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Q-Cubic Ideals of Semigroups

Research Article

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Abstract: In this paper, for a set Q, the notion of Q-cubic ideals of semigroups is introduced and some properties of such ideals are investigated.

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1. Introduction

Zadeh [13] introduced the notion of fuzzy sets and fuzzy set operations. Since then the fuzzy set theory developed by Zadeh and others has evoked great interest among researchers working in different branches of mathemaics. Zadeh [14] introduced the notion interval-valued fuzzy set, where the values of the membership function are closed subintervals of [0,1] instead of a single value from it. Kuroki [5, 6] have studied several properties of fuzzy left (right) ideals, fuzzy bi-ideals and fuzzy interior ideals in semigroups. Yuan et al.[12] introduced a new fuzzy subgroup, called a S-fuzzy subgroup, of a group over a set S and they showed that a fuzzy subgroup of a group can be seen as a S-fuzzy subgroup. The idea of intuitionistic Q-fuzzy set was first discussed by Atanassov [1, 2], as a generalization of the notion of fuzzy set. Kyung Ho Kim [10] introduced on intuitionistic Q-fuzzy ideals of semigroups. Thillaigovindan et al.[11] discussed on interval valued fuzzy quasi-ideals of semigroups. Jun et al.[9] introduced a remarkable structure namely cubic sets that combines fuzzy set and interval-valued fuzzy set. Chinnadurai et al.[4] introduced cubic ideals of Γ-semigroups. In this paper we introduced Q-cubic ideals of semigroups and some properties of such ideals are investigated.

1.1. Preliminaries

Definition 1.1 ([3]). *Let X be a semigroup. A non-empty subset A of X is called a subsemigroup of X if $A^2 \subseteq A$.*

Definition 1.2 ([3]). *A non-empty subset A of X is called a left (resp. right) ideal of X if $XA \subseteq A(resp. AX \subseteq A)$.*

Definition 1.3 ([3]). *A subsemigroup A of X is called a bi-ideal of X if $AXA \subseteq A$.*

Definition 1.4 ([3]). *A non-empty subset A of X is called an interior ideal of X if $XAX \subseteq A$.*

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Definition 1.5 ([3]). *A mapping $\mu : X \to [0,1]$ is called a fuzzy subset of X.*

Definition 1.6 ([3]). *A fuzzy subset μ of X is called fuzzy subsemigroup of X if $\mu(x\,y) \geq min\{\mu(x), \mu(y)\}$ for all $x, y \in X$.*

Definition 1.7 ([3]). *A fuzzy subset μ of X is called fuzzy left (resp. right) ideal of X if $\mu(x\,y) \geq \mu(y)$ $(\mu(x\,y) \geq \mu(x))$ for all $x, y \in X$.*

Definition 1.8 ([10]). *Let X and Q be non-empty sets. A mapping $f : X \times Q \to [0,1]$ is called a Q-fuzzy set in X over Q .*

Definition 1.9 ([10]). *A Q-fuzzy set f in X is called Q-fuzzy subsemigroup of X if $f(xy, q) \geq \min\{f(x,q), f(y,q)\}$ for all $x, y \in X$ and $q \in Q$.*

Definition 1.10 ([3]). *Let X be a non-empty set. A mapping $\overline{\mu} : X \to D[0,1]$ is called interval-valued fuzzy set(in short i-v), where $D[0,1]$ denote the family of all closed sub interval of $[0,1]$ and $\overline{\mu}(x) = [\mu^-(x), \mu^+(x)]$, where μ^- and μ^+ are fuzzy subsets of X such that $\mu^-(x) \leq \mu^+(x)$ for all $x \in X$.*

Definition 1.11 ([9]). *Let X be a non-empty set. A cubic set A of X is a structure of the form $A(x) = \{(x, \overline{\mu}_A(x), \lambda_A(x)) : x \in X\}$ and denoted by $A = (\overline{\mu}_A, \lambda_A)$ where $\overline{\mu}_A = [\mu_A^-, \mu_A^+]$ is an i-v fuzzy subset(briefly IVF) in X and λ_A is a fuzzy set in X.*

Definition 1.12 ([8]). *The complement of $A = (\overline{\mu}_A, \lambda_A)$ is defined to be the cubic set $A^c(x) = \{(x, (\overline{\mu}_A)^c(x), 1 - \lambda_A(x)) : x \in X\}$.*

Definition 1.13 ([7]). *Let $A = (\overline{\mu}, \lambda)$ be a cubic set in X. Define $U(A; \bar{t}, n) = \{x \in X | \overline{\mu}(x) \geq \bar{t}, \lambda(x) \leq n\}$ where $n \in [0,1]$ and $\bar{t} \in D[0,1]$ is called the cubic level set of A.*

Definition 1.14 ([7]). *For any non-empty subset G of a set X, the characteristic cubic set of G is defined to be a structure $\chi_G = \{(x, \overline{\mu}_{\chi_G}(x), \lambda_{\chi_G}(x)) : x \in X\}$ which is briefly denoted by $\chi_G(x) = (\overline{\mu}_{\chi_G}(x), \lambda_{\chi_G}(x))$ where*

$$\overline{\mu}_{\chi_G}(x) = \begin{cases} [1,1] & \text{if } x \in G, \\ [0,0] & \text{otherwise}, \end{cases} \qquad \lambda_{\chi_G}(x) = \begin{cases} 0 & \text{if } x \in G, \\ 1 & \text{otherwise}. \end{cases}$$

2. Q-cubic Ideals of Semigroups

Definition 2.1. *Let S and Q be any non-empty sets. A Q-cubic set A is an object having the form $A = \{(x,q), \overline{\mu}(x,q), \omega(x,q) : x \in S, q \in Q\}$ which is briefly denoted by $A = (\overline{\mu}, \omega)$ with respect to Q, where $\overline{\mu} : S \times Q \to D[0,1]$ is an interval-valued Q-fuzzy set over Q and $\omega : S \times Q \to [0,1]$ is a Q-fuzzy set over Q.*

