International Journal of Mathematics And its Applications
Volume 5, Issue 2–B (2017), 223–229.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

An analysis of convergence of Bi-lateral Laplace Transform

Research Article

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Abstract: The method of Laplace transforms which is also known as operational methods; provide easy and efficient means of solving many problems arising in many fields of engineering and science. In this paper we will analysis the convergence and strip of convergence of two dimensional Unilateral and Bi-lateral integral transforms.

Keywords: Differential equation, Unilateral Laplace Transform, Bilateral Laplace Transform, Integration.

1. Introduction

The theory of Laplace Transform is being increasingly employed in mathematics, mechanics and the engineering sciences. Laplace transform have wide applications in the solution of differential and integral equations. The method of Laplace transform is also known as operational methods; provide easy and efficient means of solving many problems arising in many fields of engineering and science.

1.1. Definition of two dimensional Uni-lateral Laplace Transform

Suppose that the function $f(x,y)$ is a real or complex-valued function of two real variables x and y defined on the region $R(0 \le x < \infty,\ 0 \le y < \infty)$. Also suppose f is Lebesgue integrable over each finite rectangle $R_{AB}(0 \le x < A,\ 0 \le y < B)$. Consider the double integral

$$F(p,q;A,B) = \int_0^A \int_0^B \exp(-px-qy) f(x,y)dxdy \tag{1}$$

where and are the complex parameters determining a point (p,q) in the plane of two complex dimensions. The double integral

$$\int_0^\infty \int_0^\infty \exp(-px-qy) f(x,y)dxdy$$

is said to exist for atleast one point (p,q) if

1. The integral (1) is bounded at the point (p,q) with respect to A and B, that is $\lim_{A\to\infty,\ B\to\infty} F(p,q;A,B)$ exists. Let us denote this limit by; $F(p,q) = L2\{f(x,y);p,q\}$

$$F(p,q) = \int_0^A \int_0^B \exp(-px-qy) f(x,y)dxdy \tag{2}$$

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2. At some point (p, q), $\lim_{A\to\infty,\ B\to\infty} F(p, q; A, B)$ exists. Let us denote this limit by; $F(p, q) = L2\{f(x, y); p, q\}$

$$F(p, q) = \int_0^A \int_0^B \exp(-px - qy) f(x, y) dx dy \tag{2}$$

The integral (2) is called the two dimensional Laplace transform or Laplace integral of $f(x, y)$. If conditions (1) and (2) are satisfied simultaneously, then we shall say the integral (2) converges bounded in at least one point (p, q). If $L2\{f(x, y); p, q\}$ exists for all pairs of values (p, q) in a certain associated region of the complex p and q planes, then we shall call it the two dimensional Laplace transform of f(x,y). The correspondence between $f(x, y)$ and $F(p, q)$ may be interpreted as a transformation which transforms the function $f(x, y)$ into the function $F(p, q)$. Thus, if (2) converges bounded for every pair of values of (p, q) in a certain region of the complex p and q planes, then we call $f(x, y)$ the original function and $F(p, q)$ the corresponding image function. The one dimensional analogue of the definition is given by

$$F(p) = L\{f(x); p\} = \int_0^\infty exp(-px) f(x) dx. \tag{3}$$

1.2. Definition of two dimensional Bi-lateral Laplace Transform

Let $f(x, y)$ be a function of x and y defined for all pairs (x, y) in $-\infty < x < \infty$ and $-\infty < y < \infty$. Also let f be the Lebesgue integral in every finite rectangle $_{AB}(-A \le x \le A,\ -B \le y \le B)$. If for a pair of complex parameters $p = \alpha + i\beta$ and $q = \lambda + i\eta$, the limits of the following double integrals:

$$\lim_{\substack{A\to\infty \\ B\to\infty}} \int_0^A \int_0^B exp\left(-px - qy\right) f\left(x, y\right) dx dy \tag{4}$$

$$\lim_{\substack{A\to\infty \\ B\to\infty}} \int_0^A \int_{-B}^0 exp\left(-px - qy\right) f\left(x, y\right) dx dy \tag{5}$$

$$\lim_{\substack{A\to\infty \\ B\to\infty}} \int_{-A}^0 \int_0^B exp\left(-px - qy\right) f\left(x, y\right) dx dy \tag{6}$$

$$\lim_{\substack{A\to\infty \\ B\to\infty}} \int_{-A}^0 \int_{-B}^0 exp\left(-px - qy\right) f\left(x, y\right) dx dy \tag{7}$$

