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Some New Combination Graphs

Research Article

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Abstract: A graph $G = (V, E)$ with p vertices and q edges is said to be combination graph, if there exists an injection $f : V(G) \rightarrow \{1, 2, \dots, p\}$ such that the induced edge function $g_f : E(G) \rightarrow \mathbb{N}$ defined by $g_f(uv) = \frac{(f(u))!}{|f(u)-f(v)|!(f(v))!}(f(u) > f(v))$, for every $uv \in E(G)$ is injective. In this paper we prove that some cycle and wheel related graphs obtained with the use of graph operations are combination graph.

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Keywords: Combination graph, prism graph, umbrella graph, wheel graph.

1. Introduction

If the vertices or edges or both of the graph are assigned values subject to certain condition(s) then it is known as graph labeling. Rosa[1] initiated the study of graph labeling in 1967. Graph Labeling has gained a lot of popularity in the recent era due to mathematical challenges of graph labeling and also to the extensive range of applications in different field such as X-ray crystallography, coding theory, cryptography, astronomy, circuit design and communication networks design. A dynamic survey on graph labeling is regularly updated by Gallian[5] and published by *The Electronic Journal of Combinatorics*. In this paper we consider simple, finite, undirected and connected graph. A graph with order p and size q is denoted as (p, q) graph. We refer to Bondy and Murty[4] for the standard terminology and notations related to Graph theory and David M. Burton[2] for the terms related to Number theory.

1.1. Definitions

Definition 1.1 ([9]). *A (p, q) graph $G = (V, E)$ is said to be combination graph if there exists an injection mapping $f : V(G) \rightarrow \{1, 2, \dots, p\}$ such that the induced edge function $g_f : E(G) \rightarrow \mathbb{N}$ defined by $g_f(uv) = \frac{(f(u))!}{|f(u)-f(v)|!(f(v))!}$ $(f(u) > f(v))$, for every $uv \in E(G)$ is injective.*

Definition 1.2 ([4]). *The wheel graph W_n is the graph obtained by joining the graphs C_n and K_1. i.e. $W_n = C_n + K_1$. Here the vertices corresponding to C_n is called rim vertices and C_n is called rim of W_n while the vertex corresponding to K_1 is called apex vertex. $|V(W_n)| = n + 1$ and $|E(W_n)| = 2n$.*

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Definition 1.3 ([5]). *Duplication of a vertex v of a graph G produces a new graph G' by adding a vertex v' with $N(v') = N(v)$. In other words a vertex v' is said to be duplication of a vertex v if all the vertices which are adjacent to v are now adjacent to v'.*

Definition 1.4 ([5]). *For any integer $m > 2$ and $n > 1$, an umbrella graph $U(m, n)$ is the graph obtained by appending a path P_n to the central vertex of a fan $F_m = P_m + K_1$. Here vertex set $V(U(m, n)) = \{u_1, u_2 \ldots, u_m, (u = v_1), v_2, \ldots, v_n\}$ and edge set $E(U(m, n)) = \{(u_i, u_{i+1})/1 \leq i \leq m-1\} \cup \{(u_i, v_1 = u)/1 \leq i \leq m\} \cup \{(v_i, v_{i+1})/1 \leq i \leq n-1\}$.*
So, $|V(U(m, n))| = m + n$ and $|E(U(m, n))| = 2m + n - 1$.

Definition 1.5 ([3]). *A graph which is obtained by replacing each vertex of star graph $K_{1,n}$ by a graph G is called star of G. It is denoted by G^*.*

Definition 1.6 ([5]). *A chord of a cycle C_n is an edge joining two non-adjacent vertices of C_n.*

Definition 1.7 ([5]). *Concurrent chords of a cycle C_n are chords in which one end vertex is common.*

Definition 1.8 ([5]). *An armed crown is a graph in which path P_m is attached at each vertex of cycle C_n. This graph is denoted by $C_n \oplus P_m$.*

Definition 1.9 ([5]). *A triangular snake (T_n) is the graph obtained from a path $u_1, u_2, \ldots, u_n$ by joining u_i and u_{i+1} to a new vertex w_i for $i = 1, 2, \ldots, n-1$.*

1.2. Some Existing Results on Combination Graphs

The following are known results on combination labeling of graphs.