Definition 2.2. *For two Q-cubic sets $A_1 = (\overline{\mu}_1, \omega_1), A_2 = (\overline{\mu}_2, \omega_2)$ in a semigroup S, we define $A_1 \sqsubseteq A_2$ if and only if $\overline{\mu}_1 \leq \overline{\mu}_2, \omega_1 \geq \omega_2$ and the Q-cubic product of $A_1 = (\overline{\mu}_1, \omega_1)$ and $A_2 = (\overline{\mu}_2, \omega_2)$ is defined to be a Q-cubic set $A_1 \copyright A_2 = \{((x,q), (\overline{\mu}\, o\, \overline{\mu}_2)(x,q), (\omega_1 o\, \omega_2)(x,q)) : x \in S$ and $q \in Q\}$ which is briefly denoted by $A_1 \copyright A_2 = (\overline{\mu}_1\, o\, \overline{\mu}_2; \omega_1 o\, \omega_2)$ with respect to Q, where*

$$(\overline{\mu}_1 o\, \overline{\mu}_2)(x,q) = \begin{cases} \sup\limits_{x=yz} \min\{\overline{\mu}_1(y,q), \overline{\mu}_2(z,q) & \text{for all } x,y,z \in S \text{ and } q \in Q \\ [0,0] & \text{otherwise} \end{cases}$$

$$(\omega_1 o\, \omega_2)(x,q) = \begin{cases} \inf\limits_{x=yz} \max\{\omega_1(y,q), \omega_2(z,q) & \text{for all } x,y,z \in S \text{ and } q \in Q \\ 1 & \text{otherwise} \end{cases}$$

Let A_1 and A_2 be two Q-cubic sets in S. The intersection of A_1 and A_2 denoted by $A_1 \sqcap A_2$ is the Q-cubic set.
$A_1 \sqcap A_2 = (\overline{\mu}_1 \cap \overline{\mu}_2, \omega_1 \cup \omega_2)$ *with respect to Q, where*

$$(\overline{\mu}_1 \cap \overline{\mu}_2)(x,q) = \min\{\overline{\mu}_1(x,q), \overline{\mu}_2(x,q)\} \text{ and } (\omega_1 \cup \omega_2)(x,q) = \max\{\omega_1(x,q), \omega_2(x,q)\}$$

The union of A_1 and A_2 denoted by $A_1 \sqcup A_2$ is the Q-cubic set.
$A_1 \sqcup A_2 = (\overline{\mu}_1 \cup \overline{\mu}_2, \omega_1 \cap \omega_2)$ *with respect to Q, where*

$$(\overline{\mu}_1 \cup \overline{\mu}_2)(x,q) = \max\{\overline{\mu}_1(x,q), \overline{\mu}_2(x,q)\} \text{ and } (\omega_1 \cap \omega_2)(x,q) = \min\{\omega_1(x,q), \omega_2(x,q)\}.$$

Definition 2.3. *For any non-empty subset H of a semigroup S is defined to be a structure*
$\chi_H = \{((x,q), \overline{\mu}_{\chi_H}(x,q), \omega_{\chi_H}(x,q)) : x \in S, q \in Q\}$ *which is briefly denoted by* $\chi_H = (\overline{\mu}_{\chi_H}, \omega_{\chi_H})$ *with respect to Q, where*

$$\overline{\mu}_{\chi_H}(x,q) = \begin{cases} \overline{1} & \text{if } x \in H \\ \overline{0} & \text{otherwise} \end{cases} \qquad \omega_{\chi_H}(x,q) = \begin{cases} 0 & \text{if } x \in H \\ 1 & \text{otherwise} \end{cases}$$

Definition 2.4. *Let $A = (\overline{\mu}, \omega)$ be a Q-cubic set of S. Define $U(A;\overline{t},n) = \{x \in X \text{ and } q \in Q \mid \overline{\mu}(x,q) \geq \overline{t}, \omega(x,q) \leq n\}$ where $n \in [0,1]$ and $\overline{t} \in D[0,1]$ is called the Q-cubic level set of A.*

Definition 2.5. *A Q-cubic set $A = (\overline{\mu}, \omega)$ of S is called a Q-cubic subsemigroup of S if $\overline{\mu}(xy,q) \geq \min\{\overline{\mu}(x,q), \overline{\mu}(y,q)\}$ and $\omega(xy,q) \leq \max\{\omega(x,q), \omega(y,q)\}$ for all $x, y \in S$ and $q \in Q$.*

Example 2.6. *Consider a semigroup $S = \{k,l,m,n,o,p\}$ and $Q = \{q\}$ with the following Cayley table:*

$\cdot$	k	l	m	n	o	p
k	n	n	n	n	n	k
l	o	o	o	o	o	l
m	n	n	n	n	o	m
n	n	n	n	n	n	n
o	o	o	o	o	o	o
p	n	n	n	n	n	p

Define a Q-cubic set $A = (\overline{\mu}, \omega)$ in S as follows:

$S:$	k	l	m	n	o	p
$\overline{\mu}(x,q):$	[0.5,0.7]	[0.3,0.5]	[0.2,0.4]	[0.6,0.8]	[0.4,0.6]	[0,0]
$\omega(x,q):$	0.6	0.7	0.7	0.3	0.4	0.8

Thus $A = (\overline{\mu}, \omega)$ is a Q-cubic subsemigroup of S.

Definition 2.7. *A Q-cubic set $A = (\overline{\mu}, \omega)$ of S is called Q-cubic left (resp.right) ideal of S if $\overline{\mu}(xy,q) \geq \overline{\mu}(y,q)$ $(\overline{\mu}(xy,q) \geq \overline{\mu}(x,q))$ and $\omega(xy,q) \leq \omega(y,q)\,(\omega(xy,q) \leq \omega(x,q))$, for all $x, y \in S$ and $q \in Q$.*
A Q-cubic set $A = (\overline{\mu}, \omega)$ of S is called a Q-cubic ideal of S if it is both Q-cubic left ideal and Q-cubic right ideal of S.

Example 2.8. *Let $S = \{0,1,2,3,4\}$ be a semigroup and $Q = \{q\}$ with the following Cayley table:*

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	0
2	0	0	2	2	4
3	0	0	2	3	4
4	0	0	2	2	4

Define a Q-cubic set $A = (\overline{\mu}, \omega)$ in S as follows

S	$\overline{\mu}(x,q)$	$\omega(x,q)$
0	[0.6,0.8]	0.1
1	[0,0.2]	0.7
2	[0.2,0.4]	0.2
3	[0.1,0.3]	0.6
4	[0.4,0.6]	0.4

It is easy to verify that $A = (\overline{\mu}, \omega)$ is a Q-cubic ideal of S.

Note that every Q-cubic left (resp.right) ideal of S is a Q-cubic subsemigroup of S but the converse is not true as seen in the following example.