Exist, then the double integral the

$$LB2\left\{f\left(x, y\right); p, q\right\} = \int_{-\infty}^\infty \int_{-\infty}^\infty \exp\left(-px - qy\right) f\left(x, y\right) dx dy \tag{8}$$

exists and we call it the two dimensional Bi-Lateral Laplace integral for $f(x, y)$ for the pair of values p and q. If $L_2\{f(x, y); p, q\}$ exist for every pairs of values (p, q) in a certain associated region of the complex planes of p and q, we call it the two dimensional Bi-lateral Laplace Transform of $f(x, y)$.

1.3. Convergence

Whether the Laplace transform $x(s)$ of a signal $x(t)$ exists or not depends on the complex variable as well as the signal itself. All complex values of for which the integral in the definition converges form a region of convergence (ROC) in the s-plane. $x(s)$ exists if and only if the argument is inside ROC. As the imaginary part $w = Im(s)$ of the complex variable $s = \sigma + i\omega$ has no effect in terms of the convergence, the ROC is determine solely by the real part σ.

1.4. Types of convergence

Since we are mainly interested in obtaining results involving multidimensional Laplace Transforms, we give a brief discussion on the convergence of double Laplace transform as well as the strip of convergence. It is known that if $L\{f(x);\ p\} = \int_0^\infty exp(-px)\,f(x)\,dx$ is ordinarily convergent, the section or part-integral $\int_0^A exp(-px)\,f(x)\,dx$ is bounded for all $A \geq 0$. If

$$L2\{f(x,y);p,q\} = \int_0^\infty \int_0^\infty \exp(-px-qy)\,f(x,y)\,dxdy \tag{9}$$

Converges ordinarily, we cannot in general take the section

$$\int_0^A \int_0^B exp(-px-qy)\,f(x,y)\,dxdy \tag{10}$$

As bounded for all $A \geq 0$, $B \geq 0$. However, if we restrict ourselves to absolute convergence of the double Laplace integral (9), then the part-integral (10) is bounded for all $A \geq 0$, $B \geq 0$, and it can be shown that from the convergence of (7) in any point (p_0, q_0) follows the convergence in all other points with $Re(p) \geq Re(Po)$, $Re(q) \geq Re(qo)$. Hence we may write the Bilateral double integral

$$LB2\{f(x,y);p,q\} = \int_{-\infty}^\infty \int_{-\infty}^\infty \exp(-px-qy)\,f(x,y)\,dxdy$$

as the sum of four unilateral double integrals of the form

$$\begin{aligned}
I_1 &= \int_0^\infty \int_0^\infty \exp(-px-qy)\,f(x,y)\,dxdy\\
I_2 &= \int_0^\infty \int_{-\infty}^0 \exp(-px-qy)\,f(x,y)\,dxdy\\
I_3 &= \int_{-\infty}^0 \int_0^\infty \exp(-px-qy)\,f(x,y)\,dxdy\\
I_4 &= \int_{-\infty}^0 \int_{-\infty}^0 \exp(-px-qy)\,f(x,y)\,dxdy
\end{aligned}$$