Theorem 1.10 ([9]). *The complete graph K_n is combination graph iff $n \leq 2$.*

Theorem 1.11 ([9]). *The cycle graph C_n is combination graph iff $n \geq 4$.*

Theorem 1.12 ([9]). *The necessary condition for a graph $G = (p, q)$ become a combination graph is*

$$\begin{cases} 4q \leq p^2; & \text{if } p \text{ is even.} \\ 4q \leq (p^2 - 1); & \text{if } p \text{ is odd.} \end{cases}$$

Theorem 1.13 ([6]). *Two copies of cycle C_n sharing a common edge; graph consisting of two cycles C_n joined by a path; corona $(T_n \odot K_1)$ are combination graphs.*

Theorem 1.14 ([8]). *Wheel graph W_n is a combination graph for $n \geq 7$.*

Theorem 1.15 ([7]). *The graph $G = (p, q)$ is a non-combination graph if it has more than one vertex of degree $p - 1$.*

We use the following simple number theory results in this paper.

Lemma 1.16 ([2]). *If $n > k > 0$ then $\binom{n+1}{k+1} > \binom{n}{k}$.*

Lemma 1.17 ([2]). *If $n > 1$ then $\binom{2n}{2} > \binom{n+3}{3}$.*

Lemma 1.18 ([2]). *If $n > 1$ and $0 \leq k \leq n$ then $\binom{n}{k} = \binom{n}{n-k}$.*

Lemma 1.19 ([2]). *If $n > 1$ then $\binom{n}{2} < \binom{n+2}{2}$.*

Lemma 1.20 ([2]). *If $n > 12$ then $\binom{n}{1} < \binom{n/2}{2}$.*

2. Some New Combination Graphs

The following are the results investigated in this paper.

Theorem 2.1. *Prism $C_n \times P_2$ is combination graph for $n \geq 6$.*

Proof. Let $V(C_n \times P_2) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$ and $E(C_n \times P_2) = \{v_i v_{i+1} | 1 \leq i \leq n-1\} \cup \{v_n v_1\} \cup \{u_i u_{i+1} | 1 \leq i \leq n-1\} \cup \{u_n u_1\} \cup \{v_i u_i | 1 \leq i \leq n\}$. Without loss of generality we consider $\{v_i | 1 \leq i \leq n\}$ as vertices of inner cycle and $\{u_i | 1 \leq i \leq n\}$ as vertices of outer cycle.

$|V(C_n \times P_2)| = 2n$ and $|E(C_n \times P_2)| = 3n$. Let us define a bijection $f : V(G) \to \{1, 2, \ldots, 2n\}$ as

$$f(v_i) = i; \qquad 1 \leq i \leq n-2.$$

$$f(v_{n-1}) = n.$$

$$f(v_n) = n-1.$$

$$f(u_i) = n+i; \quad 1 \leq i \leq n-2.$$

$$f(u_{n-1}) = 2n.$$

$$f(u_n) = 2n-1.$$

For the edge labels produced in $C_n \times P_2$, we have the following.

- The labels of edge set $\{v_i v_{i+1} / 1 \leq i \leq n-3\} \cup \{v_n v_1\} \cup \{v_{n-1} v_n\}$ are increasing natural numbers $2, 3, \ldots, n-1, n$.
- The label of edge $\{u_1 v_1\}$ is $n+1$.
- The labels of edge set $\{u_i u_{i+1} / 1 \leq i \leq n-3\}$ are increasing natural numbers $n+2, n+3, \ldots, 2n-2$.
- The label of edge $\{u_{n-1} u_n\}$ is $2n$.
- The labels of edge $\{v_{n-2} v_{n-1}\}$ is $\binom{n}{n-2} = \binom{n}{2}$ which is greater than $2n$ (See Lemma 1.18 and Lemma 1.20).
- The labels of edge set $\{u_i v_i / 1 \leq i \leq n\}$ are increasing natural numbers greater than $\binom{n}{2}$ (See Lemma 1.19 and Lemma 1.16).