Example 2.9. *Let $S = \{k, l, m, n\}$ be a semigroup and $Q = \{q\}$ with the following Cayley table:*

$\cdot$	k	l	m	n
k	k	k	k	k
l	k	k	k	k
m	k	k	k	l
n	k	k	l	m

Define a Q-cubic set $A = (\overline{\mu}, \omega)$ in S as follows:

$S:$	k	l	m	n
$\overline{\mu}(x,q):$	[0.5,0.8]	[0.3,0.6]	[0.5,0.8]	[0.2,0.4]
$\omega(x,q):$	0.2	0.6	0.4	0.6

It is easy to verify that $A = (\overline{\mu}, \omega)$ is a cubic subsemigroup of S, but it is not a Q-cubic left ideal of S. Since $\overline{\mu}(nm, q) = \overline{\mu}(l, q) = [0.3, 0.6] < [0.5, 0.8] = \overline{\mu}(m, q)$ or $\omega(nm, q) = \omega(l, q) = 0.6 > 0.4 = \omega(m, q)$.

Definition 2.10. *Let A Q-cubic set $A = (\overline{\mu}, \omega)$ of S is called Q-cubic bi-ideal of S if*

(i) $\overline{\mu}(xy, q) \geq \min\{\overline{\mu}(x, q), \overline{\mu}(y, q)\}$ *and* $\omega(xy, q) \leq \max\{\omega(x, q), \omega(y, q)\}$

(ii) $\overline{\mu}(xyz, q) \geq \min\{\overline{\mu}(x, q), \overline{\mu}(z, q)\}$ *and* $\omega(xyz, q) \leq \max\{\omega(x, q), \omega(z, q)\}$ *for all* $x, y, z \in S$ *and* $q \in Q$.

Definition 2.11. *Let A Q-cubic set $A = (\overline{\mu}, \omega)$ of S is called Q-cubic interior ideal of S if*

(i) $\overline{\mu}(xy, q) \geq \min\{\overline{\mu}(x, q), \overline{\mu}(y, q)\}$ *and* $\omega(xy, q) \leq \max\{\omega(x, q), \omega(y, q)\}$

(ii) $\overline{\mu}(xay, q) \geq \overline{\mu}(a, q)$ *and* $\omega(xay, q) \leq \omega(a, q)$ *for all* $x, y, a \in S$ *and* $q \in Q$.

Proposition 2.12. *For any Q-cubic sets $A_1 = (\overline{\mu}_1, \omega_1), A_2 = (\overline{\mu}_2, \omega_2)$ and $A_3 = (\overline{\mu}_3, \omega_3)$ in a semigroup S, we have*

(i) $A_1 \sqcup (A_2 \sqcap A_3) = (A_1 \sqcup A_2) \sqcap (A_1 \sqcup A_3)$

(ii) $A_1 \sqcap (A_2 \sqcup A_3) = (A_1 \sqcap A_2) \sqcup (A_1 \sqcap A_3)$

(iii) $A_1 \copyright (A_2 \sqcup A_3) = (A_1 \copyright A_2) \sqcup (A_1 \copyright A_3)$

(iv) $A_1 \copyright (A_2 \sqcap A_3) = (A_1 \copyright A_2) \sqcap (A_1 \copyright A_3)$

Proof. (i) and (ii) are straight forward. Let $x \in S$ and $q \in Q$.

(iii) If x is expressed as $x = yz$. For all $y, z \in S$. Then

$$\begin{aligned}
(\overline{\mu}_1 o (\overline{\mu}_2 \cup \overline{\mu}_3))(x,q) &= \sup_{x=yz} \min\{\overline{\mu}_1(y,q), (\overline{\mu}_2 \cup \overline{\mu}_3)(z,q)\} \\
&= \sup_{x=yz} \min\{\overline{\mu}_1(y,q), \max\{\overline{\mu}_2(z,q), \overline{\mu}_3(z,q)\}\} \\
&= \sup_{x=yz} \max\{\min\{\overline{\mu}_1(y,q), \overline{\mu}_2(z,q)\}, \min\{\overline{\mu}_1(y,q), \overline{\mu}_3(z,q)\}\} \\
&= \max\{\sup_{x=yz} \min\{\overline{\mu}_1(y,q), \overline{\mu}_2(z,q)\}, \sup_{x=yz} \min\{\overline{\mu}_1(y,q), \overline{\mu}_3(z,q)\}\} \\
&= ((\overline{\mu}_1 o \overline{\mu}_2) \cup (\overline{\mu}_1 o \overline{\mu}_3))(x,q) \\
(\omega_1 o (\omega_2 \cap \omega_3))(x,q) &= \inf_{x=yz} \max\{\omega_1(y,q), (\omega_2 \cap \omega_3)(z,q)\} \\
&= \inf_{x=yz} \max\{\omega_1(y,q), \min\{\omega_2(z,q), \omega_3(z,q)\}\} \\
&= \inf_{x=yz} \min\{\max\{\omega_1(y,q), \omega_2(z,q)\}, \max\{\omega_1(y,q), \omega_3(z,q)\}\} \\
&= \min\{\inf_{x=yz} \max\{w_1(y,q), \omega_2(z,q)\}, \inf_{x=yz} \max\{\omega_1(y,q), \omega_3(z,q)\}\} \\
&= ((\omega_1 o \omega_2) \cap (\omega_1 o \omega_3))(x,q).
\end{aligned}$$

If x is not expressed as $x = yz$, then $(\overline{\mu}_1 o (\overline{\mu}_2 \cup \overline{\mu}_3))(x,q) = [0,0] = ((\overline{\mu}_1 o \overline{\mu}_2) \cup (\overline{\mu}_1 o \overline{\mu}_3))(x,q)$ and $(\omega_1 o (\omega_2 \cap \omega_3))(x,q) = 1 = ((\omega_1 o \omega_2) \cap (\omega_1 o \omega_3))(x,q)$.

Thus $A_1 \copyright (A_2 \sqcup A_3) = (A_1 \copyright A_2) \sqcup (A_1 \copyright A_3)$.