We note that if in I_1, the function $\exp(-px-qy)f(x,y)$ is summable for a pair of values (p_0, q_0) of (p,q) over quadrant $Q(0 \leq x \leq \infty, 0 \leq y \leq \infty)$ that is if I_1 is absolutely integral in the Lebesgue sense, then

$$\int\int_Q \exp(-px-qy)f(x,y)dxdy \ \text{ and } \ \int\int_Q |\exp(-px-qy)f(x,y)dxdy|$$

exist for $Re(p) \geq Re(Po)$, $Re(q) \geq Re(qo)$ directly and also as limit in the sense of double integral. Further, the inner integrals in

$$\begin{aligned}
\int_0^A \int_0^B \exp(-px-qy)f(x,y)dxdy &= \int_0^A \exp(-px)dx \int_0^B \exp(-qy)f(x,y)dy\\
&= \int_0^B \exp(-qy)dy \int_0^A \exp(-px)f(x,y)dx \quad \text{and} \qquad (11)\\
\int_0^\infty \int_0^\infty \exp(-px-qy)f(x,y)dxdy &= \int_0^\infty \exp(-px)dx \int_0^\infty \exp(-qy)f(x,y)dy\\
&= \int_0^\infty \exp(-qy)dy \int_0^\infty \exp(-px)f(x,y)dx \qquad (12)
\end{aligned}$$

converges for almost all fixed x or y and converge absolutely, Equation (11) and (12) holds. If I_1 converges absolutely for a particular complex pair of values P_0 and q_0, then it converges absolutely even in a pair of complex half-planes,

$Re\,(p) \geq Re\,(Po)$, $Re\,(q) \geq Re\,(qo)$. The above conclusion does not hold for ordinary convergence of I_1. For our work we will assume absolute convergence for I_1, I_2, I_3 and I_4. Hence, in the case of real p and q and assuming a point (P_0, Q_0) absolute convergence, if we consider I_2, I_3 and I_4 as integrals similar to I_1 but with parameters $(p, -q)$, $(-p, q)$ and $(-p, -q)$ respectively, we obtain

$$\int_0^\infty \int_0^\infty \exp\left(-px - qy\right) f\left(x, y\right) dxdy \text{ converge for } p \geq p_0, q \geq q_0$$
$$\int_0^\infty \int_{-\infty}^0 \exp\left(-px - qy\right) f\left(x, y\right) dxdy \text{ converge for } p \geq p_0, q \leq q_0$$
$$\int_{-\infty}^0 \int_0^\infty \exp\left(-px - qy\right) f\left(x, y\right) dxdy \text{ converge for } p \leq p_0, q \geq q_0$$
$$\int_{-\infty}^0 \int_{-\infty}^0 \exp\left(-px - qy\right) f\left(x, y\right) dxdy \text{ converge for } p \leq p_0, q \leq q_0$$

1.5. Region of Convergence

Bilateral transform requirements for convergence are more difficult than for unilateral transforms. The region of convergence will be normally smaller. If f is a locally inferable function (or more generally a Borel measure locally of bounded variation), then the Laplace transform $F(s)$ of f converges provided that the limit $\lim_{R\to\infty} \int_0^R f(t)e^{-st}dt$ exists. The Laplace transform converges absolutely if the integral $\int_0^\infty \left|f(t)e^{-st}\right| dt$ exists (as a proper Lebesgue integral). The Laplace transform is usually understood as conditionally convergent, meaning that it converges in the former instead of the latter sense.

The set of values for which $F(s)$ converges absolutely is either of the form $Re(s) > a$ or else $Re(s) \geq a$, where a is an extended real constant, $-\infty \leq a \leq \infty$. (This follows from the dominated convergence theorem.) The constant a is known as the abscissa of absolute convergence, and depends on the growth behavior of $f(t)$. [10] **Analogously**, the two-sided transform converges absolutely in a strip of the form $a < Re(s) < b$, and possibly including the lines $Re(s) = a$ or $Re(s) = b$. [11] The subset of values of s for which the Laplace transform converges absolutely is called the region of absolute convergence or the domain of absolute convergence. In the two-sided case, it is sometimes called the strip of absolute convergence. The Laplace transform is analytic in the region of absolute convergence.