So, the induced function $g_f : E(G) \to \mathbb{N}$ defined by $g_f(uv) = \frac{(f(u))!}{|f(u)-f(v)|!(f(v))!}$ (where $f(u) > f(v)$ for every $uv \in E(G)$) is injective. Hence Prism $C_n \times P_2$ is a combination graph for $n \geq 5$. □

Example 2.2. *Combination labeling of Prism $C_{12} \times P_2$ is shown in the following Figure 1.*

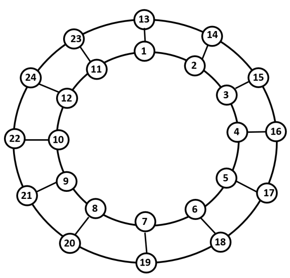

Figure 1.

Corollary 2.3. *Ladder $L_{n,2} = P_n \times P_2$ is a combination graph.*

Theorem 2.4. *The umbrella graph $U(m,n)$ is a combination graph for $m,n>2$.*

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertices of path P_n which is attached to the central vertex of the fan F_m with vertex set $\{u, u_1, u_2, \ldots, u_m\}$ $(m,n>2)$.

$V(U(m,n)) = \{u_1, u_2 \ldots, u_m, v_1(=u), v_2, \ldots, v_n\}$,

$E(U(m,n)) = \{u_iu_{i+1} | i = 1, 2, \ldots, m-1\} \cup \{u_iv_1 | i = 1, 2, \ldots, m\} \cup \{v_iv_{i+1} | i = 1, 2, \ldots, n-1\}$.

$|V(U(m,n))| = m+n$ and $|E(U(m,n))| = 2m+n-2$.

Let us define a bijection $f : V(U(m,n)) \to \{1, 2, \ldots, m+n\}$ as per the following two cases.

Case 1: n is even.

$$f(v_i) = i; \qquad 1 \le i \le n.$$
$$f(u_{2i-1}) = n+i; \qquad 1 \le i \le \left[\frac{m}{2}\right].$$
$$f(u_{2i}) = n+\frac{m}{2}+i; \; 1 \le i \le \left[\frac{m}{2}\right].$$

Case 2: n is odd.

$$f(v_i) = i; \qquad 1 \le i \le n.$$
$$f(u_{2i-1}) = n+i; \qquad 1 \le i \le \left[\frac{m+1}{2}\right].$$
$$f(u_{2i}) = n+\frac{m+1}{2}+i; \; 1 \le i \le \left[\frac{m-1}{2}\right].$$

So from Lemma 1.16 to 1.20, in above both the cases the induced function $g_f : E(G) \to \mathbb{N}$ defined by $g_f(uv) = \frac{(f(u))!}{|f(u)-f(v)|!(f(v))!}$, where $f(u) > f(v)$ for every $uv \in E(G)$ is injective.

Hence $U(m,n)$ is a combination graph for $m,n>2$. □

Example 2.5. *Combination labeling of umbrella graph $U(7,4)$ is shown in the following Figure 2.*

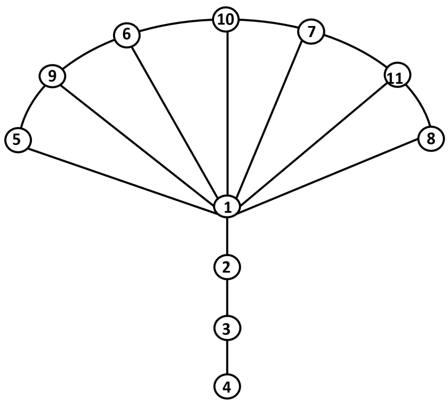

Figure 2.

