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# The Triple $\chi^3$ Sequence Spaces

Research Article

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**Abstract:** Let $\chi^3$ denote the space of all triple gai sequences and $\Lambda^3$ the space of all triple analytic sequences. This paper some theorems on general properties of $\chi^3$ sequences have been established.

**MSC:** 40A05,40C05,40D05.

**Keywords:** Gai sequence, analytic sequence, triple sequence, dual space, AK space, solid, $\delta-$ dual.


## 1. Introduction

Throughout $w, \chi$ and $\Lambda$ denote the classes of all, gai and analytic scalar valued single sequences, respectively. We write $w^3$ for the set of all complex sequences $(x_{mnk})$, where $m, n, k \in \mathbb{N}$, the set of positive integers. Then, $w^3$ is a linear space under the coordinate wise addition and scalar multiplication. Let $(x_{mnk})$ be a triple sequence of real or complex numbers. Then the series $\sum_{m,n,k=1}^{\infty} x_{mnk}$ is called a triple series. The triple series $\sum_{m,n,k=1}^{\infty} x_{mnk}$ is said to be convergent if and only if the triple sequence $(S_{mnk})$is convergent, where

$$S_{mnk} = \sum_{i,j,q=1}^{m,n,k} x_{ijq}(m, n, k = 1, 2, 3, ...).$$

A sequence $x = (x_{mnk})$is said to be triple analytic if

$$\sup_{m,n,k} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty.$$

The vector space of all triple analytic sequences are usually denoted by $\Lambda^3$. A sequence $x = (x_{mnk})$ is called triple entire sequence if

$$|x_{mnk}|^{\frac{1}{m+n+k}} \to 0 \text{ as } m, n, k \to \infty.$$

The vector space of all triple entire sequences are usually denoted by $\Gamma^3$. The space $\Lambda^3$ and $\Gamma^2$ is a metric space with the metric

$$d(x, y) = sup_{m,n,k} \left\{ |x_{mnk} - y_{mnk}|^{\frac{1}{m+n+k}} : m, n, k : 1, 2, 3, ... \right\}, \tag{1}$$

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forall $x = \{x_{mnk}\}$ and $y = \{y_{mnk}\} in\Gamma^3$. Let $\phi = \{finite\ sequences\}$. Consider a double sequence $x = (x_{mnk})$. The $(m, n, k)^{th}$ section $x^{[m,n,k]}$ of the sequence is defined by $x^{[m,n,k]} = \sum_{i,j,q=0}^{m,n,k} x_{ijq}\delta_{ijq}$ for all $m, n, k \in \mathbb{N}$,

$$\delta_{mnk} = \begin{pmatrix} 0 & 0 & \ldots 0 & 0 & \ldots \\ 0 & 0 & \ldots 0 & 0 & \ldots \\ \vdots \\ 0 & 0 & \ldots 1 & 0 & \ldots \\ 0 & 0 & \ldots 0 & 0 & \ldots \end{pmatrix}$$

with 1 in the $(m, n, k)^{th}$ position and zero other wise. A sequence $x = (x_{mnk})$ is called triple gai sequence if $((m + n + k)!\,|x_{mnk}|)^{\frac{1}{m+n+k}} \to 0$ as $m, n, k \to \infty$. The triple gai sequences will be denoted by $\chi^3$. Consider a triple sequence $x = (x_{mnk})$. The $(m, n, k)^{th}$ section $x^{[m,n,k]}$ of the sequence is defined by $x^{[m,n,k]} = \sum_{i,j,q=0}^{m,n,k} x_{ijq}\Im_{ijq}$ for all $m, n, k \in \mathbb{N}$; where $\Im_{ijq}$ denotes the triple sequence whose only non zero term is a $\frac{1}{(i+j+k)!}$ in the $(i, j, k)^{th}$ place for each $i, j, k \in \mathbb{N}$. An FK-space(or a metric space)$X$ is said to have AK property if $(\Im_{mnk})$ is a Schauder basis for $X$, or equivalently $x^{[m,n,k]} \to x$. An FDK-space is a triple sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings are continuous. If $X$ is a sequence space, we give the following definitions:

(1). $X'$ is continuous dual of $X$;

