our work. Section 3 deals with the Superior escape criterions for quadratic, cubic and n^{th} degree polynomials by using SP orbit. In Sections 4 and 5, we generate Superior Mandelbrot sets and Superior Julia sets respectively. Applications of fracals generated via SP-orbit are also given at the end of section 5. Finally, the conclusion of the paper has been given in Section 6.

2. Preliminaries

Definition 2.1 (Picard Orbit [7]). *Let X be a non-empty set and $T : X \rightarrow X$ be a mapping. For a point x_0 in X , the Picard orbit (generally called the orbit of T) is the set of all iterates of a point x_0 , i.e., $O(T, x_0) = \{x_n : x_n = Tx_{n-1}, n = 1, 2, \dots\}$ where the orbit $O(T, x_0)$ of T at the initial point x_0 is the sequence $\{T^n x_0\}$.*

Definition 2.2 (Julia Set [7]). *The filled in Julia set of the function P is defined as $K(P) = \{z \in C : P^k(z) \nrightarrow \infty\}$, where C is the complex plane, $P^k(z)$ is k^{th} iterate of function P and $K(P)$ denotes the filled Julia set. The Julia set of the function P is defined to be the boundary of $K(P)$, i.e. $J(P) = \partial K(P)$, where $J(P)$ denotes the Julia set. The set of points whose orbits are bounded under the Picard orbit $P_c(z) = z^2 + c$ is called the Julia set. We choose the initial point 0 , as 0 is the only critical point of P_c .*

Definition 2.3 (Mandelbrot Set [17]). *The Mandelbrot set M is a set consisting of all complex parameters c for which the filled Julia set of P_c is connected, that is $M = \{c \in C : K(P_c) \text{ is connected}\}$. In fact, M contains a lot of information about the structure of Julia sets. The Mandelbrot set M for the Quadratic polynomial $P_c(z) = z^2 + c$ is defined as the collection of all $c \in C$ for which the orbit of the point 0 is bounded, that is $M = \{c \in C : \{P_c^n(0)\}; n = 0, 1, 2, \dots \text{ is bounded}\}$. We choose the initial point 0 as 0 is the only critical point of P_c .*

Definition 2.4 (SP Orbit [22]). *Let us consider a sequence $\{z_n\}$ of iterates for initial point $z_0 \in X$ such that*

$$\{z_{n+1} : z_{n+1} = (1 - \alpha_n)u_n + \alpha_n T u_n; u_n = (1 - \beta_n)v_n + \beta_n T v_n; v_n = (1 - \gamma_n)z_n + \gamma_n T z_n; n = 0, 1, 2, \dots\},$$

where $\alpha_n, \beta_n, \gamma_n \in [0, 1]$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences of positive numbers. The above sequence of iterates is called as SP orbit, which is a function of five tuples $(T, z_0, \alpha_n, \beta_n, \gamma_n)$.

Definition 2.5 (Superior Julia Set [8]). *The filled in Julia set in SP orbit is known as the filled in Superior Julia set of the function P , which is defined as $SK(P) = \{z \in C : P^k(z) \nrightarrow \infty\}$, where C is the complex plane, $P^k(z)$ is k^{th} iterate of function P and $SK(P)$ denotes the filled Superior Julia set. The Superior Julia set of the function P is defined to be the boundary of $SK(P)$, i.e., $SJ(P) = \partial SK(P)$, where $SJ(P)$ denotes the Superior Julia set. The set of points whose orbits are bounded under the SP orbit $P_c(z) = z^2 + c$ is called the Superior Julia set.*

Definition 2.6 (Superior Mandelbrot Set [9]). *The Mandelbrot set in SP orbit is known as Superior Mandelbrot set SM, which is a set consisting of all complex parameters c for which the filled Superior Julia set of P_c is connected, that is $SM = \{c \in C : SK(P_c) \text{ is connected}\}$.*

In fact, SM contains a lot of information about the structure of Superior Julia sets. The Superior Mandelbrot set SM for the Quadratic polynomial $P_c(z) = z^2 + c$ is defined as the collection of all $c \in C$ for which the orbit of the point 0 is bounded, that is $SM = \{c \in C : \{P_c^n(0); n = 0, 1, 2, \dots\} \text{ is bounded}\}$. For the sake of simplicity, we take $\alpha_n = \alpha$, $\beta_n = \beta$ and $\gamma_n = \gamma$.

3. Superior Escape Criteria For Complex Polynomials in the SP Orbit

The escape criterion is very important in the computation and analysis of Superior Julia sets, Superior Mandelbrot sets and their variants. We have to use the following Superior escape criteria for the quadratics, cubic and higher degree polynomials for generating Superior Julia sets and Superior Mandelbrot sets.

3.1. Superior Escape Criteria for Quadratic Polynomials

Theorem 3.1. *Assume that $|z| \geq |c| > 2/\alpha$, $|z| \geq |c| > 2/\beta$ and $|z| \geq |c| > 2/\gamma$, where $\alpha, \beta, \gamma \in [0, 1]$ and $c \in \mathbb{C}$ (set of complex numbers). Define*

$$\begin{aligned} z_1 &= (1 - \alpha)u + \alpha P_c(u) \\ z_2 &= (1 - \alpha)u_1 + \alpha P_c(u_1) \\ &\dots \\ z_n &= (1 - \alpha)u_{n-1} + \alpha P_c(u_{n-1}) \end{aligned}$$

where $P_c(u)$ can be a quadratic, cubic or higher degree polynomial in terms of β and $n = 1, 2, 3, \dots$, then $|z_n| \rightarrow \infty$ as $n \rightarrow \infty$.

Proof. Consider

$$\begin{aligned} |v| &= |(1 - \gamma)z + \gamma P_c(z)|, \text{ for } P_c(z) = z^2 + c, \\ &= |(1 - \gamma)z + \gamma(z^2 + c)| \\ &= |(1 - \gamma)z + \gamma z^2 + c\gamma| \\ &\geq |\gamma z^2 + (1 - \gamma)z| - |c\gamma| \\ &\geq |\gamma z^2 + (1 - \gamma)z| - \gamma|z| \quad (\because |z| \geq |c|) \\ &\geq \gamma|z^2| - |z| + \gamma|z| - \gamma|z| \\ &= \gamma|z^2| - |z| \\ &\geq |z|(\gamma|z| - 1), \\ |v| &\geq |z|(\gamma|z| - 1) \end{aligned} \tag{1}$$

Also for

$$\begin{aligned}
 |u| &= |(1 - \beta)v + \beta P_c(v)|, \\
 &= |(1 - \beta)v + \beta(v^2 + c)| \\
 &\geq |(1 - \beta)|z|(\gamma|z| - 1) + \beta[{|z|(\gamma|z| - 1)}^2 + c]|
 \end{aligned} \tag{2}$$

Since $|z| \geq (2/\gamma)$ implies $\gamma|z| - 1 > 1$ so

$$|z|(\gamma|z| - 1) > |z|. \tag{3}$$

Using (3) in (2), we have

$$\begin{aligned}
 |u| &\geq |(1 - \beta)|z| + \beta(|z|^2 + c)| \\
 &\geq |\beta|z|^2 + (1 - \beta)|z|| - |\beta c| \\
 &\geq |\beta|z|^2 + (1 - \beta)|z|| - \beta|z| \quad (\because |z| \geq |c|) \\
 &\geq |z|(\beta|z| - 1) \\
 \text{i.e. } |u| &\geq |z|(\beta|z| - 1).
 \end{aligned} \tag{4}$$

Now for $z_n = (1 - \alpha)u_{n-1} + \alpha P(u_{n-1})$, we have

$$\begin{aligned}
 |z_1| &= |(1 - \alpha)u + \alpha P(u)| \\
 &= |(1 - \alpha)u + \alpha(u^2 + c)| \\
 &\geq |(1 - \alpha)|z|(\beta|z| - 1) + \alpha[{|z|(\beta|z| - 1)}^2 + c]|
 \end{aligned} \tag{5}$$