Theorem 2.6. *An armed crown $C_n \oplus P_m$ is a combination graph for $n \ge 4$ and $m \ge 1$.*

Proof. Let $u_1, u_2 \ldots, u_n$ be the vertices of cycle C_n. The armed crown $C_n \oplus P_m$ is obtained by attaching a pendant vertex of a path P_m to each vertex of cycle C_n. Let $v_{i,1}, v_{i,2}, \ldots, v_{i,m}$ be the vertices of i^{th} copy of path P_m.

Then $V(C_n \oplus P_m)$= $\{u_1, u_2 \ldots, u_n, v_{1,2}, \ldots, v_{1,m}, \ldots, v_{n,2}, \ldots, v_{n,m}\}$, where $u_i = v_{i,1}$ for $1 \le i \le n$,

$E(C_n \oplus P_m) = \{u_i u_{i+1}/i = 1 \dots n-1\} \cup \{u_n u_1\} \cup \{v_{i,j} v_{i,j+1}/1 \le i \le n, 1 \le j \le m-1\}$.

So, $|V(C_n \oplus P_m)| = mn$ and $|E(C_n \oplus P_m)| = mn$.

Let us define a bijective function $f : V(G) \to \{1, 2, \dots, mn\}$ as

$$f(u_i) = i; \qquad 1 \le i \le n-2.$$
$$f(u_{n-1}) = n.$$
$$f(u_n) = n-1.$$
$$f(v_{j,i}) = in + j; \qquad 1 \le j \le n-2.$$
$$f(v_{n-1,i}) = in + n.$$
$$f(v_{n,i}) = in + n - 1.$$

So from Lemma 1.16 to 1.20, the induced function $g_f : E(G) \to \mathbb{N}$ defined by $g_f(uv) = \frac{(f(u))!}{|f(u)-f(v)|!(f(v))!}$ is injective.

Hence $C_n \oplus P_m$ is a combination graph for $n \ge 4$ and $m \ge 1$. $\square$

Example 2.7. *Combination labeling of armed crown $C_8 \oplus P_3$ is shown in the following Figure 3.*

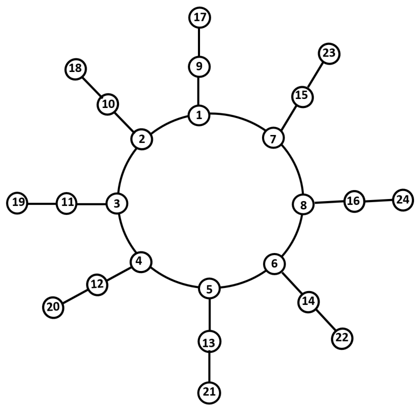

Figure 3.

Theorem 2.8. *The graph obtained by joining cycle C_m (m is even and $m \ge 4$) to each pendant vertex of $K_{1,n} (n \ge 2)$ is combination graph.*

Proof. Let u be apex and $u_1, u_2 \dots, u_n$ be n pendant vertices of $K_{1,n}$.

Let $u_{i,1}, u_{i,2} \dots, u_{i,m}$ be successive vertices of cycles $C_m^{(i)}$, $1 \le i \le n$.

Let G be a graph obtained by attached cycle $C_m^{(i)}$ to pendant vertex u_i. Here, $u_i = u_{i,1}$, for $1 \le i \le n$.

$|V(G)| = mn + 1$ and $|E(G)| = mn + n$.

Let us define a bijection $f : V(G) \to \{1, 2, \dots, mn+1\}$ as

$$f(u) = 1$$
$$f(u_i) = i + 1; \qquad 1 \le i \le n.$$
$$f(u_{i,j}) = 2nj - 3n + 2i; \qquad 1 \le i \le n, 2 \le j \le \frac{m}{2}.$$
$$f(u_{i,j}) = nm + 1 - n + i; \qquad 1 \le i \le n, j = \frac{m+2}{2}.$$
$$f(u_{i,j}) = 2nm + n + 2i + 1 - 2nj; 1 \le i \le n, \frac{m+2}{2} < j \le m.$$