(2). $X^{\alpha} = \left\{a = (a_{mnk}) : \sum_{m,n,k=1}^{\infty} |a_{mnk}x_{mnk}| < \infty,\ for\ \ each\ \ x \in X\right\}$;

(3). $X^{\beta} = \left\{a = (a_{mnk}) : \sum_{m,n,k=1}^{\infty} a_{mnk}x_{mnk}\ \ is\ \ convergent,\ \ for\ \ each\ \ x \in X\right\}$;

(4). $X^{\gamma} = \left\{a = (a_{mn}) : \sup_{m,n\geq 1} \left|\sum_{m,n,k=1}^{M,N,K} a_{mnk}x_{mnk}\right| < \infty,\ for\ \ each\ \ x \in X\right\}$;

(5). Let X be an FK-space $\supset \phi$; then $X^{f} = \left\{f(\Im_{mnk}) : f \in X'\right\}$;

(6). $X^{\delta} = \left\{a = (a_{mnk}) : \sup_{m,n,k} |a_{mnk}x_{mnk}|^{1/m+n+k} < \infty,\ for\ \ each\ \ x \in X\right\}$;

$X^{\alpha}.X^{\beta}$, $X^{\gamma}$ are called $\alpha-$(or Köthe-Toeplitz) dual of $X, \beta-$(or generalized-Köthe-Toeplitz) dual of $X, \gamma-$dual of $X, \delta-$dual of $X$ respectively. $X^{\alpha}$ is defined by Gupta and Kamptan. It is clear that $X^{\alpha} \subset X^{\beta}$ and $X^{\alpha} \subset X^{\gamma}$, but $X^{\alpha} \subset X^{\gamma}$ does not hold.

## 2. Definitions and Preliminaries

A sequence $x = (x_{mnk})$is said to be triple analytic if $sup_{mnk}\,|x_{mnk}|^{\frac{1}{m+n+k}} < \infty$. The vector space of all triple analytic sequences is usually denoted by $\Lambda^3$. A sequence $x = (x_{mnk})$ is called triple entire sequence if $|x_{mnk}|^{\frac{1}{m+n+k}} \to 0$ as $m, n, k \to \infty$. The vector space of triple entire sequences is usually denoted by $\Gamma^3$. A sequence $x = (x_{mnk})$ is called triple gai sequence if $((m + n + k)!\,|x_{mnk}|)^{1/m+n+k} \to 0$ as $m, n, k \to \infty$. The vector space of triple gai sequences is usually denoted by $\chi^3$. The space $\chi^3$ is a metric space with the metric

$$d(x, y) = sup_{m,n,k}\left\{((m + n + k)!\,|x_{mnk} - y_{mnk}|)^{\frac{1}{m+n+k}} : m, n, k : 1, 2, 3, ...\right\} \quad (2)$$

for all $x = \{x_{mnk}\}$ and $y = \{y_{mnk}\}$in $\chi^3$. In this chapter we study the general properties of $\chi^3$ establishing the following theorems:

# 3. Main Results

**Proposition 3.1.** *$\chi^3 \subset \Gamma^3$ with the hypothesis that $|x_{mnk}|^{\frac{1}{m+n+k}} \leq ((m+n+k)!\,|x_{mnk}|)^{\frac{1}{m+n+k}}$.*

*Proof.* Let $x \in \chi^3$. Then we have the following implications

$$\left(((m+n+k)!\,|x_{mn}|)^{\frac{1}{m+n+k}}\right) \, as\, m,n,k \to \infty \tag{3}$$

But $|x_{mnk}|^{\frac{1}{m+n+k}} \leq ((m+n+k)!\,|x_{mn}|)^{\frac{1}{m+n+k}}$ ; by our assumption, implies that $\Rightarrow \left(|x_{mnk}|^{\frac{1}{m+n+k}}\right) \to 0$ as $m,n,k \to \infty$, by (3) $\Rightarrow x \in \Gamma^3 \Rightarrow \chi^3 \subset \Gamma^3$. □