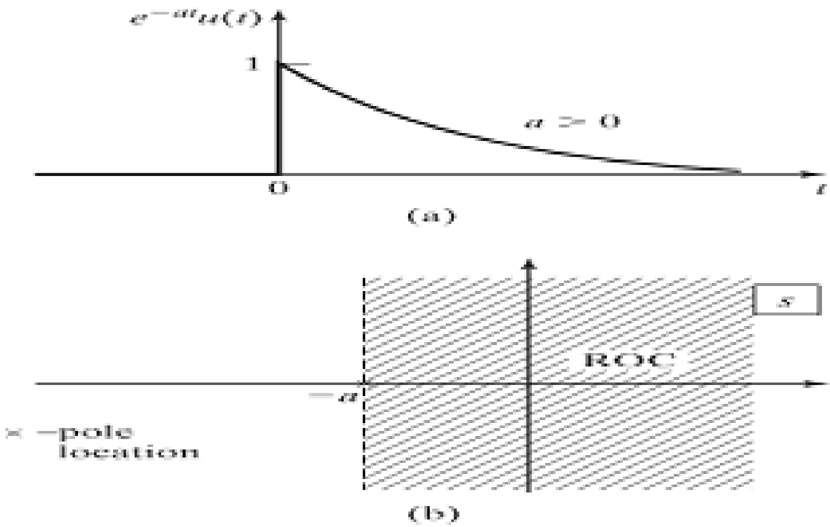
Similarly, the set of values for which $F(s)$ converges (conditionally or absolutely) is known as the region of conditional convergence, or simply the region of convergence (ROC). If the Laplace transform converges (conditionally) at $s = s_0$, then it automatically converges for all s with $Re(s) > Re(s_0)$. Therefore the region of convergence is a half-plane of the form $Re(s) > a$, possibly including some points of the boundary line $Re(s) = a$. In the region of convergence $Re(s) > Re(s_0)$, the Laplace transform of f can be expressed by integrating by parts as the integral

$$f(s) = (s - s_0)\int_0^\infty e^{-(S-S_0)t}\beta(t)dt,\ \beta(u) = \int_0^u e^{-S_0 t}\beta(t)dt$$

That is, in the region of convergence $F(s)$ can effectively be expressed as the absolutely convergent Laplace transform of some other function. In particular, it is analytic. There are several Paley–Wiener theorems concerning the relationship between the decay properties of f and the properties of the Laplace transform within the region of convergence. In engineering applications, a function corresponding to a linear time-invariant (LTI) system is *stable* if every bounded input produces a bounded output. This is equivalent to the absolute.

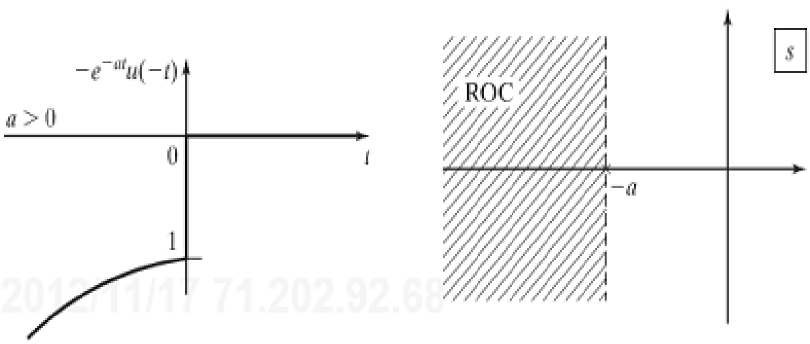
Example 1.1.

$$
\begin{aligned}
f(t) &= e^{-at}u(t);\\
F(s) &= \int_{0+}^{\infty} f(t)e^{-st}dt\\
&= \int_{0+}^{\infty} e^{-at}e^{-st}dt = \frac{1}{-s-a}e^{(-s-a)t}\bigg|_{0+}^{\infty};\;\; note: s=\delta+j\omega\\
&= \frac{-1}{s+a}e^{-(\delta+j\omega+a)t}\bigg|_{0+}^{\infty} = \frac{-1}{s+a}e^{-(\delta+a)t}e^{-j\omega t}\bigg|_{0+}^{\infty} \Rightarrow Re(s+a)>0\\
&= \frac{1}{s+a}; Re(s)>-a
\end{aligned}
$$



