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A Note on the Asymptotic Behaviour of Some Difference Equation of Second Order

Research Article

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Abstract: In this paper we study the asymptotic behaviour of some difference equations of second order.

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1. Introduction

In the qualitative theory of ordinary differential equations, the Lié nard equation

$$x'' + f(x)x' + g(x) = 0$$

is a paradigmatic case of second order differential equation, named after the French physicist Alfred-Marie Liénard. During the development of radio and vacuum tube technology, Liénard equations were intensely studied as they can be used to model oscillating circuits. Under certain additional assumptions Liénard's theorem guarantees the uniqueness and existence of a limit cycle for such a system.

The classical method of Lyapunov for studying stability and asymptotic stability is based in a suitable function satisfying some properties (called Lyapunov's Functions). This method, usually named Direct o Second Method, was originated in the fundamental memoir of the russian mathematician Alexander Mijailovich Lyapunov, published in Russian in 1892, translated into French in 1907 (reprinted in the forty) and in English years later, see Liapounov (1949). Since that time this area has been extensively (perhaps even exhaustively) investigated. Statements and proofs of mathematical results underlying the method and numerous examples and references can be found in many books and papers.

The theory of discrete dynamical systems has grown in the last decade. Difference equations can arise in many modelization problems or they may be a discrete approximation of a continuous process.This growth has been strongly promoted by the advanced technology in computation and the large number of applications in biology, engineering and other fields. Non-linear

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difference equations of second order are very important; such equations appear naturally as discrete analogues of differential and delay differential equations which are model various phenomena [7], so it becomes important the asymptotic study of the solutions of these equations. However the Second Lyapunov Method is underdeveloped in the case of the difference equations [3, 5, 8, 13], so that it arises naturally, the need to develop for these equations, in particular, for a case as relevant as the Liénard Equation.

In this note we present some results about asymptotic behaviour of certain difference equation of second order of Liénard type.

2. Results

2.1. On the monotonic character of positive solutions.

Consider the second order difference equation

$$x(n+2) + f(x(n))x(n+1) + b(n)g(x(n+1), x(n)) = 0 \tag{1}$$

Special cases of this difference equation have appeared in the classical theories of the business cycle since 1939 [2, 9, 10]. Depending on the choice of the function f, this equation exhibits a remarkable variety of dynamical behaviours as discussed in [4, 11]. The results obtained are motived by those of [12].

Lemma 2.1. *If $0 < a \leq f(x(n)) \leq A < +\infty$, $0 \leq \frac{g(z,u)}{|z-u|} \leq k$, $0 < b \leq b(n) \leq B < +\infty$ for all z, $n \in \mathbb{N}$ and $c < A < 1$ for some c, then every positive solution of (1) is eventually decreasing.*

Proof. Let $\{x_n\}$ be a positive solution of (1). Then the ratios $r_n = \frac{x(n)}{x(n-1)}, n \geq 0$ and satisfy

$$\begin{aligned} \frac{x(n+2)}{x(n+1)} &\leq A + b\frac{g(x(n))}{x(n+1)} \\ r_{n+2} &\leq A + c\left|1 - \frac{1}{r_{n+1}}\right|. \end{aligned}$$

Since it is also true that $r_{n+2} \geq A$. From this we have

$$A \leq r_{n+2} \leq A + c\left|1 - \frac{1}{r_{n+1}}\right|, n \geq 0$$

If $r_1 \leq 1$ we have

$$A \leq r_2 \leq A + c\left(1 - \frac{1}{r_{n+1}}\right) < 1$$

Because $c + A < 1$.Inductively, if for $k \geq 2$

$$0 \leq r_n < 1, n < k$$

then $0 < r_k < 1$ so that

$$r_1 \leq 1 \text{ then } r_n < 1 \text{ for all } n > 1. \tag{2}$$

Now suppose that $r_1 > 1$. Then

$$A \leq r_2 \leq A + c - \frac{A}{r_1}$$

If $A + c \leq 1$, then $r_2 < 1$ and (2) holds. Assume that $A + c > 1$ and $r_2 > 1$. Then

$$0 \leq r_3 \leq A + c - \frac{c}{r_2} < r_2$$

The last inequality holds because for every $u > 1$, $A + c - \frac{c}{u} < u$ if and only if $u^2 - (A+c)u + c > 0$. This is true because $u^2 - (A+c)u + c = 0$, with zeros $u_{1,2} = \frac{A+c \pm \sqrt{(A+c)^2 - 4c}}{2}$, only if $u_{1,2} < 1$. Now, if $r_3 < 1$, then (2) holds for $n > 2$. Otherwise, using the last fact, we can show inductively that $r_1 > r_2 > r_3 > \dots$. So, there is some $k \geq 1$ such that $r_k \leq 1$ and (2) holds for $n > k$. Hence, we have shown that for any choice of r_0, the sequence r_n is eventually less than 1; i.e., the sequence $\{x_n\}$ is decreasing for all n sufficiently large. □

In a similar way, we can prove the following lemma.