**Proposition 3.2.** *The dual space of $\chi^3$ is $\Lambda^3$. In other words $(\chi^3)^* = \Lambda^3$.*

*Proof.* We recall that

$$\Im_{mnk} = \begin{pmatrix} 0, & 0, & ...0, & 0, & ... \\ 0, & 0, & ...0, & 0, & ... \\ . & & & & \\ . & & & & \\ . & & & & \\ 0, & 0, & ...\frac{1}{(m+n+k)!}, & 0, & ... \\ 0, & 0, & ...0, & 0, & ... \end{pmatrix}$$

with $\frac{1}{(m+n+k)!}$ in the $(m,n,k)th$ position and zero's else where. With

$$x = \Im_{mnk}, (|x_{mnk}|)^{\frac{1}{m+n+k}} = \begin{pmatrix} 0^{\frac{1}{3}}, . & . & . \; 0^{\frac{1}{m+n+k}} \\ . & & \\ . & & \\ . & & \\ 0^{\frac{1}{m+n+k}}, & \left(\frac{1}{(m+n+k)!}\right)^{\frac{1}{m+n+k}}, & . \; 0^{\frac{1}{m+n+k}} \\ & (m,n)^{th} & \\ 0^{\frac{1}{m+2+k}}, & . & . \; 0^{\frac{1}{m+n+2+k}} \end{pmatrix} = \begin{pmatrix} 0, . & . & . \; 0 \\ . & & \\ . & & \\ . & & \\ 0, & \left(\frac{1}{(m+n+k)!}\right)^{\frac{1}{m+n+k}}, & . \; 0 \\ & (m,n,k)^{th} & \\ 0, & . & . \; 0 \end{pmatrix}$$

which is a triple gai sequence. Hence $\Im_{mnk} \in \chi^3$. We have $f(x) = \sum_{m,n,k=1}^{\infty} x_{mnk} y_{mnk}$. With $x \in \chi^3$ and $f \in (\chi^3)^*$ the dual space of $\chi^3$. Take $x = (x_{mnk}) = \Im_{mnk} \in \chi^3$. Then

$$|y_{mnk}| \leq \|f\|\, d(\Im_{mnk}, 0) < \infty \;\; \forall\, m,n,k \tag{4}$$

Thus $(y_{mnk})$ is a bounded sequence and hence an triple analytic sequence. In other words $y \in \Lambda^3$. Therefore $(\chi^3)^* = \Lambda^3$. This completes the proof. □

**Proposition 3.3.** $(\Gamma^3)^\beta \overset{\subset}{\neq} \Lambda^3$.

*Proof.* Let $y = (y_{mnk})$ be an arbitrary point in $\left(\Gamma^3\right)^\beta$. If $y$ is not in $\Lambda^3$, then for each natural number $p$, we can find an index $m_p n_p k_p$ such that

$$\left(|y_{m_p n_p k_p}|^{\frac{1}{m_p+n_p+k_p}}\right) > p, (p = 1, 2, 3, \cdots) \tag{5}$$

Define $x = \{x_{mnk}\}$ by

$$(x_{mnk}) = \begin{cases} \frac{1}{p^{m+n+k}}, & \text{for } (m,n,k) = (m_p, n_p, k_p) \text{ for some} p \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

Then we have $x$ is in $\Gamma^3$, but for infinitely $mnk$,

$$(|y_{mnk} x_{mnk}|) > 1. \tag{7}$$

Consider the sequence $z = \{z_{mnk}\}$, where $M\left(\frac{z_{111}}{\rho}\right) = M\left(\frac{x_{111}}{\rho}\right) - s$ with

$$s = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(x_{mnk}), (z_{mnk}) = (x_{mnk}). \tag{8}$$

Then $z$ is a point of $\Gamma^3$. Also, $\sum\sum\sum(z_{mnk}) = 0$. Hence, $z$ is in $\Gamma^3$; but, by (7), $\sum\sum\sum(z_{mnk} y_{mnk})$ does not converge:

$$\Rightarrow \sum\sum\sum x_{mnk} y_{mnk} \; diverges. \tag{9}$$

Thus, the sequence $y$ would not be in $\left(\Gamma^3\right)^\beta$. This contradiction proves that

$$\left(\Gamma^3\right)^\beta \subset \Lambda^3. \tag{10}$$

If we now choose $y_{1nk} = x_{1nk} = 1$ and $y_{mnk} = x_{mnk} = 0\,(m > 1)$ for all $n, k$ then obviously $x \in \Gamma^3$ and $y \in \Lambda^3$, but