From Lemma 1.16 to 1.20, the induced function $g_f : E(G) \to \mathbb{N}$ defined by $g_f(uv) = \frac{(f(u))!}{|f(u)-f(v)|!(f(v))!}$ is injective. Hence graph obtained by joining cycle C_m (m is even and $m \geq 4$) to each pendant vertex of $K_{1,n}(n \geq 2)$ is combination graph. □

Example 2.9. *Combination labeling in graph obtained by joining cycle C_8 to each pendant vertex of $K_{1,4}$ is shown in the following Figure 4.*

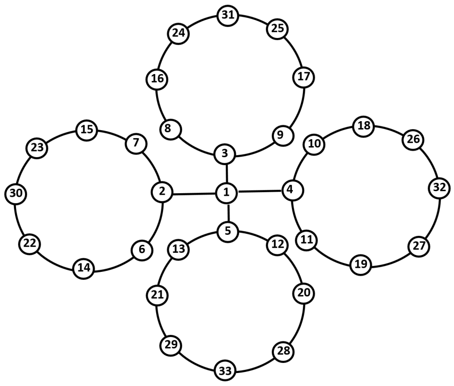

Figure 4.

Theorem 2.10. *The graph obtained by duplication of any rim vertex of wheel graph W_n is a combination graph for $n \geq 7$.*

Proof. Let v_0 be apex and $v_1, v_2, \ldots, v_n$ be rim vertices of W_n and let DW_n denote the resultant graph obtained by duplicating an arbitrary rim vertex of W_n. Without loss of generality let v_1' be the duplication of the vertex v_1.
$|V(DW_n)| = n + 2$, $|E(DW_n)| = 2n + 3$.
Let us define a bijection $f : V(G) \to \{1, 2, \ldots, n + 2\}$ as per the following two cases.
Case 1: n is even.

$$f(v_0) = 1.$$
$$f(v_1') = 2.$$
$$f(v_{2i-1}) = 2 + i; \quad 1 \leq i \leq \frac{n}{2}.$$
$$f(v_{2i}) = 2 + \frac{n}{2} + i;\ 1 \leq i \leq \frac{n}{2} \text{ except when } f(v_{2i-1}) + f(v_{2i+1}) = f(v_{2i}).$$

When $f(v_{2i-1}) + f(v_{2i+1}) = f(v_{2i})$, interchange $f(v_{2i})$ and $f(v_{2i+2})$.

Case 2: n is odd.

$$f(v_0) = 1.$$
$$f(v_1') = 2.$$
$$f(v_{2i-1}) = 2 + i; \quad 1 \leq i \leq \frac{n+1}{2}.$$
$$f(v_{2i}) = 2 + \frac{n+1}{2} + i;\ 1 \leq i \leq \frac{n-1}{2} \text{ except when } f(v_{2i-1}) + f(v_{2i+1}) = f(v_{2i}).$$

When $f(v_{2i-1}) + f(v_{2i+1}) = f(v_{2i})$, interchange $f(v_{2i})$ and $f(v_{2i+2})$.

From Lemma 1.16 to 1.20, in above both the cases the induced function $g_f : E(G) \to \mathbb{N}$ defined by $g_f(uv) = \frac{(f(u))!}{|f(u)-f(v)|!(f(v))!}$ is injective.

Hence duplication of any rim vertex in wheel graph W_n is combination graph for $n \geq 7$. □

Example 2.11. *Combination labeling of duplication of rim vertex in W_8 is shown in the following Figure 5.*

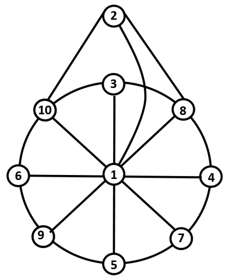

Figure 5.