Lemma 2.2. *Under assumptions of Lemma 2.1 if $g(.,.) \geq 0$ and $g(0,0) = 0$ then every non-positive solution of (1) is non-decreasing and converges to zero.*

Remark 2.3. *These results are consistent with those of [12].*

2.2. On the stability character and Lyapunov functions

In this section let us consider the autonomous vector difference equation

$$x(n+1) = f(x(n)), x(n_0) = x_0, \tag{3}$$

A point x^* in $\mathbb{R}^k$ is called an *equilibrium point* of (4) if $f(x^*) = x^*$ for all $n \geq n_0$. Without loss of generality we can assume that $x^* \equiv 0$. We are now to introduce the stability definition of the equilibrium point x^* of (3).

Definition 2.4 ([1])**.** *The equilibrium point x^* of (4) is said to be stable (S) if given $\varepsilon > 0$ and $n_0 \geq 0$ there exists $\delta = \delta(\varepsilon, n_0)$ such that $\|x_0 - x^*\| < \delta$ implies $\|x(n, n_0, x_0) - x^*\| < \varepsilon$ for all $n \geq n_0$, uniformly stable (US) if δ may be chosen independent of n_0, unstable if it is not stable.*

Remark 2.5. *For autonomous system the stability and uniform stability are equivalent [1].*

In this section we applies the Lyapunov's direct method to difference equation (3). So we have some precisions. Let $V : \mathbb{R}^k \to \mathbb{R}$ be defined as a real-valued function. The variation of V relative to (3) would then be defined as

$$\triangle V(x) = V(f(x)) - V(x)$$

and

$$\triangle V(x(n)) = V(f(x(n))) - V(x(n)) = V(x(n+1)) - V(x(n)).$$

Notice that if $\triangle V(x) \leq 0$, then V is non-increasing along solutions of (3). The function V is said to be a *Lyapunov function* on a subset $H \in \mathbb{R}^k$ if:

(1) V is continuous on H, and

(2) $\triangle V(x) \leq 0$, whenever x and $f(x)$ belong to H.

The definitions of *positive (negative) definite* and *semi positive (negative) definite function* are similar to ordinary differential equations case and we can see in [1]. The following result is the basic tool [1].

Theorem 2.6 (Lyapunov Stability Theorem)**.** *If V is a Lyapunov function for (3) in a neighbourhood H of the equilibrium point x^*, and V is positive definite with respect to x^*, then x^* is stable. If, in addition, $\triangle V(x) < 0$ whenever $x, f(x) \in H$ and $x = x^*$, then x^* is asymptotically stable. Moreover, if $G = H = \mathbb{R}^k$ and*

$$V(x) \to \infty \ as \ x \to \infty,$$

then x^* is globally asymptotically stable. We consider the following system

$$x(n+1) = y(n)$$
$$y(n+1) = -g(x(n))$$

with $g(0) = 0$ and we can take as Lyapunov function the following $V(x, y) = x^2 + y^2$. This is clearly continuous and positive definite on $\mathbb{R}^2$ and we have

$$\triangle V(x(n), y(n)) = g^2(x(n)) - x^2(n) = x^2(n)\left[\left(\frac{g(x(n))}{x(n)}\right)^2 - 1\right] \tag{4}$$

Taking into account (4) and (5) we have the following result.

Lemma 2.7. *If in (5), $g(.)$ is a continuous function on $\mathbb{R}$ such that $0 < \frac{g(x(n))}{x(n)} < 1$ and $g(0) = 0$, then the origin is stable (and uniformly stable).*

Remark 2.8. *In the linear case*

$$x(n+1) = y(n),$$
$$y(n+1) = -\frac{x(n)}{4}.$$

this results are coincident with those knowns. This shows the consistency of Lemma 2.7.

Remark 2.9. *This results remain true if the function $g(x(n))$ is replaced by any function $f(n, x(n))$ satisfying $f(n, 0) = 0$, $x(n)f(n, x(n)) > 0$ and $|f(n, x(n))| \leq g(x(n))$.*

Now we consider the following system

$$x(n+1) = y(n),$$
$$y(n+1) = \frac{ax(n)}{(1 + y^2(n))}.$$

It is easy we obtain the following result.

Lemma 2.10. *If in (6), $g(.)$ and $h(.)$ are continuous function on $\mathbb{R}$ such that $0 < \frac{g(x(n))}{x(n)} < \alpha < 1$, $g(0) = 0$ and $h(y(n)) \geq 1 + \beta y^2(n)$, then the origin is stable (and uniformly stable).*

Proof. It is enough take the same Lyapunov function. □

Remark 2.11. *The Remark 2.9 it remains valid.*

3. Concluding Remarks

Summarizing our results, the Lemmas 2.1 and 2.2 shows the asymptotic behavior of solutions of certain Liénard type difference equation with some "signum condition" of g(.,.). If in (1) we consider:

$$f(x(n)) = -c,\ c < 0$$
$$g(x(n+1), x(n)) = -f(x(n+1) - x(n))$$
$$b(n) \equiv 1$$

our results completes those obtained in [4, 12] refer the asymptotic behavior of (1) with $0 \leq c < 1$ and $a \leq c-1$ or $a < \frac{2-c}{3-c}$, $c \neq 0$. The Lemmas 3 and 4 are obtained under "limit" case $f(x(n)) \equiv 0$ and $g(x(n+1), x(n)) = g(x(n))$ in (1) and they provide results on stability of solutions of (3) building an appropiatte Lyapunov Function. The main obstacle to generalize these results is the obatining the above function and the application of Chain Rule [1].

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