Since $|z| \geq (2/\beta)$ implies $\beta|z| - 1 > 1$ so

$$|z|(\beta|z| - 1) > |z| \tag{6}$$

Using (6) in (5), we get

$$\begin{aligned}
 |z_1| &\geq |(1 - \alpha)|z| + \alpha(|z|^2 - 1)| \\
 &\geq |\alpha|z|^2 + (1 - \alpha)|z|| - \alpha|z| \\
 &\geq |z|(\alpha|z| - 1) \\
 |z_1| &\geq |z|(\alpha|z| - 1).
 \end{aligned}$$

Since $|z| \geq |c| > (2/\alpha)$, $|z| \geq |c| > (2/\beta)$ and $|z| \geq |c| > (2/\gamma)$ exist. Therefore we have $\alpha|z| - 1 > 1$. Hence there exists a $\lambda > 0$ such that $\alpha|z| - 1 > \lambda + 1$. Consequently, we have $|z_1| > (1 + \lambda)|z|$. Particularly, $|z_n| > |z|$. So, repeating the same argument n times we obtain, $|z_n| > (1 + \lambda)^n |z|$. Thus, the orbit of z tends to infinity. Hence the result. \square

From the above theorem, we obtain the following corollaries:

Corollary 3.2. *Assume that $|c| > 2/\alpha$, $|c| > 2/\beta$ and $|c| > 2/\gamma$, then the orbit $SP(P_c, 0, \alpha, \beta, \gamma)$ escapes to infinity.*

In the proof of the theorem, the superior escape criterion used, actually gives us a little more information. In the proof, we used the only fact that $|z| > |c|$ and $|c| > 2/\alpha$, $|c| > 2/\beta$, $|c| > 2/\gamma$. Hence we have the following refinement of the superior escape criterion:

Corollary 3.3 (Superior Escape Criterion). *Suppose $|z| > \max\{|c|, 2/\alpha, 2/\beta, 2/\gamma\}$, then $|z_n| > (1 + \lambda)^n |z|$ and $|z_n| \rightarrow \infty$ as $n \rightarrow \infty$.*

We notice that we may apply Corollary 3.3 to $|z_k|$ for some $k \geq 0$ to have the following result.

Corollary 3.4. *Suppose $|z_k| > \max\{|c|, 2/\alpha, 2/\beta, 2/\gamma\}$, for some $k \geq 0$, then we have, $|z_{k+1}| > (1 + \lambda)^n |z_k|$ and $|z_n| \rightarrow \infty$ as $n \rightarrow \infty$.*

Using this corollary, we obtain an algorithm for computing the filled Superior Julia sets of quadratic P_c , for any complex number ‘ c ’. For given any point ‘ z ’ satisfying $|z| \leq |c|$, we obtain the orbit of ‘ z ’. If, for some ‘ n ’, $|z_n|$ lies outside the circle of radius $\max\{|c|, 2/\alpha, 2/\beta, 2/\gamma\}$, then we observe that the orbit escapes to infinity, which means that ‘ z ’ does not lie in the filled superior Julia set. Otherwise, if $|z_n|$ never exceeds this bound, then by definition ‘ z ’ is in the filled superior Julia set.

3.2. Superior Escape Criteria for Cubic Polynomials

Now, we prove the following theorem for a cubic polynomial $P_c(z) = z^3 + c$, where c is any complex number.

Theorem 3.5. *Suppose that $|z| > |c| > (2/\alpha)^{1/2}$, $|z| > |c| > (2/\beta)^{1/2}$ and $|z| > |c| > (2/\gamma)^{1/2}$, where $0 < \alpha < 1$, $0 < \beta < 1$, $0 < \gamma < 1$ and c is in the complex plane. Define*

$$\begin{aligned} z_1 &= (1 - \alpha)u + \alpha P_c(u), \\ z_2 &= (1 - \alpha)u_1 + \alpha P_c(u_1), \\ &\dots \\ z_n &= (1 - \alpha)u_{n-1} + \alpha P_c(u_{n-1}), n = 1, 2, 3, 4, \dots \end{aligned}$$

where $P_c(u)$ is the function of β , then $|z_n| \rightarrow \infty$ as $n \rightarrow \infty$.

Proof. Consider

$$\begin{aligned} |v| &= |(1 - \gamma)z + \gamma P(z)|, \text{ for } P_c(z) = z^3 + c, \\ &= |(1 - \gamma)z + \gamma(z^3 + c)| \\ &= |(1 - \gamma)z + \gamma z^3 + c\gamma| \\ &\geq |\gamma z^3 + (1 - \gamma)z| - |c\gamma| \\ &\geq |\gamma z^3 + (1 - \gamma)z| - \gamma |z| \quad (\because |z| \geq |c|) \\ &\geq \gamma |z^3| - |z| + \gamma |z| - \gamma |z| \\ &= \gamma |z^3| - |z| \geq |z| (\gamma |z^2| - 1), \\ \text{i.e. } |v| &\geq |z| (\gamma |z^2| - 1) \end{aligned} \tag{7}$$

Also for

$$\begin{aligned} |u| &= |(1 - \beta)v + \beta P_c(v)| \\ &= |(1 - \beta)v + \beta (v^3 + c)| \\ &\geq \left| (1 - \beta) |z| (\gamma |z|^2 - 1) + \beta \left[\left\{ |z| (\gamma |z|^2 - 1) \right\}^3 + c \right] \right| \end{aligned} \tag{8}$$

Since $|z| \geq (2/\gamma)^{1/2}$ implies $\gamma|z|^2 - 1 > 1$ so

$$|z|(\gamma|z|^2 - 1) > |z|. \quad (9)$$

Using (9) in (8), we have

$$\begin{aligned} |u| &\geq |(1 - \beta)|z| + \beta(|z|^3 + c)| \\ &\geq |\beta|z|^3 + (1 - \beta)|z|| - |\beta c| \\ &\geq |\beta|z|^3 + (1 - \beta)|z|| - \beta|z| \quad (\because |z| \geq |c|) \\ &\geq |z|(\beta|z|^2 - 1) \\ \text{i.e. } |u| &\geq |z|(\beta|z|^2 - 1) \end{aligned} \quad (10)$$

Now for $z_n = (1 - \alpha)u_{n-1} + \alpha P_c(u_{n-1})$, we have

$$\begin{aligned} |z_1| &= |(1 - \alpha)u + \alpha P_c(u)| \\ &= |(1 - \alpha)u + \alpha(u^3 + c)| \\ &\geq \left| (1 - \alpha)|z|(\beta|z|^2 - 1) + \alpha \left[\{|z|(\beta|z|^2 - 1)\}^3 + c \right] \right| \end{aligned} \quad (11)$$

Since $|z| \geq (2/\beta)^{1/2}$ implies $\beta|z|^2 - 1 > 1$ so

$$|z|(\beta|z|^2 - 1) > |z|. \quad (12)$$

Using (12) in (11), we get

$$\begin{aligned} |z_1| &\geq |(1 - \alpha)|z| + \alpha(|z|^3 - 1)| \\ &\geq |\alpha|z|^3 + (1 - \alpha)|z|| - \alpha|z| \\ &\geq |z|(\alpha|z|^2 - 1) \\ \text{i.e. } |z_1| &\geq |z|(\alpha|z|^2 - 1). \end{aligned}$$

Since $|z| > (2/\alpha)^{1/2}$, $|z| > (2/\beta)^{1/2}$ and $|z| > (2/\gamma)^{1/2}$ exist. Therefore, we have $\alpha|z|^2 - 1 > 1$. Hence, there exists a $\lambda > 1$ such that $|z_1| > \lambda|z|$. Repeating this inequality n times, we obtain $|z_n| > \lambda^n|z|$. Therefore, the orbit of z under the cubic polynomial $P_c(z)$ tends to infinity. Hence the result. \square

Using above result, we obtain the following corollaries.