Remark 2.12. *The graph obtained by duplication of apex vertex in W_n $(n \geq 3)$ is not combination graph because the degree of apex and its duplicate vertex become one less the number of vertices which is not possible in combination graph as per Theorem 1.15.*

Theorem 2.13. *Cycle C_n with $[\frac{n-4}{2}]$ concurrent chords is a combination graph for all $n \geq 6$.*

Proof. Let $v_1, v_2, \ldots v_n$ be the vertices of C_n, where v_i is adjacent to v_{i+1} for $1 \leq i \leq n-1$ and v_n is adjacent to v_1. Let us consider $v_{n-1}v_3, v_{n-1}v_4, \ldots v_{n-1}v_{[\frac{n}{2}]}$ to be $[\frac{n-4}{2}]$ concurrent chords of C_n.

Let us define a bijection $f : V(G) \to \{1, 2, \ldots, n\}$ as

$$f(v_i) = i; \qquad 1 \leq i \leq n-2.$$

$$f(v_{n-1}) = n.$$

$$f(v_n) = n-1.$$

So, the induced function $g_f : E(G) \to \mathbb{N}$ defined by $g_f(uv) = \frac{(f(u))!}{|f(u)-f(v)|!(f(v))!}$ is injective.

Hence C_n with $[\frac{n-4}{2}]$ concurrent chords is a combination graph for all $n \geq 6$. □

Example 2.14. *Combination labeling of C_{12} with 4 concurrent chords is shown in the following Figure 6.*

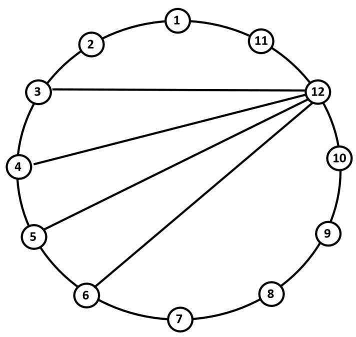

Figure 6.

Theorem 2.15. *Duplication of vertex in cycle C_n is a combination graph for $n \geq 5$.*

Proof. Let $v_1, v_2, \ldots v_n$ be the vertices of C_n, where v_i is adjacent to v_{i+1} for $1 \leq i \leq n-1$ and v_n is adjacent to v_1. Without loss of generality let v'_{n-1} be the duplication of the vertex v_{n-1}.
$|V(DC_n)| = n+1$ and $|E(DC_n)| = n+2$.
Let us define a bijective function $f : V(G) \rightarrow \{1, 2, \ldots, n+1\}$ as

$$f(v_i) = i; \qquad 1 \leq i \leq n-2.$$

$$f(v_{n-1}) = n.$$

$$f(v'_{n-1}) = n+1.$$

$$f(v_n) = n-1.$$

From Lemma 1.16 to 1.20, the induced function $g_f : E(G) \rightarrow \mathbb{N}$ defined by $g_f(uv) = \frac{(f(u))!}{|f(u)-f(v)|!(f(v))!}$ (where $f(u) > f(v)$ for every $uv \in E(G)$) is injective.
Hence duplication of any vertex in C_n is a combination graph for $n \geq 5$. □

Example 2.16. *Combination labeling of C_{12} with duplication of vertex u_{11} is shown in the following Figure 7.*

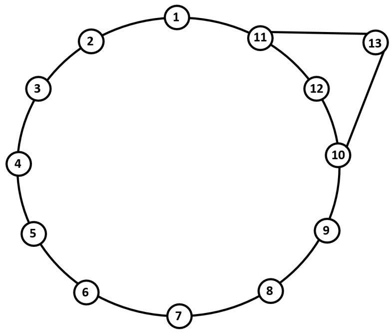

Figure 7.

Remark 2.17. *Duplication of vertex in C_3 does not satisfy the necessary condition for combination graph (See Theorem 1.12) and by trial and error we can check duplication of vertex in C_4 is not combination graph even though it satisfies the necessary condition for combination graph.*

3. Conclusion

It is very interesting to study graphs which admit combination labeling. Here we proved that prism $C_n \times P_2$, umbrella, armed crown, cycle C_m is attached to each pendant vertex of $K_{1,n}$, cycle C_n with $[\frac{n-4}{2}]$ concurrent chords, duplication of rim vertex in wheel W_n and duplication of any vertex in cycle C_n are combination graph. To investigate equivalent results for different graph families is an open area of research.

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