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{n=1}^{\infty} x_{mnk} y_{mnk} = \infty. \text{ Hence, } \; y \notin \left(\Gamma^3\right)^\beta \tag{11}$$

From (10) and (11), we are granted $\left(\Gamma^3\right)^\beta \overset{\subset}{\neq} \Lambda^3$. □

**Proposition 3.4.** *The $\beta-$ dual space of $\chi^3$ is $\Lambda^3$.*

*Proof.* First, we observe that $\chi^3 \subset \Gamma^3$, by Proposition 3.1. Theorefore $\left(\Gamma^3\right)^\beta \subset \left(\chi^3\right)^\beta$. But $\left(\Gamma^3\right)^\beta \overset{\subset}{\neq} \Lambda^3$, by Proposition 3.2. Hence

$$\Lambda^3 \subset \left(\chi^3\right)^\beta \tag{12}$$

Next we show that $\left(\chi^3\right)^\beta \subset \Lambda^3$. Let $y = (y_{mnk}) \in \left(\chi^3\right)^\beta$. Consider $f(x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} x_{mnk} y_{mnk}$ with $x = (x_{mnk}) \in \chi^3$. Here

$$\Im_{mnk} = \begin{pmatrix} 0 & 0 & ...0 & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \\ . & & & & \\ . & & & & \\ . & & & & \\ 0 & 0 & ...\frac{1}{(m+n+k)!} & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \\ & & & & \end{pmatrix}$$

with $\frac{1}{(m+n+k)!}$ in the $(m,n,k)^{th}$ position and zero otherwise.

$$x=[(\Im_{mn}-\Im_{mn+1}-\Im_{mn+2})-(\Im_{m+1n}-\Im_{m+1n+1}-\Im_{m+1n+2})-(\Im_{m+2n}-\Im_{m+2n+1}-\Im_{m+2n+2})]$$

$$=\begin{pmatrix} 0 & 0 & ...0 & 0 & ... & 0 \\ 0 & 0 & ...0 & 0 & ... & 0 \\ . \\ . \\ . \\ 0 & 0 & ...\frac{1}{(m+n+k)!} & \frac{-1}{(m+n+k)!} & \frac{-1}{(m+n+k)!} & ...\ 0 \\ 0 & 0 & ...0 & 0 & ... & 0 \end{pmatrix}-\begin{pmatrix} 0 & 0 & ...0 & 0 & ... & 0 \\ 0 & 0 & ...0 & 0 & ... & 0 \\ . \\ . \\ . \\ 0 & 0 & ...\frac{1}{(m+n+k)!} & \frac{-1}{(m+n+k)!} & \frac{-1}{(m+n+k)!} & ...\ 0 \\ 0 & 0 & ...0 & 0 & ... & 0 \end{pmatrix}$$

$\left(((m+n+k)!\,|x_{mnk}|)^{\frac{1}{m+n+k}}\right)=\begin{pmatrix} 0 & 0 & ...0 & 0 & ... & 0 \\ 0 & 0 & ...0 & 0, & ... & 0 \\ . \\ . \\ . \\ 0 & 0 & ...\frac{1}{(m+n+k)!} & \frac{-1}{(m+n+k)!} & \frac{-1}{(m+n+k)!} & ...\ 0 \\ 0 & 0 & ...\frac{-1}{(m+n+k)!} & \frac{1}{(m+n+k)!} & \frac{1}{(m+n+k)!} & ...\ 0 \\ 0 & 0 & ...0 & 0, & ... & 0 \end{pmatrix}$. Hence converges to zero. Therefore $[(\Im_{mn}-\Im_{mn+1}-\Im_{mn+2})-(\Im_{m+1n}-\Im_{m+1n+1}-\Im_{m+1n+2})-(\Im_{m+2n}-\Im_{m+2n+1}-\Im_{m+2n+2})] \in \chi^3$. Hence $d\left((\Im_{mn}-\Im_{mn+1}-\Im_{mn+2})-(\Im_{m+1n}-\Im_{m+1n+1}-\Im_{m+1n+2})-(\Im_{m+2n}-\Im_{m+2n+1}-\Im_{m+2n+2}),0\right)=1$. But $|y_{mnk}|\leq d\left((\Im_{mn}-\Im_{mn+1}-\Im_{mn+2})-(\Im_{m+1n}-\Im_{m+1n+1}-\Im_{m+1n+2})-(\Im_{m+2n}-\Im_{m+2n+1}-\Im_{m+2n+2}),0\right)\leq \|f\|\cdot 1<\infty$ for each $m,n,k$. Thus $(y_{mnk})$ is a triple bounded sequence and hence an analytic sequence. In other words $y\in\Lambda^3$. But $y=(y_{mnk})$ is arbitrary in $\left(\chi^3\right)^\beta$. Therefore

