# Diophantine Equations and its Applications in Real Life

**Research Article**

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**Abstract:** Diophantine equations can be used in various fields. In this paper, we have tried to study the origin of Diophantine Equations and how they can be applied in Real life.

**Keywords:** Diophantine Equations, Balancing Chemical Equations, Network flow, Pythagorean Triples, Fermat Last Theorem.
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## 1. Introduction

The word 'Diophantine' came from 'Diophantus' who was a Mathematician in Alexandria sometimes around 250AD. The mathematics in his personal life motivated his study. His Boyhood lasted $1/6^{th}$ of his life, his beard grew after $1/12^{th}$ more, after $1/7^{th}$ more he got married and his son was born 5 years later. The son lived half his father's age and the father died 4 years after his son. If x was the age at which Diophantus died then, $\frac{1}{6}x + \frac{1}{12}x + \frac{1}{7}x + 5 + \frac{1}{2}x + 4 = x$, which solves to x = 84 (his age). The motivation to study more equations in one or more unknowns having only integral solutions lead to the origin of *Diophantine Equation which is defined as* a Polynomial equation with Integral coefficients which is solvable in Integers.

### 1.1. Forms (Types) of Diophantine Equations

The Simplest form of Diophantine Equation is a Linear Diophantine Equation in One variable, namely, $ax = b$; when $x = \frac{b}{a}$ must be integral.

*For instance*, If $2/5^{th}$ of a number is 4 more than $1/3^{rd}$ of the number then what is the number?

$$\frac{2}{5}x - \frac{x}{3} = 4 \Rightarrow x = 60 \text{ (Integral)}$$

Extending this, we move to linear Diophantine Equation in Two variables which is of the form : $ax + by = c$. We observe that not all Diophantine equations are solvable. The following result tells us the condition of solvability.

**Result 1.1.** *The Linear Diophantine equation $ax + by = c$ has a solution if and only if $d \mid c$ where $d = \gcd(a, b)$. If $x_0$, $y_0$ is any particular solution of this equation, then the other solutions are given by*

$$x = x_0 + \left(\frac{b}{d}\right)t, \qquad y = y_0 - \left(\frac{a}{d}\right)t$$

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*where t is an arbitrary integer.*

*Proof.* **First Assertion:** We know that there are integers r and s for which a=dr and $b = ds$. If a solution of $ax + by = c$ exists, so that $ax_0 + by_0 = c$ for suitable $x_0$ and $y_0$, then

$$c = ax_0 + by_0 = drx_0 + dsy_0 = d(rx_0 + sy_0)$$

which simply says that $d \mid c$.

Conversely, Assume that $d \mid c$, say $c = dt$. Then clearly integers $x_0$ and $y_0$ can be found satisfying $d = ax_0 + by_0$. Multiplying by t,

$$c = dt = (ax_0 + by_0)\, t = a(tx_0) + b(ty_0).$$

Hence, Diophantine equation $ax + by = c$ has $x = tx_0$ and $y = ty_0$ as a particular solution.

**Second Assertion:** Let us suppose that a solution $x_0$, $y_0$ of the given equation is known. If $x'$, $y'$ is any other solution, then $ax_0 + by_0 = c = ax' + by'$ which is equivalent to $a(x' - x_0) = b(y_0 - y')$. There exist relatively prime integers r and s such that $a = dr$, $b = ds$. Substituting these values into last written equation we get

$$r(x' - x_0) = s(y_0 - y')$$

Now we have, $r|s(y_0 - y')$ with $gcd(r, s) = 1$. Using Euclid's Lemma (If a prime p divides product ab of 2 integers a and b, p must divide atleast one of those integers a and b), it must be the case that $r \mid (y_0 - y')$ or $y_0 - y' = rt$, for some integer t. Substituting, we get $x' - x_0 = st$ which leads to the formulas

$$x' = x_0 + st = x_0 + \left(\frac{b}{d}\right) t,$$
$$y' = y_0 - rt = y_0 - \frac{a}{d} t$$

It is easy to see that these values satisfy the Diophantine equation, regardless of choice of integer t; for

$$ax' + by' = a\left[x_0 + \left(\frac{b}{d}\right) t\right] + b\left[y_0 - \left(\frac{a}{d}\right) t\right]$$
$$= (ax_0 + by_0) + \left(\frac{ab}{d} - \frac{ab}{d}\right) t$$
$$= c + 0.t$$
$$= c$$

Therefore, there are infinite number of solutions of the given equation, one for each value of $t$. $\square$

*Note:* Diophantine equations can be extended in both linear and non linear forms for finite and infinitely many variables.

## 2. Applications of Diophantine Equations in Real Life

**Word Problem on Age :** $1/8^{th}$ of Ross' age 1 year ago plus $1/9^{th}$ of his age 1 year from now is 4 years. What is his present age?

$$\frac{H - 1}{8} + \frac{H + 1}{9} = 4 \Rightarrow H = 17 \text{ (his present age)}.$$

***Note:*** When simultaneous Linear Diophantine Equations are given in more than 1 variable then the system it generates is equivalent to the system of linear equations.