Corollary 3.6 (Superior Escape Criterion). *Let $P_c(z) = z^3 + c$, where c is any complex number. If $|z| > \max\{|c|, (2/\alpha)^{1/2}, (2/\beta)^{1/2}, (2/\gamma)^{1/2}\}$, then $|z_n| \rightarrow \infty$ as $n \rightarrow \infty$. This gives the Superior escape criterion for a cubic polynomial.*

Corollary 3.7. *For some $k \geq 0$, let us suppose $|z_k| > \max\{|c|, (2/\alpha)^{1/2}, (2/\beta)^{1/2}, (2/\gamma)^{1/2}\}$. Then $|z_{k+1}| > \lambda|z_k|$ and $|z_n| \rightarrow \infty$ as $n \rightarrow \infty$.*

We find that Corollary 3.7 gives the algorithm to generate Superior Julia sets for cubic polynomial $P_c(z)$.

3.3. A General Superior Escape Criterion

Now, we will obtain a general Superior escape criterion for polynomials of the form $G_c(z) = z^n + c$.

Theorem 3.8. *For a general function $G_c(z) = z^n + c$, $n = 1, 2, 3, \dots$, where $0 < \alpha < 1$, $0 < \beta < 1$, $0 < \gamma < 1$, and c is a complex number. Define*

$$\begin{aligned} z_1 &= (1 - \alpha)u + \alpha G_c(u) \\ z_2 &= (1 - \alpha)u_1 + \alpha G_c(u_1) \\ &\dots \\ z_n &= (1 - \alpha)u_{n-1} + \alpha G_c(u_{n-1}), \quad n = 1, 2, 3, 4, \dots \end{aligned}$$

Then, the general Superior escape criterion is $\max\{|c|, (2/\alpha)^{1/n-1}, (2/\beta)^{1/n-1}, (2/\gamma)^{1/n-1}\}$.

Proof. For proving the theorem, we shall use the method of induction.

For $n = 1$, we have $G_c(z) = z + c$, and this implies

$$|z| > \max\{|c|, 0, 0, 0\}.$$

For $n = 2$, we have $G_c(z) = z^2 + c$, so by Theorem 3.1 the superior escape criterion is

$$|z| > \max\{|c|, 2/\alpha, 2/\beta, 2/\gamma\}.$$

Similarly, for $n = 3$, we get $G_c(z) = z^3 + c$. Then the superior escape criterion from the above theorem is

$$|z| > \max\{|c|, (2/\alpha)^{1/2}, (2/\beta)^{1/2}, (2/\gamma)^{1/2}\}.$$

Hence the theorem is true for $n = 1, 2, 3, \dots$. Now, suppose that theorem is true for any n . We prove that the result is true for $n + 1$. Let $G_c(z) = z^{n+1} + c$ and $|z| \geq |c| > (2/\alpha)^{1/n}$, $|z| \geq |c| > (2/\beta)^{1/n}$ and $|z| \geq |c| > (2/\gamma)^{1/n}$. Then, consider

$$\begin{aligned} |v| &= |(1 - \gamma)z + \gamma G_c(z)|, \quad \text{where } G_c(z) = z^{n+1} + c \\ &= |(1 - \gamma)z + \gamma(z^{n+1} + c)| \\ &= |z| |\gamma |z^n| - \gamma + 1| - \gamma |c| \\ &= |z| (\gamma |z^n| - 1) + \gamma |z| - \gamma |z| \quad (\because |z| \geq |c|) \\ \text{i.e. } |v| &= |z| (\gamma |z^n| - 1) \end{aligned} \tag{13}$$

Also,

$$\begin{aligned} |u| &= |(1 - \beta)v + \beta G_c(v)| \\ &= |(1 - \beta)v + \beta(v^{n+1} + c)| \\ &\geq \left| (1 - \beta) |z| (\gamma |z^n| - 1) + \beta \left[\{ |z| (\gamma |z^n| - 1) \}^{n+1} + c \right] \right| \end{aligned} \tag{14}$$

Since $|z| \geq (2/\gamma)^{1/n}$ implies $\gamma |z|^n - 1 > 1$ so

$$|z| (\gamma |z|^n - 1) > |z|. \tag{15}$$

Using (15) in (14), we have

$$\begin{aligned}
 |u| &\geq |(1 - \beta)|z| + \beta(|z|^{n+1} + c)| \\
 &\geq |\beta|z|^{n+1} + (1 - \beta)|z|| - |\beta c| \\
 &\geq |\beta|z|^{n+1} + (1 - \beta)|z|| - \beta|z| \quad (\because |z| \geq |c|) \\
 &\geq |z|(\beta|z|^n - 1) \\
 \text{i.e. } |u| &\geq |z|(\beta|z|^n - 1)
 \end{aligned} \tag{16}$$

Now for $z_n = (1 - \alpha)u_{n-1} + \alpha G_c(u_{n-1})$, we have

$$\begin{aligned}
 |z_1| &= |(1 - \alpha)u + \alpha G_c(u)| \\
 &= |(1 - \alpha)u + \alpha(u^{n+1} + c)| \\
 &\geq \left| (1 - \alpha)|z|(\beta|z|^n - 1) + \alpha \left[|z|(\beta|z|^n - 1) \right]^{n+1} + c \right|
 \end{aligned} \tag{17}$$

Since $|z| \geq (2/\beta)^{1/n}$ implies $\beta|z|^n - 1 > 1$ so

$$|z|(\beta|z|^n - 1) > |z|. \tag{18}$$

Using (18) in (17), we get

$$\begin{aligned}
 |z_1| &\geq |(1 - \alpha)|z| + \alpha(|z|^{n+1} - 1)|, \\
 &\geq |\alpha|z|^{n+1} + (1 - \alpha)|z|| - \alpha|z| \\
 &\geq |z|(\alpha|z|^n - 1) \\
 \text{i.e. } |z_1| &\geq |z|(\alpha|z|^n - 1).
 \end{aligned}$$

Since $|z| > (2/\alpha)^{1/n}$, $|z| > (2/\beta)^{1/n}$ and $|z| > (2/\gamma)^{1/n}$ exist, we have $\alpha|z|^n - 1 > 1 + \lambda > 1$. In particular,

$$\begin{aligned}
 |z_1| &> (1 + \lambda)|z| \\
 &\vdots \\
 |z_n| &> (1 + \lambda)^n |z|
 \end{aligned}$$

Hence, $|z_n| \rightarrow \infty$ as $n \rightarrow \infty$. This completes the proof. \square

Corollary 3.9. Suppose $|c| > (2/\alpha)^{1/n-1}$, $|c| > (2/\beta)^{1/n-1}$ and $|c| > (2/\gamma)^{1/n-1}$ exists. Then the orbit $SP(G_c, 0, \alpha, \beta, \gamma)$ escapes to infinity.

Corollary 3.10 (Superior Escape Criterion). Let us assume that for some $k \geq 0$,

$|z_k| > \max\{|c|, (2/\alpha)^{1/k-1}, (2/\beta)^{1/k-1}, (2/\gamma)^{1/k-1}\}$, then $|z_k| > \lambda|z_{k-1}|$ and $|z_n| \rightarrow \infty$ as $n \rightarrow \infty$.

This corollary gives an algorithm to generate Superior Julia sets for the functions of the type $G_c(z) = z^n + c$, $n = 2, 3, \dots$

Remark 3.11.

- (1). Gujar et al.[20, 21] followed the escape criterion of the quadratic polynomial $z^2 + c$ to generate the Julia and Mandelbrot sets for $n \in \mathbb{R}$ and $n \geq 0$.
- (2). Using Mathematica 10.0, we have computed a few Superior Julia sets and Superior Mandelbrot sets for quadratic, cubic and higher degree polynomials. In 1992, Gujar et al. [20] observed that the fractals generated by the mapping $P_c(z) = z^n + c$ contain $(n - 1)$ lobes. They have reflectional and rotational symmetry with an angle $2\pi/(n - 1)$.