$$\left(\chi^3\right)^\beta\subset\Lambda^3 \tag{13}$$

From (12) and (13) we get $\left(\chi^3\right)^\beta=\Lambda^3$. □

**Proposition 3.5.** *$\chi^3$ has AK.*

*Proof.* Let $x=(x_{mnk})\in\chi^3$ and take $x^{[m,n,k]}=\sum_{i,j,u=0}^{m,n,k}x_{iju}\Im_{iju}$ for all $m,n,k\in\mathbb{N}$. Hence $d\left(x,x^{[r,\,s\,t]}\right)=sup_{mnk}\left((m+n+k)!\,|x_{mnk}|\right)^{\frac{1}{m+n+k}}: m\geq r+1, n\geq s+1, k\geq t+1\to 0$ as $m,\,n,\,k\to\infty$. Therefore, $x^{[r,\,s,\,t]}\to x$ as $r,\,s,\,t\to\infty$ in $\chi^3$ Thus $\chi^3$ has AK. This completes the proof. □

**Proposition 3.6.** *$\chi^3$ is solid.*

*Proof.* Let $|x_{mnk}|\leq|y_{mnk}|$ and let $y=(y_{mnk})\in\chi^3$. We have $\left(((m+n+k)!\,|x_{mnk}|)^{\frac{1}{m+n+k}}\right)\leq\left(((m+n+k)!\,|y_{mnk}|)^{\frac{1}{m+n+k}}\right)$. But $\left(((m+n+k)!\,|y_{mnk}|)^{\frac{1}{m+n+k}}\right)\in\chi^3$, because $y\in\chi^3$. That is $\left(((m+n+k)!\,|y_{mnk}|)^{\frac{1}{m+n+k}}\right)\to 0\Rightarrow\left(((m+n+k)!\,|x_{mnk}|)^{\frac{1}{m+n+k}}\right)\to 0\,as\,m,n,k\to\infty$. Therefore $x=(x_{mnk})\in\chi^3$. This completes the proof. □

**Proposition 3.7.** *$\delta-$ dual of $\chi^3$ is $\Lambda^3$.*

*Proof.* Let $y \in \delta-$ dual of $\chi^3$. Then $|x_{mnk}y_{mnk}| \leq M^{m+n+k}$ for some constant $M > 0$ and for each $x \in \chi^3$. Therefore $|y_{mnk}| \leq M^{m+n+k}$ for each $m, n, k$ by taking

$$x = \Im_{mnk} = \begin{pmatrix} 0 & 0 & ...0 & 0 & ... & 0 \\ 0 & 0 & ...0 & 0 & ... & 0 \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ 0 & 0 & ...\frac{1}{(m+n+k)!} & 0 & ... & 0 \\ 0 & 0 & ...0 & 0 & ... & 0 \\ & & & & & \end{pmatrix}.$$