(iv) If x is expressed as $x = yz$. Then

$$\begin{aligned}
(\overline{\mu}_1 o (\overline{\mu}_2 \cap \overline{\mu}_3))(x,q) &= \sup_{x=yz} \min\{\overline{\mu}_1(y,q), (\overline{\mu}_2 \cap \overline{\mu}_3)(z,q)\} \\
&= \sup_{x=yz} \min\{\overline{\mu}_1(y,q), \min\{\overline{\mu}_2(z,q), \overline{\mu}_3(z,q)\}\} \\
&= \sup_{x=yz} \min\{\min\{\overline{\mu}_1(y,q), \overline{\mu}_2(z,q)\}, \min\{\overline{\mu}_1(y,q), \overline{\mu}_3(z,q)\}\} \\
&= \min\{\sup_{x=yz} \min\{\overline{\mu}_1(y,q), \overline{\mu}_2(z,q)\}, \sup_{x=yz} \min\{\overline{\mu}_1(y,q), \overline{\mu}_3(z,q)\}\} \\
&= ((\overline{\mu}_1 o \overline{\mu}_2) \cap (\overline{\mu}_1 o \overline{\mu}_3))(x,q). \\
(\omega_1 o (\omega_2 \cup \omega_3))(x,q) &= \inf_{x=yz} \max\{\omega_1(y,q), (\omega_2 \cup \omega_3)(z,q)\} \\
&= \inf_{x=yz} \max\{\omega_1(y,q), \max\{\omega_2(z,q), \omega_3(z,q)\}\} \\
&= \inf_{x=yz} \max\{\max\{\omega_1(y,q), \omega_2(z,q)\}, \max\{\omega_1(y,q), \omega_3(z,q)\}\} \\
&= \max\{\inf_{x=yz} \max\{w_1(y,q), \omega_2(z,q)\}, \inf_{x=yz} \max\{\omega_1(y,q), \omega_3(z,q)\}\} \\
&= ((\omega_1 o \omega_2) \cup (\omega_1 o \omega_3))(x,q).
\end{aligned}$$

If x is not expressed as $x = yz$, then $(\overline{\mu}_1 o (\overline{\mu}_2 \cap \overline{\mu}_3))(x,q) = [0,0] = ((\overline{\mu}_1 o \overline{\mu}_2) \cap (\overline{\mu}_1 o \overline{\mu}_3))(x,q)$ and $(\omega_1 o (\omega_2 \cup \omega_3))(x,q) = 1 = ((\omega_1 o \omega_2) \cup (\omega_1 o \omega_3))(x,q)$.

Thus $A_1 \copyright (A_2 \sqcap A_3) = (A_1 \copyright A_2) \sqcap (A_1 \copyright A_3)$. □

Theorem 2.13. *Let S be a semigroup. Then the following are equivalent:*

(i) The intersection of two Q-cubic subsemigroup of S is a Q-cubic subsemigroup of S.

(ii) The intersection of two Q-cubic left (resp.right) ideal of S is a Q-cubic left (resp.right) ideal of S.

Proof. Let $A_1 = (\overline{\mu}_1, \omega_1)$ and $A_2 = (\overline{\mu}_2, \omega_2)$ be Q-cubic subsemigroup of S. Let $x, y \in S$ and $q \in Q$. Then

(i)

$$\begin{aligned}
(\overline{\mu}_1 \cap \overline{\mu}_2)(xy, q) &= \min\{\overline{\mu}_1(xy, q), \overline{\mu}_2(xy, q)\} \\
&\geq \min\{\min\{\overline{\mu}_1(x, q), \overline{\mu}_1(y, q)\}, \min\{\overline{\mu}_2(x, q), \overline{\mu}_2(y, q)\}\} \\
&= \min\{\min\{\overline{\mu}_1(x, q), \overline{\mu}_2(x, q)\}, \min\{\overline{\mu}_1(y, q), \overline{\mu}_2(y, q)\}\} \\
&= \min\{(\overline{\mu}_1 \cap \overline{\mu}_2)(x, q), (\overline{\mu}_1 \cap \overline{\mu}_2)(y, q)\} \\
(\omega_1 \cup \omega_2)(xy, q) &= \max\{\omega_1(xy, q), \omega_2(xy, q)\} \\
&\leq \max\{\max\{\omega_1(x, q), \omega_1(y, q)\}, \max\{\omega_2(x, q), \omega_2(y, q)\}\} \\
&= \max\{\max\{\omega_1(x, q), \omega_2(x, q)\}, \max\{\omega_1(y, q), \omega_2(y, q)\}\} \\
&= \max\{(\omega_1 \cup \omega_2)(x, q), (\omega_1 \cup \omega_2)(y, q)\}.
\end{aligned}$$

$\therefore A_1 \sqcap A_2 = (\overline{\mu}_1 \cap \overline{\mu}_2, \omega_1 \cup \omega_2)$ is a Q-cubic subsemigroup of S.

(ii)

$$\begin{aligned}
(\overline{\mu}_1 \cap \overline{\mu}_2)(xy, q) &= \min\{\overline{\mu}_1(xy, q), \overline{\mu}_2(xy, q)\} \\
&\geq \min\{\overline{\mu}_1(y, q), \overline{\mu}_2(y, q)\} \\
&= (\overline{\mu}_1 \cap \overline{\mu}_2)(y, q) \\
(\omega_1 \cup \omega_2)(xy, q) &= \max\{\omega_1(xy, q), \omega_2(xy, q)\} \\
&\leq \max\{\omega_1(y, q), \omega_2(y, q)\} \\
&= (\omega_1 \cup \omega_2)(y, q)
\end{aligned}$$

$\therefore A_1 \sqcap A_2 = (\overline{\mu}_1 \cap \overline{\mu}_2, \omega_1 \cup \omega_2)$ is a Q-cubic left (resp.right) ideal of S. □

Lemma 2.14. *If $A = (\overline{\mu}, \omega)$ is a Q-cubic subemigroup of a semigroup S, then the set $S_A = \{A(x, q) = A(0, q) \mid x \in S,\ q \in Q\}$ is a subsemigroup of S over Q.*

Proof. Let $x, y \in S_A$ and $q \in Q$. Then $\bar{\mu}(x, q) = \bar{\mu}(y, q) = \bar{\mu}(0, q)$ and $\omega(x, q) = \omega(y, q) = \omega(0, q)$. Since $A = (\bar{\mu}, \omega)$ is a Q-cubic subsemigroup of S. It follows that

$$\bar{\mu}(xy, q) \geq \min\{\bar{\mu}(x, q), \bar{\mu}(y, q)\} = \bar{\mu}(0, q)\, and$$
$$\omega(xy, q) \leq \max\{\omega(x, q), \omega(y, q)\} = \omega(0, q)$$

Thus $xy \in S_A$ and consequently S_A is a subsemigroup of S over Q. □

Lemma 2.15. *Let $A = (\overline{\mu}, \omega)$ be a Q-cubic subgroup of a semigroup S such that $\bar{\mu}(x, q) \geq \bar{\mu}(y, q)$ (or $\bar{\mu}(y, q) \geq \bar{\mu}(x, q)$) and $\omega(x, q) \leq \omega(y, q)$ (or $\omega(y, q) \leq \omega(x, q)$) for all $x, y \in S$ and $q \in Q$. Then $A = (\overline{\mu}, \omega)$ is a Q-cubic left (or right) ideal of S.*