4. Superior Mandelbrot Sets in the SP Orbit

The Superior Mandelbrot sets computed in SP orbit for quadratic, cubic and higher degree polynomials are analyzed as follows:

4.1. Superior Mandelbrot Sets for the Quadratic Polynomial $P_c(z) = z^2 + c$

Following results have been obtained and analyzed from the graphical patterns of Superior quadratic Mandelbrot sets:

- (1). In Figs. 1-9, it is observed that the Superior Mandelbrot sets have reflection symmetry about x-axis.
- (2). Figs. 4-6 show that as we decrease the value of β , the number of loops increase and the main cardioid becomes slimmer.
- (3). In figs. 7-9, we observed that by decreasing the values of α , β and γ the Superior Mandelbrot sets become fatter.
- (4). In figs. 10-12, the Superior Mandelbrot sets are symmetrical about x-axis and y-axis.

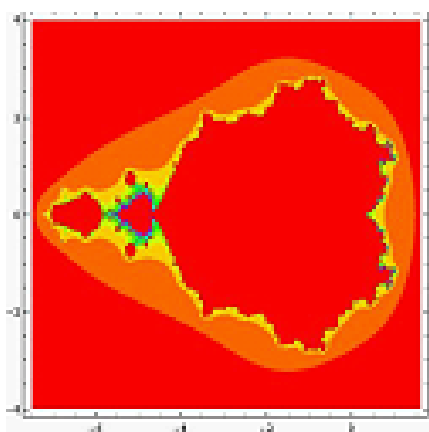


Figure 1: Superior Mandelbrot set for $\alpha = 0.7, \beta = 0.2, \gamma = 0.4$

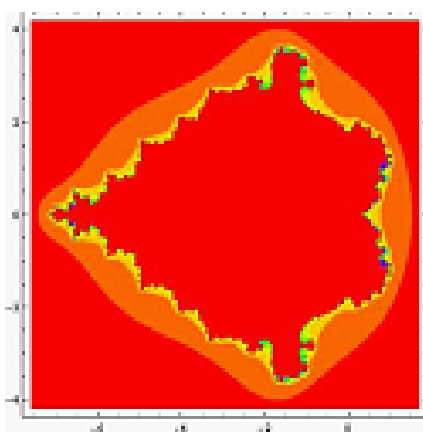


Figure 2: Superior Mandelbrot set for $\alpha = 0.4, \beta = 0.2, \gamma = 0.7$

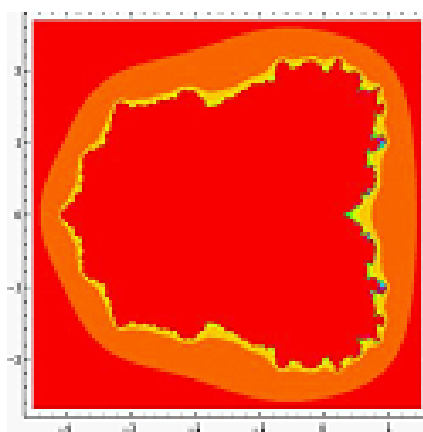


Figure 3: Superior Mandelbrot set for $\alpha = 0.7, \beta = 0.4, \gamma = 0.2$

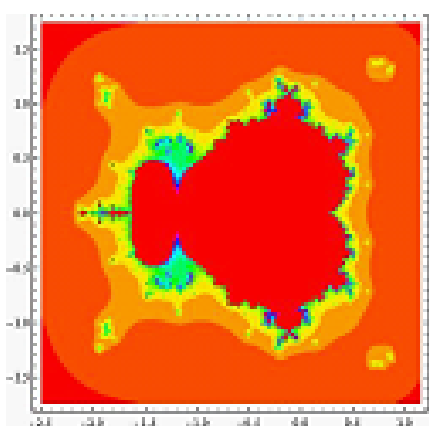


Figure 4: Superior Mandelbrot set for $\alpha = \beta = 0.9, \gamma = 0.1$

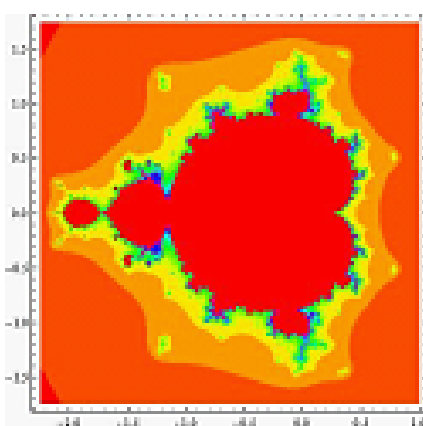


Figure 5: Superior Mandelbrot set for $\alpha = 0.1, \beta = \gamma = 0.9$

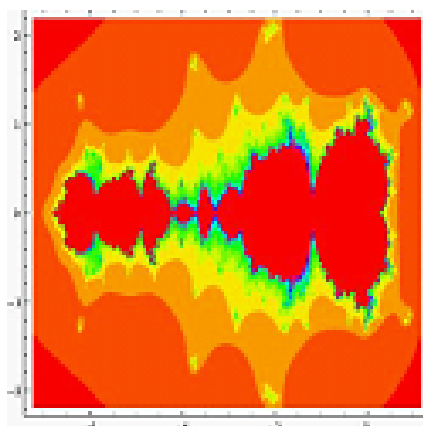


Figure 6: Superior Mandelbrot set for $\alpha = 0.9, \beta = 0.1, \gamma = 0.9$

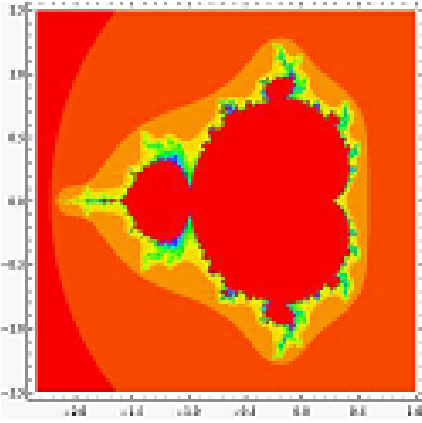


Figure 7: Superior Mandelbrot set for $\alpha = \beta = \gamma = 0.9$

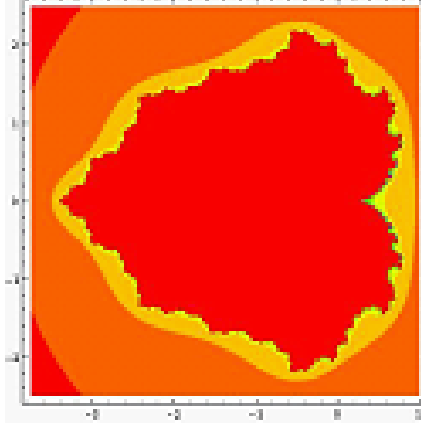


Figure 8: Superior Mandelbrot set for $\alpha = \beta = \gamma = 0.5$

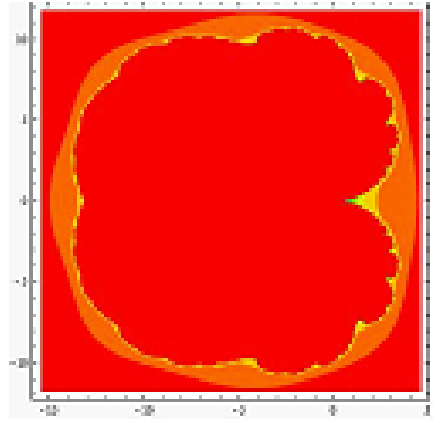


Figure 9: Superior Mandelbrot set for $\alpha = \beta = \gamma = 0.1$

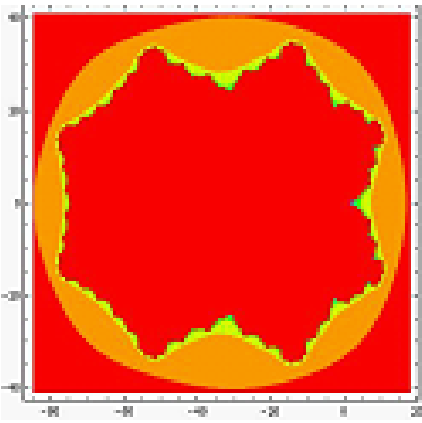


Figure 10: Superior Mandelbrot set for $\alpha = \gamma = 0.5, \beta = 0.015$