This shows that $y \in \Lambda^3$. Then

$$\left(\chi^3\right)^\delta \subset \Lambda^3 \tag{14}$$

On the other hand, let $y \in \Lambda^3$. Let $\epsilon > 0$ be given. Then $|y_{mnk}| < M^{m+n+k}$ for each $m, n, k$ and for some constant $M > 0$. But $x \in \chi^3$. Hence $((m+n+k)! \, |x_{mnk}|) < \epsilon^{m+n+k}$ for each $m, n, k$ and for each $\epsilon > 0$. i.e $|x_{mnk}| < \frac{\epsilon^{m+n+k}}{(m+n+k)!}$. Hence

$$|x_{mnk}y_{mnk}| = |x_{mnk}| \, |y_{mnk}| < \frac{\epsilon^{m+n+k}}{(m+n+k)!} M^{m+n+k} = \frac{(\epsilon M)^{m+n+k}}{(m+n+k)!} \Rightarrow y \in \left(\chi^3\right)^\delta$$

$$\Lambda^3 \subset \left(\chi^3\right)^\delta \tag{15}$$

From (14) and (15) we get $\left(\chi^3\right)^\delta = \Lambda^3$. □

**Proposition 3.8.** $\left(\Lambda^3\right)^\beta = \Lambda^3$.

*Proof.* Step 1: Let $(x_{mnk}) \in \Lambda^3$ and let $(y_{mnk}) \in \Lambda^3$. Then we get $|y_{mnk}|^{\frac{1}{m+n+k}} \leq M$ for some constant $M > 0$. Also

$$\begin{aligned} (x_{mnk}) \in \chi^3 &\Rightarrow ((m+n+k)! \, |x_{mnk}|)^{\frac{1}{m+n+k}} \leq \epsilon = \frac{1}{2M} \\ &\Rightarrow |x_{mnk}| \leq \frac{1}{2^{m+n+k} M^{m+n+k} \, (m+n+k)!}. \end{aligned}$$

Hence

$$\begin{aligned} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |x_{mnk}y_{mnk}| &\leq \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |x_{mnk}| \, |y_{mnk}| \\ &< \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{2^{m+n+k}} \frac{1}{M^{m+n+k}} M^{m+n+k} \frac{1}{(m+n+k)!} \\ &< \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{2^{m+n+k}} \frac{1}{(m+n+k)!} < \infty. \end{aligned}$$

Therefore, we get that $(x_{mnk}) \in \left(\Lambda^3\right)^\beta$ and so we have

$$\chi^3 \subset \left(\Lambda^3\right)^\beta \tag{16}$$

Step 2: Let $(x_{mnk}) \in \left(\Lambda^3\right)^\beta$. This says that

$$\Rightarrow \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |x_{mnk}y_{mnk}| < \infty \, for \, each \, (y_{mnk}) \in \Lambda^3 \tag{17}$$

Assume that $(x_{mnk}) \notin \chi^3$, then there exists a sequence of positive integers $(m_p + n_p + k_p)$ strictly increasing such that

$$\left|x_{m_p+n_p+k_p}\right| > \frac{1}{2^{m_p+n_p+k_p}} \frac{1}{(m+n+k)!}, (p = 1, 2, 3, \cdots)$$

Take

$$y_{m_p,n_p,k_p} = 2^{m_p+n_p+k_p} (m+n+k)! \, (p = 1, 2, 3, \cdots)$$

and $y_{mnk} = 0$ otherwise. Then $(y_{mnk}) \in \Lambda^3$. But

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |x_{mnk} y_{mnk}| = \sum\sum\sum_{p=1}^{\infty} \left|x_{m_p n_p k_p} y_{m_p n_p k_p}\right| > 1 + 1 + 1 + \cdots .$$

We know that the infinite series $1 + 1 + 1 + \cdots$ diverges. Hence $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |x_{mnk} y_{mnk}|$ diverges. This contradicts (17). Hence $(x_{mnk}) \in \chi^3$. Therefore

$$\left(\Lambda^3\right)^{\beta} \subset \chi^3 \tag{18}$$

From (16) and (18) we get $\left(\Lambda^3\right)^{\beta} = \chi^3$. □

**Definition 3.9.** *Let $p = (p_{mnk})$ is a triple sequence of positive real numbers. Then*

$$\chi^3(p) = \left\{x = (x_{mnk}) : \left(((m+n+k)! \, |x_{mnk}|)^{\frac{1}{m+n+k}}\right)^{p_{mnk}} \to 0 \, as \, m, n, k \to \infty\right\}$$

*suppose that $p_{mnk}$ is a constant for each $m, n, k$ then $\chi^3(p) = \chi^3$.*

**Proposition 3.10.** *Let $0 \le p_{mnk} \le q_{mnk}$ and let $\left\{\frac{q_{mnk}}{p_{mnk}}\right\}$ be bounded. Then $\chi^3(q) \subset \chi^3(p)$.*