Proof. Let $\bar{\mu}(x, q) \geq \bar{\mu}(y, q)$ and $\omega(x, q) \leq \omega(y, q)$ for all $x, y \in S$ and $q \in Q$. Then we have

$$\bar{\mu}(xy, q) \geq \min\{\bar{\mu}(x, q), \bar{\mu}(y, q)\} = \bar{\mu}(y, q)\, and$$
$$\omega(xy, q) \leq \max\{\omega(x, q), \omega(y, q)\} = \omega(y, q)$$

Hence $A = (\overline{\mu}, \omega)$ is a Q-cubic left ideal of S. Similarly, if we let $\bar{\mu}(y,q) \geq \bar{\mu}(x,q)$ and $\omega(y,q) \leq \omega(x,q))$ for all $x, y \in S$ and $q \in Q$, then it is easy to prove that $A = (\overline{\mu}, \omega)$ is a Q-cubic right ideal of S. □

Theorem 2.16. *A Q-cubic set $A = (\overline{\mu}, \omega)$ of a semigroup S is a Q-cubic bi-ideal of S if and only if μ^-, μ^+ and ω are Q-fuzzy ideals of S.*

Proof. Let $A = (\overline{\mu}, \omega)$ be a Q-cubic bi-ideal of S. For any $x, y \in S$ and $q \in Q$ then we have

$$\begin{aligned}[\mu^-(xy,q), \mu^+(xy,q)] &= \overline{\mu}(xy,q)\\ &\geq \min\{\overline{\mu}(x,q), \overline{\mu}(y,q)\}\\ &= \min\{[\mu^-(x,q), \mu^+(x,q)], [\mu^-(y,q), \mu^+(y,q)]\}\\ &= [\min\{\mu^-(x,q), \mu^-(y,q)], [\mu^+(x,q), \mu^+(y,q)\}]\end{aligned}$$

It follows that $\mu^-(xy,q) \geq \min\{\mu^-(x,q), \mu^-(y,q)\}$ and $\mu^+(xy,q) \geq \min\{\mu^+(x,q), \mu^+(y,q)\}$. Clearly, $\omega(xy,q) \leq \max\{\omega(x,q), \omega(y,q)\}$. Therefore μ^-, μ^+ and ω are Q-fuzzy ideals of S.

Conversely, suppose that μ^-, μ^+ and ω are Q-fuzzy ideals of S. Let $x, y \in S, q \in Q$.

$$\begin{aligned}\overline{\mu}(xy,q) &= [\mu^-(xy,q), \mu^+(xy,q)]\\ &\geq [\min\{\mu^-(x,q), \mu^-(y,q)\}, \min\{\mu^+(x,q), \mu^+(y,q)\}]\\ &= \min\{[\mu^-(x,q), \mu^+(x,q)], [\mu^-(y,q), \mu^+(y,q)]\}\\ &= \min\{\overline{\mu}(x,q), \overline{\mu}(y,q)\}\end{aligned}$$

Clearly, $\omega(xy,q) \leq \max\{\omega(x,q), \omega(y,q)\}$. Therefore $A = (\overline{\mu}, \omega)$ is a Q-cubic subsemigroup of S.

$$\begin{aligned}\overline{\mu}(xyz,q) &= [\mu^-(xyz,q), \mu^+(xyz,q)]\\ &\geq [\min\{\mu^-(x,q), \mu^-(z,q)\}, \min\{\mu^+(x,q), \mu^+(z,q)\}]\\ &= \min\{[\mu^-(x,q), \mu^+(x,q)], [\mu^-(z,q), \mu^+(z,q)]\}\\ &= \min\{\overline{\mu}(x,q), \overline{\mu}(z,q)\}\end{aligned}$$

Clearly, $\omega(xyz,q) \leq \max\{\omega(x,q), \omega(z,q)\}$. Therefore $A = (\overline{\mu}, \omega)$ is a Q-cubic bi-ideal of S. □

Theorem 2.17. *If $\{A_i\}_{i \in \curlywedge}$ is a family of Q-cubic bi-ideals of a semigroup S then $\sqcap A_i$ is a Q-cubic bi-ideal of S, where $\sqcap A_i = (\cap\overline{\mu}_i, \cup\omega_i)$, $\cap\overline{\mu}_i = inf\{\overline{\mu}_i(x,q) | i \in \curlywedge, x \in S$ and $q \in Q\}, \cup\omega_i = sup\{\omega_i(x,q) | i \in \curlywedge, x \in S$ and $q \in Q\}$ and $i \in \curlywedge$ is any index set.*

Proof. Since $A_i = (\overline{\mu}_i, \omega_i | i \in \curlywedge)$ is a family of Q-cubic bi-ideals of S. Let $x, y, z \in S$ and $q \in Q$.

$$\begin{aligned}\cap\overline{\mu}_i(xy,q) &= \inf\{\overline{\mu}_i(xy,q) | i \in \curlywedge, x, y \in S \text{ and } q \in Q\}\\ &\geq \inf\min\{\overline{\mu}_i(x,q), \overline{\mu}_i(y,q)\}\\ &= \min\{\inf\overline{\mu}_i(x,q), \inf\overline{\mu}_i(y,q)\}\\ &= \min\{\cap\overline{\mu}_i(x,q), \cap\overline{\mu}_i(y,q)\}\\ \cup\omega_i(xy,q) &= \sup\{\omega_i(xy,q) | i \in \curlywedge, x, y \in S \text{ and } q \in Q\}\\ &\leq \sup\max\{\omega_i(x,q), \omega_i(y,q)\}\\ &= \max\{\sup\omega_i(x,q), \sup\omega_i(y,q)\}\\ &= \max\{\cup\omega_i(x,q), \cup\omega_i(y,q)\}\end{aligned}$$

Hence $\sqcap A_i = (\cap\overline{\mu}_i, \cup\omega_i)$ is a Q-cubic subsemigroup of S.