**Word Problem on Business:** James invested a part of his investment in 10% bond A and a part in 20% bond B. His interest income during first year is Rs.4,000. If he invests 60% more in 10% bond A and 10% more in 20% bond B, his income during second year increases by Rs. 2,000. Find his initial investments.

*Solution.* Let his investment be Rs.x and Rs.y in Bond A and Bond B respectively. Then

$$0.10x + 0.20y = 4000$$

and in second year,

$$0.10(1.60x) + 0.20(1.10y) = 6000$$

$$0.16x + 0.22y = 6000$$

We can solve these as

$$\begin{bmatrix} 0.10 & 0.20 \\ 0.16 & 0.22 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4000 \\ 6000 \end{bmatrix}$$
$$\quad A \qquad\qquad X \qquad B$$

since $|A| = -0.01 \neq 0$, so A is invertible. Then

$$X = A^{-1}B$$

$$= \frac{1}{-0.01} \begin{bmatrix} 0.22 & -0.20 \\ -0.16 & 0.10 \end{bmatrix} \begin{bmatrix} 4000 \\ 6000 \end{bmatrix}$$

$$= \frac{1}{-0.01} \begin{bmatrix} -320 \\ -40 \end{bmatrix}$$

$$X = \begin{bmatrix} 3,20,000 \\ 40,000 \end{bmatrix}$$

i.e. $x = Rs.3,20,000$, $y = Rs.40,000$.

**Network Flow :** The Traffic flow, in vehicles per hour, over several one way streets in Delhi during a typical early afternoon is given in the following diagram. Determine the general flow patterns for the network.
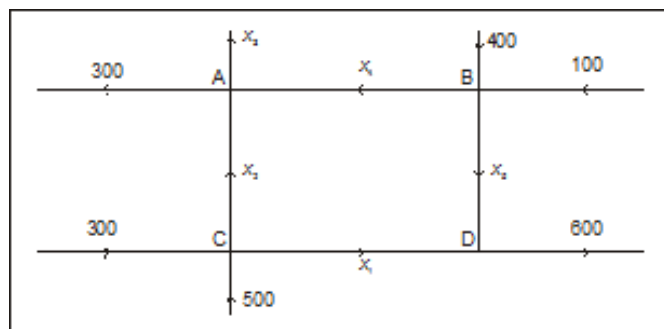


The system generated by the above condition is

| Intersection | Inflow | Outflow |
|---|---|---|
| A | $x_2 + x_4$ | $= 300 + x_3$ |
| B | $400 + 100$ | $= x_4 + x_5$ |
| C | $300 + 500$ | $= x_1 + x_2$ |
| D | $x_1 + x_5$ | $= 600$ |

The Total Inflow = Total Outflow

$$\Rightarrow x_2 + x_4 + 400 + 100 + 300 + 500 + x_1 + x_5 = 300 + x_3 + x_4 + x_5 + x_1 + x_2 + 600$$

$$\Rightarrow 1300 = x_3 + 900$$

$$\Rightarrow x_3 = 400$$

for a simultaneous solution we express the above conditions as

$$x_2 - x_3 + x_4 = 300$$

$$x_4 + x_5 = 500$$

$$x_1 + x_2 = 800$$

$$x_1 + x_5 = 600$$

$$x_3 = 400$$

As $x_1$, $x_2$, $x_3$, $x_4$, $x_5$ represents vehicles, $\therefore$ $x_1$, $x_2$,..., $x_5$ must be whole numbers. $\Rightarrow$ The above generated system is a system of Linear Diophantine equations in 5 variables. A solution on simplification is obtained as follows:

$$x_1 = 600 - x_5$$

$$x_2 = 200 + x_5$$

$$x_3 = 400$$

$$x_4 = 500 - x_5$$

$$x_5 = \text{ free variable.}$$

A negative flow in Network Branch corresponds to flow in opposite Direction to the above shown model, since the streets in the problem are one way, so none of the variables can be negative. This fact leads to certain limitations on the possible variables. For Instance $x_5 < 500$ (as $x_4$ cannot be negative). $\therefore$ the solutions depend upon the choice of $x_5$ from 0 to 500 (whole numbers).