*Proof.* Let

$$x \in \chi^3(q) \tag{19}$$

Therefore we have

$$\left(((m+n+k)! \, |x_{mnk}|)^{\frac{1}{m+n+k}}\right)^{q_{mnk}} \to 0 \, as \, m, n, k \to \infty. \tag{20}$$

Let $t_{mnk} = \left(((m+n+k)! \, |x_{mnk}|)^{\frac{1}{m+n+k}}\right)^{q_{mnk}}$ and $\lambda_{mnk} = \frac{p_{mnk}}{q_{mnk}}$. Since $p_{mnk} \le q_{mnk}$, we have $0 \le \lambda_{mnk} \le 1$. Take $0 < \lambda < \lambda_{mnk}$. Define

$$u_{mnk} = \begin{cases} t_{mnk}, & \text{if } (t_{mnk} \ge 1) \\ 0, & \text{if } (t_{mnk} < 1) \end{cases}; v_{mnk} = \begin{cases} 0, & \text{if } (t_{mnk} \ge 1) \\ t_{mnk}, & \text{if } (t_{mnk} < 1) \end{cases} \tag{21}$$

$t_{mnk} = u_{mnk} + v_{mnk}; t_{mnk}^{\lambda_{mnk}} = u_{mnk}^{\lambda_{mnk}} + v_{mnk}^{\lambda_{mnk}}$. Now it follows that

$$u_{mnk}^{\lambda_{mnk}} \le u_{mnk} \le t_{mnk}; v_{mnk}^{\lambda_{mnk}} \le u_{mnk}^{\lambda}.$$

Since $t_{mnk}^{\lambda_{mnk}} = u_{mnk}^{\lambda_{mnk}} + v_{mnk}^{\lambda_{mnk}}$, then $t_{mnk}^{\lambda_{mnk}} \le t_{mnk} + v_{mnk}^{\lambda}$.

$$\begin{aligned}
&\left(\left(((m+n+k)! \, |x_{mnk}|)^{\frac{1}{m+n+k}}\right)^{q_{mnk}}\right)^{\lambda_{mnk}} \le \left(((m+n+k)! \, |x_{mnk}|)^{\frac{1}{m+n+k}}\right)^{q_{mnk}} \\
\Rightarrow &\left(\left(((m+n+k)! \, |x_{mnk}|)^{\frac{1}{m+n+k}}\right)^{q_{mnk}}\right)^{\frac{p_{mnk}}{q_{mnk}}} \le \left(((m+n+k)! \, |x_{mnk}|)^{\frac{1}{m+n+k}}\right)^{q_{mnk}} \\
\Rightarrow &\left(\left(((m+n+k)! \, |x_{mnk}|)^{\frac{1}{m+n+k}}\right)^{q_{mnk}}\right)^{p_{mnk}} \le \left(((m+n+k)! \, |x_{mnk}|)^{\frac{1}{m+n+k}}\right)^{q_{mnk}}.
\end{aligned}$$

But $\left(((m+n+k)! \, |x_{mnk}|)^{\frac{1}{m+n+k}}\right)^{q_{mnk}} \to 0$ as $m, n, k \to \infty$. (by (20)). Therefore $\left(((m+n+k)! \, |x_{mnk}|)^{\frac{1}{m+n+k}}\right)^{p_{mnk}} \to 0$ as $m, n, k \to \infty$. Hence

$$x \in \chi^3(p) \tag{22}$$

From (19) and (22) we get $\chi^3(q) \subset \chi^3(p)$. □

**Proposition 3.11.**

*(a) Let* $0 < inf p_{mnk} \leq p_{mnk} \leq 1$. *Then* $\chi^3(p) \subset \chi^3$

*(b) Let* $1 \leq p_{mnk} \leq supp_{mnk} < \infty$. *Then* $\chi^3 \subset \chi^3(p)$.