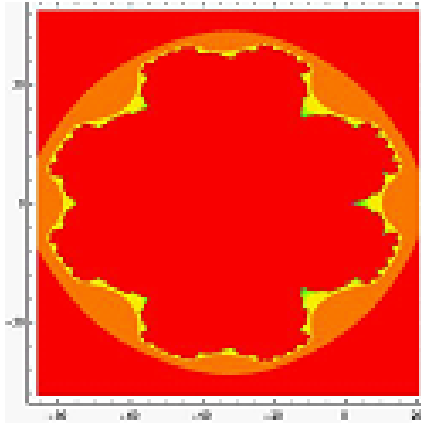


Figure 11: Superior Mandelbrot set for $\alpha = \beta = 0.015, \gamma = 0.05$

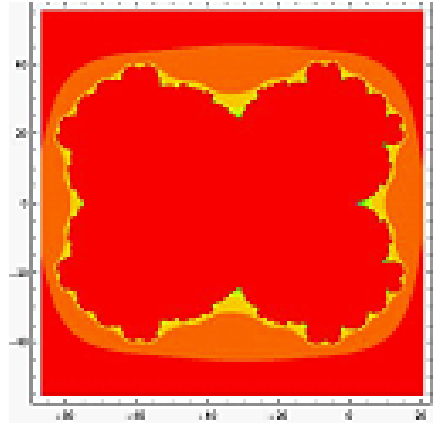


Figure 12: Superior Mandelbrot set for $\alpha = 0.05, \beta = \gamma = 0.015$

4.2. Superior Mandelbrot Sets for the Cubic Polynomial and Higher Degree Polynomials

In case of cubic and higher degree polynomials, we observe the following results:

- (1). In figs. 13, 14, 15, 20, 21, the Superior Mandelbrot sets have reflection symmetry along x -axis as well as y -axis.
- (2). Some Superior Mandelbrot sets have reflection symmetry only along x -axis.(see figs.16-19,22,23)
- (3). It is observed that in fig.13, the Superior Mandelbrot set took the shape of decorated Coupled Lightning Lamp (Diya in Hindi language).
- (4). In figs.14 & 15, it was also found that Superior Mandelbrot sets looked like decorated Coupled Urns (Kalash in Hindi language).
- (5). In figs. 16-22, Superior Mandelbrot sets took the shape of beautiful scared patterns (Rangolies in Hindi language).
- (6). The Superior Mandelbrot sets computed for higher degree polynomials have the rotational symmetry about the centre. In figs. 23-24, the shapes of Superior Mandelbrot sets are similar to Circular saw or Sudarshan Chakra.

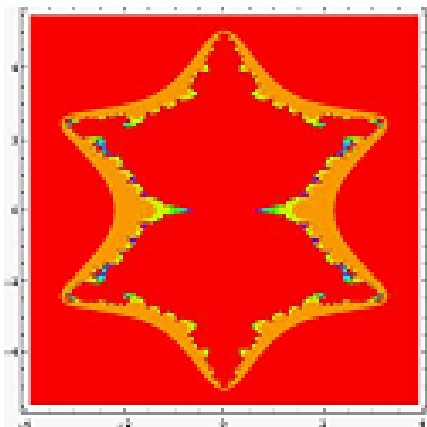


Figure 13: Superior Mandelbrot set for $n = 3, \alpha = 0.015, \beta = \gamma = 0.115$

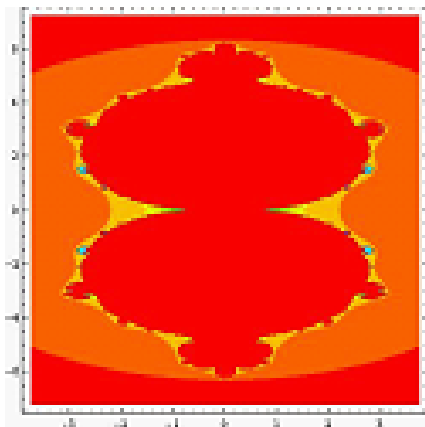


Figure 14: Superior Mandelbrot set for $n = 3, \alpha = \beta = \gamma = 0.05$

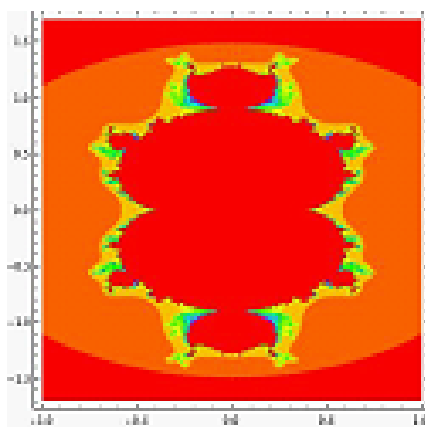


Figure 15: Superior Mandelbrot set for $n = 3, \alpha = \beta = \gamma = 0.9$

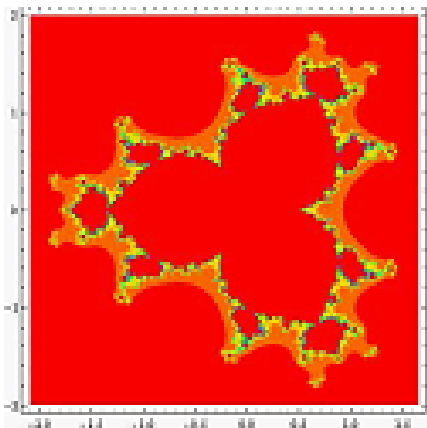


Figure 16: Superior Mandelbrot set for $n = 4, \alpha = 0.064, \beta = 0.3, \gamma = 0.99$

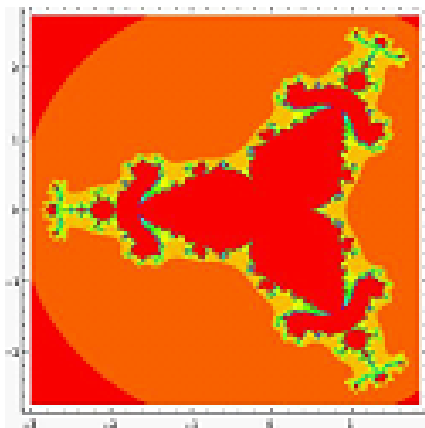


Figure 17: Superior Mandelbrot set for $n = 4, \alpha = 0.3, \beta = 0.064, \gamma = 0.99$

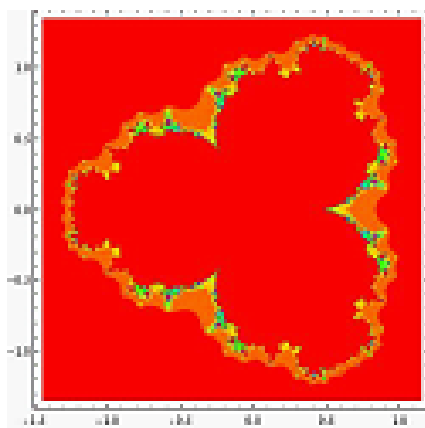


Figure 18: Superior Mandelbrot set for $n = 4, \alpha = 0.3, \beta = 0.99, \gamma = 0.064$