$$\begin{aligned}
\cap\overline{\mu}_i(xyz,q) &= \inf\{\overline{\mu}_i(xyz,q) | i \in \curlywedge, x,y,z \in S \; and \; q \in Q\} \\
&\geq \inf\min\{\overline{\mu}_i(x,q), \overline{\mu}_i(z,q)\} \\
&= \min\{\inf\overline{\mu}_i(x,q), \inf\overline{\mu}_i(z,q)\} \\
&= \min\{\cap\overline{\mu}_i(x,q), \cap\overline{\mu}_i(z,q)\} \\
\cup\omega_i(xyz,q) &= \sup\{\omega_i(xyz,q) | i \in \curlywedge, x,y,z \in S \; and \; q \in Q\} \\
&\leq \sup\max\{\omega_i(x,q), \omega_i(z,q)\} \\
&= \max\{\sup\omega_i(x,q), \sup\omega_i(z,q)\} \\
&= \max\{\cup\omega_i(x,q), \cup\omega_i(z,q)\}
\end{aligned}$$

Therefore $\sqcap A_i = (\cap\overline{\mu}_i, \cup\omega_i)$ is a Q-cubic bi-ideal of S. □

Theorem 2.18. *If $A = (\overline{\mu}, \omega)$ is a Q-cubic bi-ideal of a semigroup S, then the set $S_A = \{\bar{\mu}(x,q) = \bar{\mu}(0,q), \omega(x,q) = \omega(0,q) \mid x \in S$,*
$q \in Q\}$ is a bi-ideal of S over Q.

Proof. By the lemma 2.14 $A = (\overline{\mu}, \omega)$ is a subsemigroup of S over Q. Let $x,y,z \in S_A$ and $q \in Q$. Then $\bar{\mu}(x,q) = \bar{\mu}(z,q) = \bar{\mu}(0,q)$ and $\omega(x,q) = \omega(z,q) = \omega(0,q)$. It follows that

$$\begin{aligned}
\bar{\mu}(xyz,q) &\geq \min\{\bar{\mu}(x,q), \bar{\mu}(z,q)\} = \bar{\mu}(0,q) \; and \\
\omega(xyz,q) &\leq \max\{\omega(x,q), \omega(z,q)\} = \omega(0,q)
\end{aligned}$$

Thus $xyz \in S_A$ and consequently S_A is a bi-ideal of S over Q. □

Theorem 2.19. *Let H be any nonempty subset of a semigroup S. Then H is a bi-ideal of S over Q if and only if the characteristic Q-cubic set $\chi_H = (\overline{\mu}_{\chi_H}, \omega_{\chi_H})$ is a Q-cubic bi-ideal of S.*

Proof. Assume that H is a bi-ideal of S. Let $x,y,z \in S$ and $q \in Q$. Suppose that $\overline{\mu}_{\chi_H}(xy,q) < \min\{\overline{\mu}_{\chi_H}(x,q), \overline{\mu}_{\chi_H}(y,q)\}$. It follows that $\overline{\mu}_{\chi_H}(xy,q) = \overline{0}, \min\{\overline{\mu}_{\chi_H}(x,q), \overline{\mu}_{\chi_H}(y,q)\} = \overline{1}$ and $\omega_{\chi_H}(xy,q) > \max\{\omega_{\chi_H}(x,q), \omega_{\chi_H}(y,q)\}$. It follows that $\omega_{\chi_H}(xy,q) = 1, \max\{\omega_{\chi_H}(x,q), \omega_{\chi_H}(y,q)\} = 0$. This implies that $x,y \in H$ but $xy \notin H$, a contradicts to H. So, $\overline{\mu}(xy,q) \geq \min\{\overline{\mu}(x,q), \overline{\mu}(y,q)\}$ and $\omega(xy,q) \leq \max\{\omega(x,q), \omega(y,q)\}$.
Suppose that $\overline{\mu}_{\chi_H}(xyz,q) < \min\{\overline{\mu}_{\chi_H}(x,q), \overline{\mu}_{\chi_H}(z,q))\}$. It follows that $\overline{\mu}_{\chi_H}(xyz\,q) = \overline{0}, \min\{\overline{\mu}_{\chi_H}(x,q), \overline{\mu}_{\chi_H}(z,q)\} = \overline{1}$ and $\omega_{\chi_H}(xyz,q) > \max\{\omega_{\chi_H}(x,q), \omega_{\chi_H}(z,q)\}$. It follows that $\omega_{\chi_H}(xyz,q) = 1, \max\{\omega_{\chi_H}(x,q), \omega_{\chi_H}(z,q)\} = 0$. This implies that $x,z \in H$ but $xyz \notin H$, a contradicts to H. So, $\overline{\mu}(xyz,q) \geq \min\{\overline{\mu}(x,q), \overline{\mu}(z,q)\}$ and $\omega(xyz,q) \leq \max\{\omega(x,q), \omega(z,q)\}$. This shows that χ_H is a Q-cubic bi-ideal of S.
Conversely, $\chi_H = (\overline{\mu}_{\chi_H}, \omega_{\chi_H})$ is a Q-cubic bi-ideal of S for any subset H of S. Let $x,y \in H$ and $q \in Q$ then $\overline{\mu}_{\chi_H}(x) = \overline{\mu}_{\chi_H}(y) = \overline{1}$ and $\omega_{\chi_H}(x) = \omega_{\chi_H}(y) = 0$ since χ_H is a Q-cubic bi-ideal of S. $\overline{\mu}_{\chi_H}(xy,q) \geq \min\{\overline{\mu}_{\chi_H}(x,q), \overline{\mu}_{\chi_H}(y,q)\} \geq \min\{\overline{1}, \overline{1}\} = \overline{1}$ and $\omega_{\chi_H}(x,q) \leq \max\{\omega_{\chi_H}(x,q), \omega_{\chi_H}(y,q)\} \leq \max\{0,0\} = 0$. This implies that $xy \in H$.
Let $x,y,z \in H$ and $q \in Q$ then $\overline{\mu}_{\chi_H}(x) = \overline{\mu}_{\chi_H}(y) = \overline{\mu}_{\chi_H}(z) = \overline{1}$ and $\omega_{\chi_H}(x) = \omega_{\chi_H}(y) = \omega_{\chi_H}(z) = 0$. $\overline{\mu}_{\chi_H}(xyz,q) \geq \min\{\overline{\mu}_{\chi_H}(x,q), \overline{\mu}_{\chi_H}(z,q)\} \geq \min\{\overline{1}, \overline{1}\} = \overline{1}$ and $\omega_{\chi_H}(xyz,q) \leq \max\{\omega_{\chi_H}(x,q), \omega_{\chi_H}(z,q)\} \leq \max\{0,0\} = 0$. Which implies that $xyz \in H$. Hence H is a bi-ideal of S over Q. □