*Proof.*

(a) Let $x \in \chi^3(p)$

$$\left(((m+n+k)!\,|x_{mnk}|)^{\frac{1}{m+n+k}}\right)^{p_{mnk}} \to 0\, as\, m,n,k \to \infty \tag{23}$$

Since $0 < inf p_{mnk} \leq p_{mnk} \leq 1$,

$$\left(((m+n+k)!\,|x_{mnk}|)^{\frac{1}{m+n+k}}\right) \leq \left(((m+n+k)!\,|x_{mnk}|)^{\frac{1}{m+n+k}}\right)^{p_{mn}} \tag{24}$$

From (23) and (24) it follows that

$$x \in \chi^3 \tag{25}$$

Thus $\chi^3(p) \subset \chi^3$. This completes the proof.

(b) Let $p_{mnk} \geq 1$ for each $mnk$ and $supp_{mnk} < \infty$ and let $x \in \chi^3$.

$$\left(((m+n+k)!\,|x_{mnk}|)^{\frac{1}{m+n+k}}\right) \to 0\, as\, m,n,k \to \infty \tag{26}$$

Since $1 \leq p_{mnk} \leq supp_{mnk} < \infty$, we have

$$\left(((m+n+k)!\,|x_{mnk}|)^{\frac{1}{m+n+k}}\right)^{p_{mn}} \leq \left(((m+n+k)!\,|x_{mnk}|)^{\frac{1}{m+n+k}}\right) \tag{27}$$

$\left(((m+n+k)!\,|x_{mnk}|)^{\frac{1}{m+n+k}}\right)^{p_{mn}} \to 0\, as\, m,n,k \to \infty$ (by using (26)). Therefore $x \in \chi^3(p)$.

□

**Proposition 3.12.** *Let* $0 < p_{mnk} \leq q_{mnk} < \infty$ *for each* $m,n,k$. *Then* $x \in \chi^3(p) \subseteq x \in \chi^3(q)$.

*Proof.* Let $x \in \chi^3(p)$

$$\left(((m+n+k)!\,|x_{mnk}|)^{\frac{1}{m+n+k}}\right)^{p_{mnk}} \to 0\, as\, m,n,k \to \infty \tag{28}$$

This implies that $\left(((m+n+k)!\,|x_{mnk}|)^{\frac{1}{m+n+k}}\right) \leq 1$ for sufficiently large $m,n,k$. We get

$$\left(((m+n+k)!\,|x_{mnk}|)^{\frac{1}{m+n+k}}\right)^{q_{mnk}} \leq \left(((m+n+k)!\,|x_{mnk}|)^{\frac{1}{m+n+k}}\right)^{p_{mnk}} \tag{29}$$

$\Rightarrow \left(((m+n+k)!\,|x_{mnk}|)^{\frac{1}{m+n+k}}\right)^{q_{mnk}} \to 0\, as\, m,n,k \to \infty$ (by using (28)). Since $x \in \chi^3(q)$, hence $x \in \chi^3(p) \subseteq \chi^3(q)$. □

**Proposition 3.13.** $\left(\chi^3\right)^{\mu} = \Lambda^3$ *for* $\mu = \alpha, \beta, \gamma, f$.

*Proof.* **Step 1:** We have $\chi^3$ has AK by Proposition (3.5). Hence by Lemma (2)(ii)[Wilansky], we get $\left(\chi^3\right)^{\beta} = \left(\chi^3\right)^{f}$. But $\left(\chi^3\right)^{\beta} = \Lambda^3$. Hence

$$\left(\chi^3\right)^{\beta} = \Lambda^3 \tag{30}$$

**Step 2:** Since AK implies AD, by Lemma 2(iii)[Wilansky] we get $\left(\chi^3\right)^{\beta} = \left(\chi^3\right)^{\gamma}$. Therefore

$$\left(\chi^3\right)^{\gamma} = \Lambda^3 \tag{31}$$

**Step 3:** We have $\chi^3$ is normal by Proposition (3.6). Hence, [Kampthan, Proposition 2.7], we get

$$\left(\chi^3\right)^{\alpha} = \left(\chi^3\right)^{\gamma} = \Lambda^3 \tag{32}$$

From (30), (31) and (32), we have $\left(\chi^3\right)^{\alpha} = \left(\chi^3\right)^{\beta} = \left(\chi^3\right)^{\gamma} = \left(\chi^3\right)^{f} = \Lambda^3$. □

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