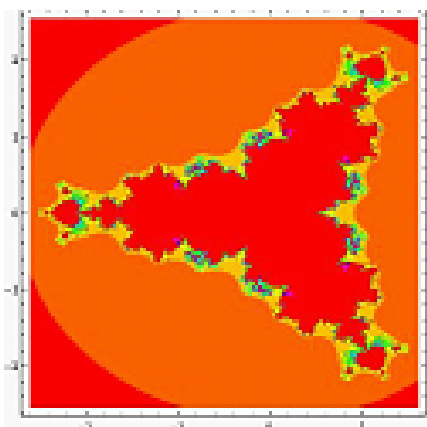


Figure 19: Superior Mandelbrot set for $n = 4, \alpha = 0.99, \beta = 0.1, \gamma = 0.1$

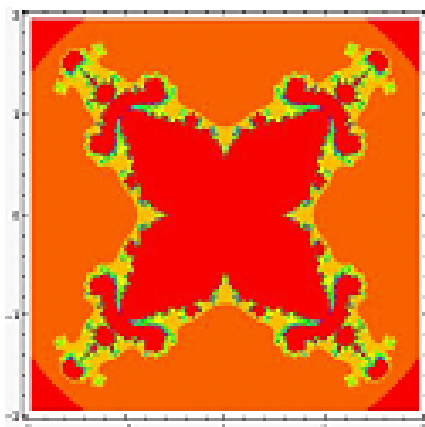


Figure 20: Superior Mandelbrot set for $n = 5, \alpha = 0.3, \beta = 0.06, \gamma = 0.98$

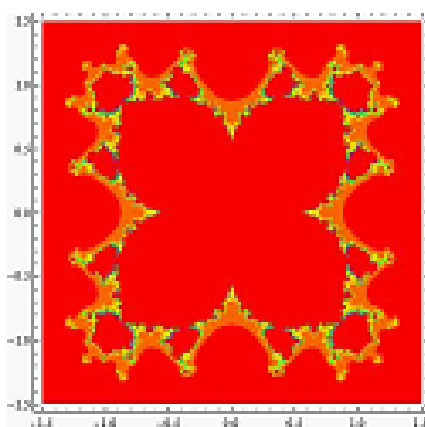


Figure 21: Superior Mandelbrot set for $n = 5, \alpha = 0.06, \beta = 0.3, \gamma = 0.98$

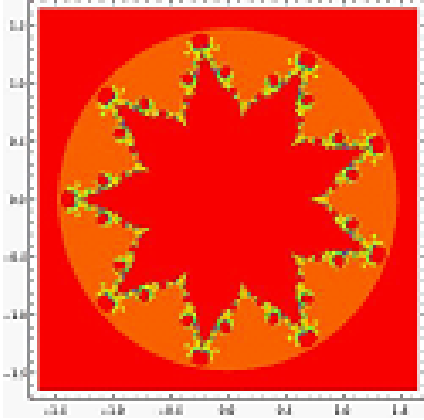


Figure 22: Superior Mandelbrot set for $n = 10, \alpha = 0.3, \beta = 0.06, \gamma = 0.98$

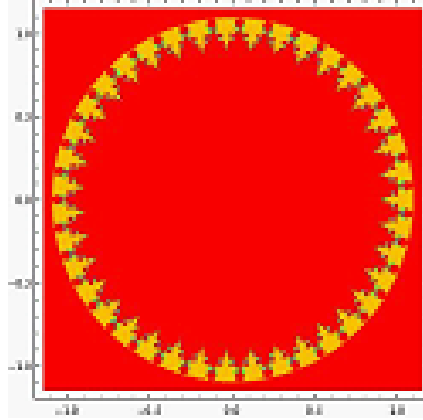


Figure 23: Superior Mandelbrot set for $n = 40, \alpha = 0.3, \beta = 0.06, \gamma = 0.98$

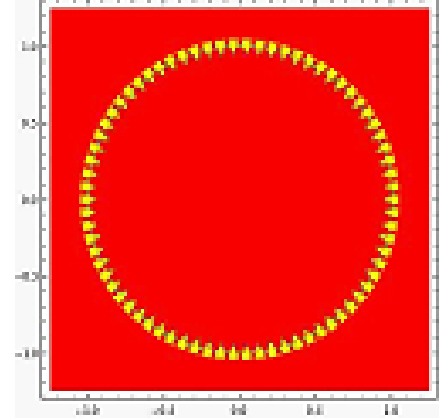


Figure 24: Superior Mandelbrot set for $n = 75, \alpha = 0.6, \beta = 0.6, \gamma = 0.1$

5. Superior Julia Sets in SP Orbit

In this section, we have generated some Superior Julia sets in the SP orbit for quadratic, cubic and higher degree polynomials. Here we used the idea of Wang et al. [26] to study the escape lines around the filled superior Julia sets. It is necessary to shade the escape line area differently to present approximate filled superior Julia sets. Therefore, they have been shaded using different color plates. In our analysis, we found that the escape points that lie near the boundary of filled Superior Julia sets (i.e., in the first escape line area) have been colored blue. The areas/points that are colored blue, green, yellow and orange are in the decreasing order of the number of iterations necessary to escape. Further, the red color indicates the orbits that escape most quickly.

5.1. Superior Julia Sets for the Quadratic Polynomial $P_c(z) = z^2 + c$

From the graphical representation of Superior Julia sets, we got the following observations:

- (1). In figs. 25 and 26, we saw that as we fix the value of c and decrease the values of α and β , the quadratic Superior Julia set becomes disconnected.
- (2). In figs. 28-30, we observed that the Superior Julia sets become slimmer as we increase the values of α, β and γ and the value of c remains fixed.

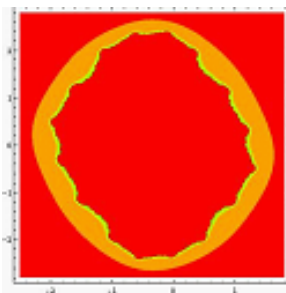


Figure 25: Superior Julia set for $c = -0.23 + 0.23I, \alpha = 0.3, \beta = 0.6, \gamma = 0.9$

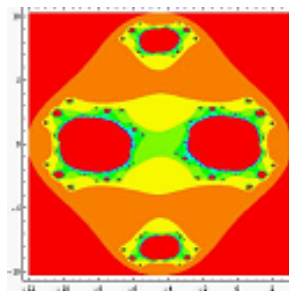


Figure 26: Superior Julia set for $c = -0.23 + 0.23I, \alpha = \beta = 0.1, \gamma = 0.9$

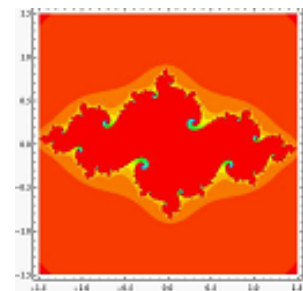


Figure 27: Superior Julia set for $c = 0.75 + 0.1I, \alpha = \beta = \gamma = 0.991$