Example 1.2.

$$
\begin{aligned}
f(t) &= -e^{-at}u(-t);\\
F(s) &= \int_{-\infty}^{\infty} f(t)e^{-st}dt\\
&= \int_{-\infty}^{0^-} -e^{-at}e^{-st}dt = \frac{1}{s+a}e^{-(s+a)t}\bigg|_{-\infty}^{0^-} = \frac{1}{s+a};\;\; Re(s+a)<0\\
&= \frac{1}{s+a}; Re(s)<-a \rightarrow \frac{1}{s+a};\;\; Re(s)<-a
\end{aligned}
$$



Example 1.3. *RCO may not always exist!*

$$f(t) = e^{2t}u(t) + e^{-3t}u(-t)$$
$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt$$
$$e^{2t}u(t) \leftrightarrow \frac{1}{s-2};\ Re(s) > 2$$
$$e^{-3t}u(-t) \leftrightarrow -\frac{1}{s+3};\ Re(s) < -3$$
$$F(s) = \frac{1}{s-2} - \frac{1}{s+3};$$

There is no common ROC → Laplace Transform cannot be applied.

1.6. Strip of Convergence

In equation (3), it is tacitly assumed that the integral is convergent for $Re(p) = p_0$. However, it can be shown that it converges for other values of p. Geometrically, we can says if the Integral in (3) is convergent at some point P_0 of the complex p-plane, then it converges everywhere In the region on the right hand side of the straight line drawn through $p = p_0$ and parallel to the imaginary axis. If we let a be the greatest lower bound of values of Re(p) for which we have convergence, then the Integral converges in the region that is bounded at the left by the straight line $Re(p) = a$, parallel to the imaginary axis. Let us call a the abscissa of convergence of the unilateral Laplace integral. Now let us consider equation $F(p) = p\int_{-\infty}^{\infty} \exp(-px)f(x)dx$ that is

$$F(p) = p\int_{-\infty}^{0} \exp(-px)f(x)dx + p\int_{0}^{\infty} \exp(-px)f(x)dx \quad \text{or} \tag{13}$$
$$F(p) = p\int_{0}^{\infty} \exp(px)f(-x)dx + p\int_{0}^{\infty} \exp(-px)f(x)dx \tag{14}$$

2. Conclusion

The second integral is a unilateral Laplace integral with abscissa of convergence a, say, and thus converges for $Re(p) > a$. The first integral is also a unilateral Laplace integral of the parameter $-p$. If we let its abscissa of convergence be $-\beta$, then it converges for $Re(-p) > -\beta$ so that $Re(p) < \beta$, This is the region extending to infinity at the left of the vertical line $Re(p) = \beta$. Therefore, if $f(x)$ is such that $a < \beta$, then the region of convergence for the individual unilateral Laplace integrals overlap, resulting in a common strip of the complex plane where (13) converges. Thus, the bilateral Laplace integral converges in a region bounded by two vertical lines $a < Re(p) < \beta$. This region is called the strip of convergence for the bilateral Laplace integral.

3. Future Work

In the above 2.4 (Example 1.3) we see that there is no common ROC for $f(t) = e^{2t}u(t) + e^{-3t}u(-t)$, $F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt$ so we will try to find a region of convergence for the function using Laplace or other integral Methods.

Acknowledgement

I would like to express a very special thanks and appreciation to Er Premprakash Gupta Chairman of PGI Group for his patience, guidance. My sincere thanks to Dr Yavuraj Bhatnagar group Director of KCMT and PGIT group Bareilly. It has been great and enjoyable working with him. I would also like to thank Dean Mewar University Professor Ojha for serving on my committee.

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