## 3. Balancing Chemical Equations

When propane gas burns, the propane ($C_3H_8$) combines with oxygen ($O_2$) to form carbon dioxide ($CO_2$) and water ($H_2O$) according to the following equation

$$x_1(C_3H_8) + x_2(O_2) \rightarrow x_3(CO_2) + x_4(H_2O)$$
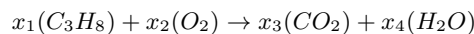
To Balance this equation, $x_1$, $x_2$, $x_3$, $x_4 \in \mathbb{N} \cup \{0\}$ such that total no. of carbon, hydrogen and oxygen atoms on the left matches the corresponding atoms on the right. Since these equations involve 3 types of atoms, so we construct a vector in

$\mathbb{R}^3$ for each reactant of the form

$$\begin{bmatrix} Carbon \\ Hydrogen \\ Oxygen \end{bmatrix}$$

We obtain

$$x_1 \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

Row reduction of the matrix so formed leads to the general solution

$$x_1 = \frac{1}{4}x_4, \qquad x_2 = \frac{5}{4}x_4, \qquad x_3 = \frac{3}{4}x_4, \qquad x_4 = \text{free variable.}$$

Since the coefficient in a chemical equation must be integers, let $x_4 = 4$. In this case, $x_1 = 1$, $x_2 = 5$, $x_3 = 3$ and Balanced Equation becomes

$$C_3H_8 + 5O_2 \rightarrow 3CO_2 + 4H_2O$$
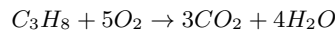
*Note:* The equation would also be Balanced if $x_4 = 4.K$, $K \in \mathbb{N}$.

But since chemist would prefer to use a Balanced equation whose coefficient are the smallest possible whole numbers. So, $x_4 = 4$ is as the optimal choice possible. The Equation $x^2 + y^2 = z^2$ : A Pythagorean triple is a set of 3 integers $x$, $y$, $z$ such that $x^2 + y^2 = z^2$, the triple is said to be primitive if $\gcd(x, y, z) = 1$.

**Result 3.1.** *All the solutions of the Pythagorean Equation $x^2 + y^2 = z^2$ satisfying the conditions*

$$\gcd(x, y, z) = 1, \ 2 \mid x, \ x > 0, \ y > 0, \ z > 0$$

*and given by the formulas $x = 2st$, $y = s^2 - t^2$, $z = s^2 + t^2$ for integers $s > t > 0$ such that $\gcd(s, t) = 1$ and $s \not\equiv t \pmod 2$.*

Fermat, who is regarded as the Father of Number theory mentioned that It is impossible to write a cube as a sum of 2 cubes, a fourth power as a sum of two fourth powers and so on. He was simply asserting that if, $n > 2$, then the Diophantine Equation (Non-Linear) $x^n + y^n = z^n$ has no solution in integers. Other than the trivial solution in which at least one of the variables is zero. The quotation just cited came to be known as *Fermat's Last Theorem*. Now we will observe some Real life application of Pythagorean triples.

**Ladder Problem :** A painter has to paint a wall. A ladder (35 foot), is leaning against, the side of wall positioned such that Base of the ladder is 21 feet from the Base of the Building. How far above the ground is the point where the ladder touches the wall. By Pythagorean triples, $x^2 = 35^2 - 21^2 \Rightarrow x = 28$ feet.
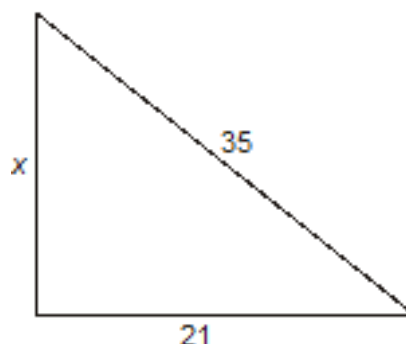
**Architecture (Interior) :**
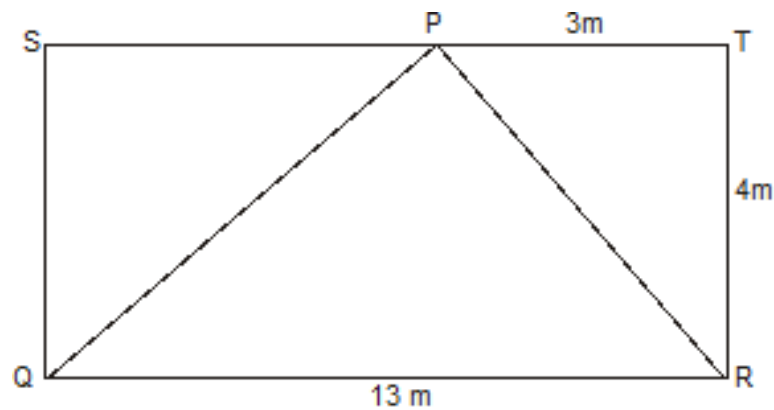


STQR is a Room with dimensions ($13m \times 4m$). An architecture has to find distance of a wall painting placed at P from corner Q and corner R of the room. Using Pythagorean Triples, $PR^2 = 3^2 + 4^2 \Rightarrow PR = 5m$. Also, $13^2 = QP^2 + PR^2$ ie: $QP = 12m$.

# 4. Conclusion

(1). Many Linear as well as Non-linear Diophantine equations in finite or infinite number of variables are solvable.

(2). When used in Real life, they can be really helpful and have wide applications.

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