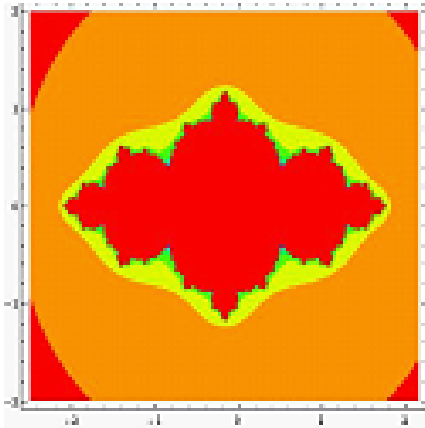


Figure 28: Superior Julia set for $c = -1.38, \alpha = \beta = \gamma = 0.75$

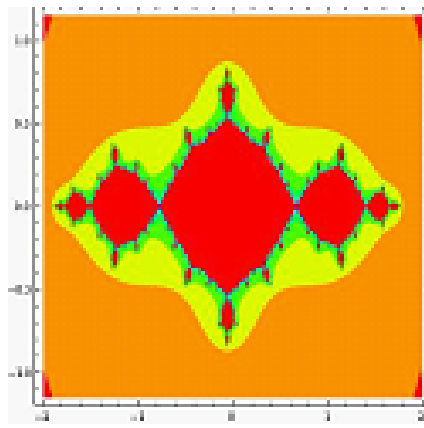


Figure 29: Superior Julia set for $c = -1.38, \alpha = \beta = \gamma = 0.9$

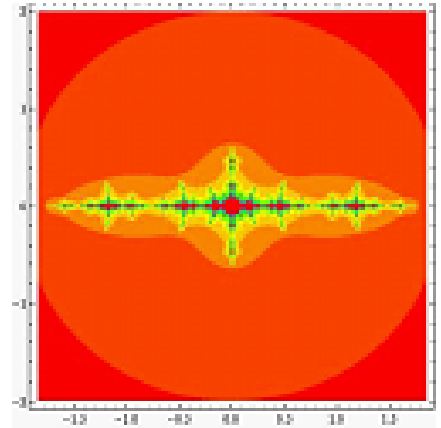


Figure 30: Superior Julia set for $c = -1.38, \alpha = \beta = \gamma = 1$

5.2. Superior Julia Sets for the Cubic Polynomial and Higher Degree Polynomials

We got the following observations from the graphical representations of Cubic and higher Superior Julia sets:

- (1). In figs 31-33, fixing the value of c , it was analyzed that when any two parameters out of α, β and γ are fixed then the number of loops in Cubic Superior Julia sets increased with the increment in the third parameter and got fatter with increase in the second parameter.
- (2). In figs. 34-35, as we put the equal value to all the three parameters and increase them monotonically we find that Cubic Superior Julia sets become more decorated.
- (3). For fixed c -values and different values of α, β and γ , we get different structures of Cubic Superior Julia sets. (figs.37-39)
- (4). Fig. 41 shows the disconnectivity of orbits of Superior Julia sets.
- (5). Further, the Superior Julia sets computed for higher degree polynomials have the rotational symmetry about the centre. The shapes of Superior Julia sets are similar to Circular saw or Sudarshan Chakra [Figs.44, 45].

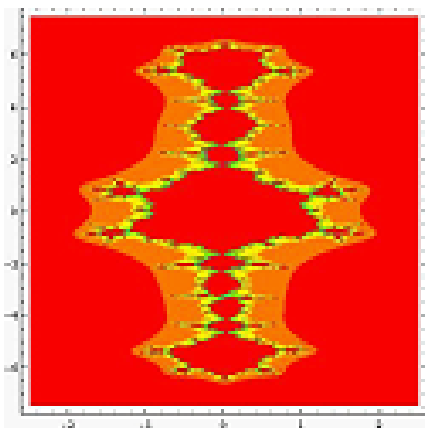


Figure 31: Cubic Superior Julia set for $c = -0.2 + 0.6I, \alpha = 0.7, \beta = \gamma = 0.03$

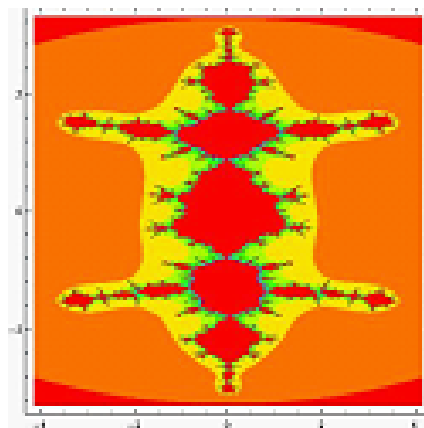


Figure 32: Cubic Superior Julia set for $c = -0.2 + 0.6I, \alpha = \beta = 0.03, \gamma = 0.7$

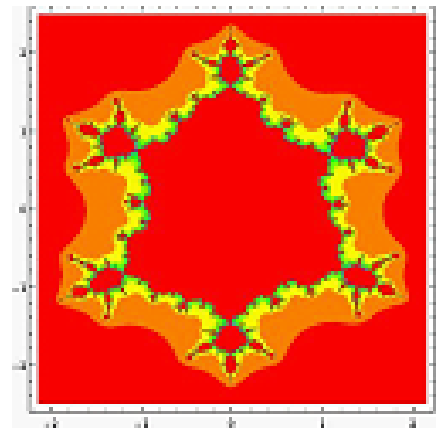


Figure 33: Cubic Superior Julia set for $c = -0.2 + 0.6I, \alpha = 0.03, \beta = 0.7, \gamma = 0.03$

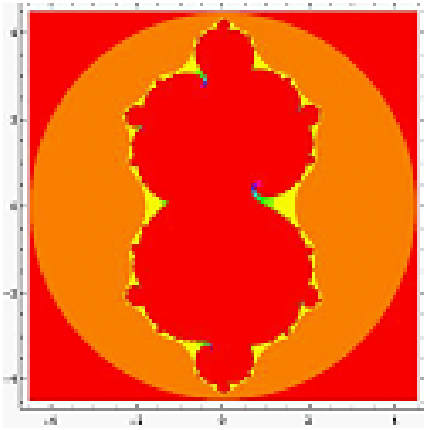


Figure 34: Cubic Superior Julia set for $c = 0.7 - 0.4I, \alpha = \beta = \gamma = 0.1$

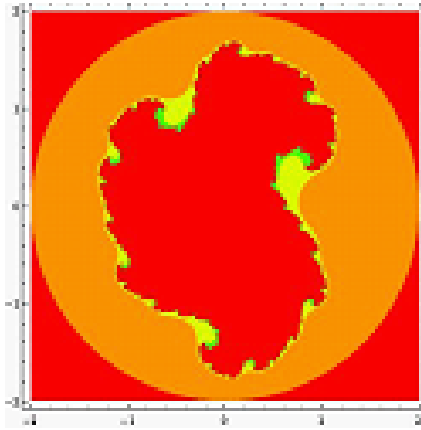


Figure 35: Cubic Superior Julia set for $c = 0.7 - 0.4I, \alpha = \beta = \gamma = 0.5$

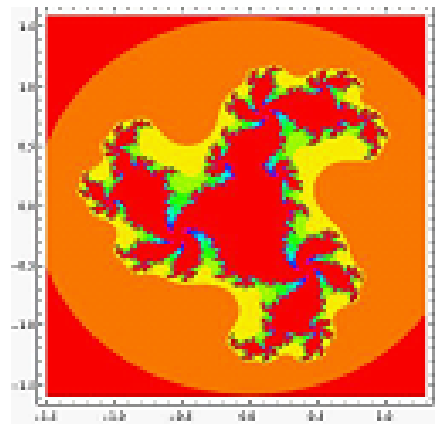


Figure 36: Cubic Superior Julia set for $c = 0.7 - 0.4I, \alpha = \beta = \gamma = 0.8$

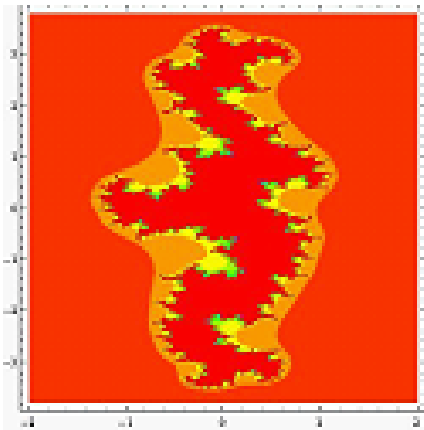


Figure 37: Cubic Superior Julia set for $c = 0.7 - 0.4I, \alpha = 1, \beta = 0.1, \gamma = 0.5$