Theorem 2.20. *If $A = (\overline{\mu}, \omega)$ be any Q-cubic set of a semigroup S then $A = (\overline{\mu}, \omega)$ is a Q-cubic bi-ideal of S if and only if the level set $U(A; \overline{t}, n)$ is a bi-ideal of S over Q when it is nonempty.*

Proof. Assume that $A = (\overline{\mu}, \omega)$ is a Q-cubic bi-ideal of S. Let $x, y, z \in U(A; \overline{t}, n), q \in Q$ for all $\overline{t} \in D[0,1]$ and $n \in [0,1]$. Then $\overline{\mu}(x,q) \geq \overline{t}$, $\overline{\mu}(y,q) \geq \overline{t}$, $\overline{\mu}(z,q) \geq \overline{t}$ and $\omega(x,q) \leq n, \omega(y,q) \leq n, \omega(z,q) \leq n$.
Now, suppose $x, y \in U(A; \overline{t}, n)$ then $\overline{\mu}(xy,q) \geq \min\{\overline{\mu}(x,q), \overline{\mu}(y,q)\} \geq \min\{\overline{t}, \overline{t}\} = \overline{t}$ and

$\omega(xy,q) \leq \max\{\omega(x,q), \omega(y,q)\} \leq \max\{n,n\} = n$. Thus $xy \in U(A; \overline{t}, n)$.
Suppose $x, y, z \in U(A; \overline{t}, n)$ then $\overline{\mu}(xyz,q) \geq \min\{\overline{\mu}(x,q), \overline{\mu}(z,q)\} \geq \min\{\overline{t}, \overline{t}\} = \overline{t}$ and

$\omega(xyz,q) \leq \max\{\omega(x,q), \omega(z,q)\} \leq \max\{n,n\} = n$. Hence $xyz \in U(A; \overline{t}, n)$. Therefore $U(A; \overline{t}, n)$ is a bi-ideal of S over Q.
Conversely, let $\overline{t} \in D[0,1]$ and $n \in [0,1]$ be such that $U(A; \overline{t}, n) \neq \emptyset$ and $U(A; \overline{t}, n)$ is a bi-ideal of S over Q.
Let us assume that $\overline{\mu}(xy,q) \ngeq \min\{\overline{\mu}(x,q), \overline{\mu}(y,q)\}$ or $\omega(xy,q) \nleq \max\{\omega(x,q), \omega(y,q)\}$. If $\overline{\mu}(xy,q) \ngeq \min\{\overline{\mu}(x,q), \overline{\mu}(y,q)\}$ then there exists $\overline{t}_1 \in D[0,1]$ such that $\overline{\mu}(xy,q) < \overline{t}_1 < \min\{\overline{\mu}(x,q), \overline{\mu}(y,q)\}$. Hence $x, y \in U(A; \overline{t}, \max\{\omega(x,q), \omega(y,q)\})$ but $xy \notin U(A; \overline{t}_1, \max\{\omega(x,q), \omega(y,q)\})$ which is a contradiction. If $\omega(xy,q) \nleq \max\{\omega(x,q), \omega(y,q)\}$ then there exists $n_1 \in [0,1]$ such that $\omega(xy,q) > n_1 > \max\{\omega(x,q), \omega(y,q)\}$. Hence $x, y \in U(A; \min\{\overline{\mu}(x,q), \overline{\mu}(y,q)\}, n_1)$ but $xy \notin U(A; \min\{\overline{\mu}(x,q), \overline{\mu}(y,q)\}, n_1)$. This gives a contradiction. Hence $\overline{\mu}(xy,q) \geq \min\{\overline{\mu}(x,q), \overline{\mu}(y,q)\}$ and $\omega(xy,q) \leq \max\{\omega(x,q), \omega(y,q)\}$.
Suppose $\overline{\mu}(xyz,q) \ngeq \min\{\overline{\mu}(x,q), \overline{\mu}(z,q)\}$ or $\omega(xyz,q) \nleq \max\{\omega(x,q), \omega(z,q)\}$. If $\overline{\mu}(xyz,q) \ngeq \min\{\overline{\mu}(x,q), \overline{\mu}(z,q)\}$ then there exists $\overline{t}_1 \in D[0,1]$ such that $\overline{\mu}(xyz,q) < \overline{t}_1 < \min\{\overline{\mu}(x,q), \overline{\mu}(z,q)\}$. Hence $x, z \in U(A; \overline{t}, \max\{\omega(x,q), \omega(z,q)\})$ but $xyz \notin U(A; \overline{t}_1, \max\{\omega(x,q), \omega(z,q)\})$ which is a contradiction. If $\omega(xyz,q) \nleq \max\{\omega(x,q), \omega(z,q)\}$ then there exists $n_1 \in [0,1]$ such that $\omega(xyz,q) > n_1 > \max\{\omega(x,q), \omega(z,q)\}$. Hence $x, z \in U(A; \min\{\overline{\mu}(x,q), \overline{\mu}(z,q)\}, n_1)$ but $xyz \notin U(A; \min\{\overline{\mu}(x,q), \overline{\mu}(z,q)\}, n_1)$. This gives a contradiction. Thus $\overline{\mu}(xyz,q) \geq \min\{\overline{\mu}(x,q), \overline{\mu}(z,q)\}$ and $\omega(xyz,q) \leq \max\{\omega(x,q), \omega(z,q)\}$. Therefore $A = (\overline{\mu}, \omega)$ is a Q-cubic bi-ideal of S. □

Theorem 2.21. *If $\{A_i\}_{i \in \curlywedge}$ is a family of Q-cubic interior ideals of a semigroup S then $\sqcap A_i$ is a Q-cubic interior ideal of S, where $\sqcap A_i = (\cap \overline{\mu}_i, \cup \omega_i)$, $\cap \overline{\mu}_i = inf\{\overline{\mu}_i(x,q) | i \in \curlywedge, x \in S$ and $q \in Q\}, \cup \omega_i = sup\{\omega_i(x,q) | i \in \curlywedge, x \in S$ and $q \in Q\}$ and $i \in \curlywedge$ is any index set.*

Theorem 2.22. *If $A = (\overline{\mu}, \omega)$ is a Q-cubic interior ideal of a semigroup S, then the set $S_A = \{\bar{\mu}(x,q) = \bar{\mu}(0,q), \omega(x,q) = \omega(0,q) \mid x \in S,\ q \in Q\}$ is a interior ideal of S over Q.*

Theorem 2.23. *Let H be any non-empty subset of a semigroup S. Then H is an interior ideal of S over Q if and only if the characteristic cubic set $\chi_H = (\overline{\mu}_{\chi_H}, \omega_{\chi_H})$ of H in S is a Q-cubic interior ideal of S.*