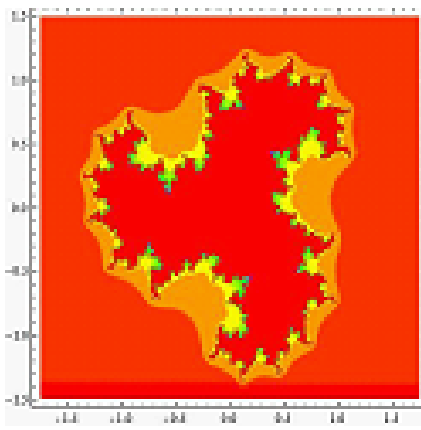


Figure 38: Cubic Superior Julia set for $c = 0.7 - 0.4I, \alpha = 0.5, \beta = 1, \gamma = 0.1$

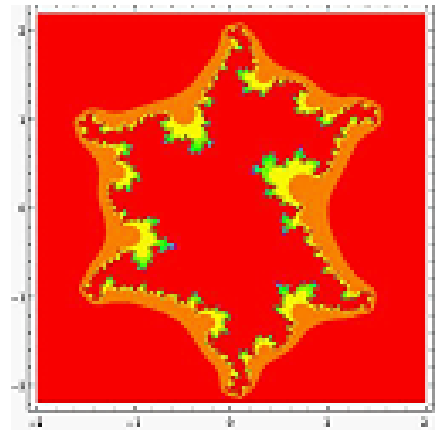


Figure 39: Cubic Superior Julia set for $c = 0.7 - 0.4I, \alpha = 0.1, \beta = 0.5, \gamma = 1$

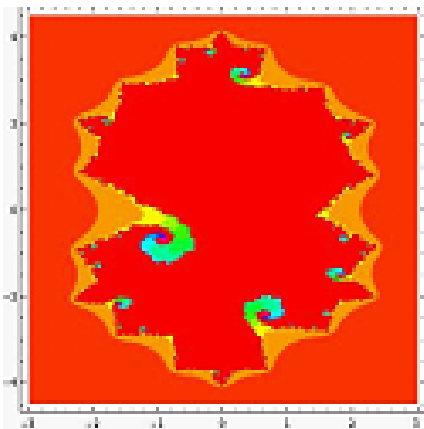


Figure 40: Cubic Superior Julia set for $c = -1.39 + 0.90I, \alpha = 0.11, \beta = 0.12, \gamma = 0.03$

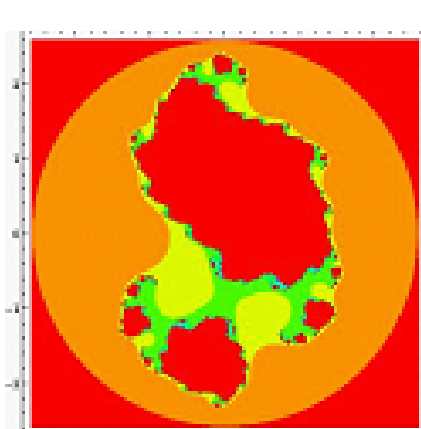


Figure 41: Cubic Superior Julia set for $c = -1.39 + 0.90I, \alpha = \beta = \gamma = 0.3$

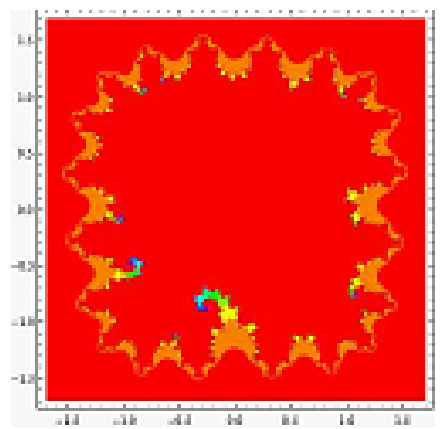


Figure 42: Superior Julia set for $n=5, c = 0.3 - 0.8I, \alpha = 0.01, \beta = 0.3, \gamma = 0.03$

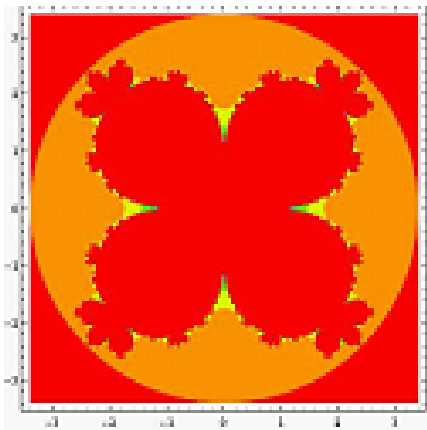


Figure 43: Superior Julia set for $n=5$,
 $c = 0.04 - 0.4I$, $\alpha = \beta = \gamma = 0.015$

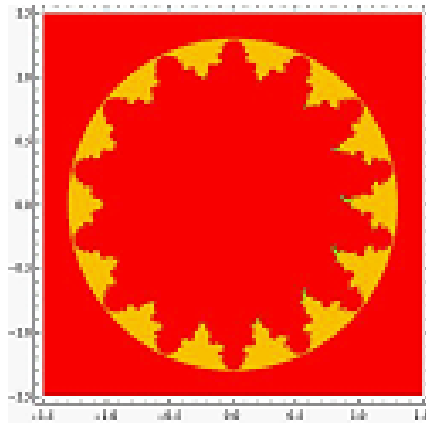


Figure 44: Superior Julia set for $n=15$,
 $c = 0.7 - 0.4I$, $\alpha = 0.1$, $\beta = 0.05$,
 $\gamma = 0.9$

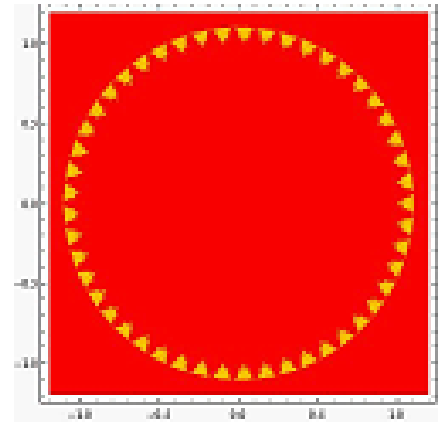


Figure 45: Superior Julia set for $n=50$,
 $c = 0.04 - 0.4I$, $\alpha = \beta = \gamma = 0.015$

5.3. Applications of Fracals Generated Via SP Orbit

The Julia sets and Mandelbrot sets have potential applications in image processing, cryptography, data transmission, image texture analysis and other useful areas in our modern era [27]. In 2014, Ashish et. al [3] introduced a four-step iterative procedure in the study of fractal theory and showed that it converges more rapidly than the Picard iterative procedure. W. Phuengrattana et.al [22] proposed the SP-iteration and proved numerically that it converges faster than the other iterations. Therefore to determine the points in the Superior Mandelbrot sets and the Superior Julia sets, the SP orbit supports more rapidly than the Noor orbit. Thus the future work on fractal compression, data transmission and in other fields may be improved by using the *SP orbit*.

6. Conclusion

In this paper, Superior Julia sets and Superior Mandelbrot sets for complex quadratic, cubic and n th degree polynomials have been generated via SP iterative procedures. In our analysis, we find the following conclusions:

1. Using SP-iterative procedure, 20-35 iterations are usually enough for a good approximation of Superior Mandelbrot sets and Superior Julia sets, while other iterative procedures Mann, Ishikawa, Noor etc. require more number of iterations for a good approximation of fractals.
2. The graphical representation of Superior Mandelbrot sets and Superior Julia sets showed that their shapes depend upon the values of α , β and γ . Using different values of α , β and γ , we found that a few Superior Mandelbrot sets took the shape of decorated coupled urns, a decorated coupled lightening lamp (Diya in Hindi language) and beautiful scared patterns (Rangolies in Hindi language).
3. For higher degree polynomials the Superior Mandelbrot sets and Superior Julia sets both have the rotational symmetry about the centre and their shapes are similar to Circular saw or Sudarshan Chakra.
4. The results obtained in this paper may provoke further research in the field of Fractal theory.

Acknowledgements

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