3. Homomorphism of Q-cubic Ideals in Semigroups

Definition 3.1. *Let X, Y be two semigroups and Q be non-empty set. A mapping $f : X \times Q \to Y \times Q$ is called a homomorphism if $f(xy,q) = f(x,q) \cdot f(y,q)$ for all $x, y \in X$ and $q \in Q$.*

Definition 3.2. *Let f be a mapping from a set $X \times Q$ to a set $Y \times Q$ and $A = (\overline{\mu}, \omega)$ be a Q-cubic set of X then the image* *of X (i.e.,) $f(A) = (f(\overline{\mu}), f(\omega))$ is a Q-cubic set of Y is defined by*

$$f(A)(x,q) = \begin{cases} f(\overline{\mu})(x,q) = \begin{cases} \sup\limits_{y \in f^{-1}(x)} \overline{\mu}(y,q) & \text{if } f^{-1}(x) \neq \emptyset \\ [0,0] & \text{otherwise} \end{cases} \\ f(\omega)(x,q) = \begin{cases} \inf\limits_{y \in f^{-1}(x)} \omega(y,q) & \text{if } f^{-1}(x) \neq \emptyset \\ 1 & \text{otherwise} \end{cases} \end{cases}$$

Let f be a mapping from a set $X \times Q$ to $Y \times Q$ and $A = (\overline{\mu}, \omega)$ be a Q-cubic set of Y then the pre image of Y (i.e.,) $f^{-1}(A) = (f^{-1}(\overline{\mu}), f^{-1}(\omega))$ is a Q-cubic set of X is defined by

$$f^{-1}(A)(x,q) = \begin{cases} f^{-1}(\overline{\mu})(x,q) = \overline{\mu}(f(x),q) \\ f^{-1}(\omega)(x,q) = \omega(f(x),q) \end{cases}$$

Theorem 3.3. *Let X, Y be semigroups, Q be any non-empty set, $f : X \times Q \to Y \times Q$ be a homomorphism of semigroups.*
(i) If $A = (\overline{\mu}, \omega)$ is a Q-cubic subsemigroup of Y then the preimage $f^{-1}(A) = (f^{-1}(\overline{\mu}), f^{-1}(\omega))$ is a Q-cubic subsemigroup of X.
(ii) If $A = (\overline{\mu}, \omega)$ is a Q-cubic left (resp.right) ideal of Y then the preimage $f^{-1}(A) = (f^{-1}(\overline{\mu}), f^{-1}(\omega))$ is a Q-cubic left (resp.right) ideal of X.

Proof. Assume that $A = (\overline{\mu}, \omega)$ is a Q-cubic subsemigroup of Y and $x, y \in X,\ q \in Q$. Then

(i)

$$\begin{aligned} f^{-1}(\overline{\mu})(xy,q) &= \overline{\mu}(f(xy),q) \\ &= \overline{\mu}(f(x)f(y),q) \\ &\geq \min\{\overline{\mu}(f(x),q), \overline{\mu}(f(y),q)\} \\ &= \min\{f^{-1}(\overline{\mu})(x,q), f^{-1}(\overline{\mu})(y,q)\} \\ f^{-1}(\omega)(xy,q) &= \omega(f(xy),q) \\ &= \omega(f(x)f(y),q) \\ &\leq \max\{\omega(f(x),q), \omega(f(y),q)\} \\ &= \max\{f^{-1}(\omega)(x,q), f^{-1}(\omega)(y,q)\} \end{aligned}$$

Hence $f^{-1}(A) = (f^{-1}(\overline{\mu}), f^{-1}(\omega))$ is a Q-cubic subsemigroup of X.

(ii)

$$\begin{aligned} f^{-1}(\overline{\mu})(xy,q) &= \overline{\mu}(f(xy),q) \\ &= \overline{\mu}(f(x)f(y),q) \\ &\geq \overline{\mu}(f(y),q) \\ &= f^{-1}(\overline{\mu})(y,q) \\ f^{-1}(\omega)(xy,q) &= \omega(f(xy),q) \\ &= \omega(f(x)f(y),q) \\ &\leq \omega(f(y),q) \\ &= f^{-1}(\omega)(y,q) \end{aligned}$$

Hence $f^{-1}(A) = (f^{-1}(\overline{\mu}), f^{-1}(\omega))$ is a Q-cubic left (resp. right) ideal of X. □

Theorem 3.4. *Let X, Y be semigroups, Q be any non-empty set and $f : X \times Q \to Y \times Q$ be a homomorphism. If $A = (\overline{\mu}, \omega)$ is a Q-cubic bi-ideal of Y then the preimage $f^{-1}(A) = (f^{-1}(\overline{\mu}), f^{-1}(\omega))$ is a Q-cubic bi-ideal of X.*

Proof. Assume that $A = (\overline{\mu}, \omega)$ is a Q-cubic bi- ideal of Y and $x, y, z \in X$, $q \in Q$. Then
By theorem 3.3. $f^{-1}(A) = (f^{-1}(\overline{\mu}), f^{-1}(\omega))$ is a Q-cubic subsemigroup of X.

$$\begin{aligned}
f^{-1}(\overline{\mu})(xyz, q) &= \overline{\mu}(f(xyz), q) \\
&= \overline{\mu}(f(x)f(y)f(z), q) \\
&\geq \min\{\overline{\mu}(f(x), q), \overline{\mu}(f(z), q)\} \\
&= \min\{f^{-1}(\overline{\mu})(x, q), f^{-1}(\overline{\mu})(z, q))\} \\
f^{-1}(\omega)(xyz, q) &= \omega(f(xyz), q) \\
&= \omega(f(x)f(y)f(z), q) \\
&\leq \max\{\omega(f(x), q), \omega(f(z), q)\} \\
&= \max\{f^{-1}(\omega)(x, q), f^{-1}(\omega)(z, q))\}
\end{aligned}$$

Hence $f^{-1}(A) = (f^{-1}(\overline{\mu}), f^{-1}(\omega))$ is a Q-cubic bi-ideal of X. □

Theorem 3.5. *Let X, Y be semigroups, Q be any non-empty set and $f : X \times Q \to Y \times Q$ be a homomorphism. If $A = (\overline{\mu}, \omega)$ is a Q-cubic interior ideal of Y then the preimage $f^{-1}(A) = (f^{-1}(\overline{\mu}), f^{-1}(\omega))$ is a Q-cubic interior ideal of X.*

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