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On Associated Lie Ring

Research Article

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Abstract: Let R be any ring and $L(R)$ be the associated Lie ring of R. In this paper, we have shown that if char $R \neq 2$ and $L(R)$ is 2-Engel, then its unit group is also 2-Engel.

Keywords: Associated lie ring, Engel group.

1. Introduction

Let R be any associative ring with identity. As we know that R can be treated as a Lie ring under the multiplication $[x, y] = xy - yx$, called associated Lie ring of R, and is denoted by $L(R)$. Also we know that a group G is called an Engel group if for each pair of $x, y \in G$, we have $(x, \underbrace{y, y, ..., y}_{ntimes}) = 1$ for some $n = n(x, y)$. If n can be chosen independently of x and y, then G is called n-Engel group. A Lie algebra L is called Engel Lie Algebra if for each ordered pair $[x, y]$, there is an integer $n(x, y)$ such that $[x, \underbrace{y, y, ..., y}_{ntimes}] = 0$. It is well known that every Lie nilpotent group is Engel Lie algebra. One of the basic results for Engel Lie algebra is Engel's theorem which states that Every finite dimensional Engel Lie algebra is nilpotent. So for finite dimensional Lie algebra Engel condition is equivalent to nilpotency. A lot of work has been done on Engel groups [5–7, 9]. Motose and Tominaga [4] and Khripta find out the conditions when $U(KG)$, the unit group of group algebra KG, is solvable or nilpotent. Fischer et al. [3] investigated when $U(KG)$ is solvable n-Engel group. Gupta and Levin proved that if R is Lie nilpotent of class n, then $U(R)$ is nilpotent of class atmost n. For the derived series Sharma and Srivastava proved that $\delta^2(U(R)) - 1 \subseteq \delta^{[2]}(L(R))R$. Working in this direction in this paper, we have shown that if $L(R)$ is 2-Engel, then its unit group is also $2-$Engel.

2. Preliminaries

In 1983 [1], Gupta and Levin proved the following result.

Theorem 2.1. *Let R be any arbitrary ring. Then for all $n \geq 1$,*

$$\gamma_n(U(R)) - 1 \subseteq \gamma_{[n]}(L(R))R.$$

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Thus if R is Lie nilpotent of class n, then U(R) is nilpotent of class atmost n. Following result was given by Sharma and Srivastava [2].

Theorem 2.2. *Let R be any arbitrary ring. Then*

$$\delta^2(U(R)) - 1 \subseteq \delta^{[2]}(L(R))R.$$

3. Main Result

Theorem 3.1. *Let R be a ring and $L(R)$ be the associated Lie ring of R. If $L(R)$ is 2-Engel, then its unit group is also 2-Engel.*

Proof. Let $L(R)$ be the associated Lie ring of R. Let $L(R)$ be 2-Engel. Then $[x, y, y] = 1$ forall $x, y \in R$. We have to prove that $U(R)$ is also $2-$Engel that means $(x, y, y) = 1$ for all $x, y \in U(R)$.

$$\begin{aligned} (x, y, y) - 1 &= (x, y)^{-1}y^{-1}[(x, y), y] \\ &= (x, y)^{-1}y^{-1}[(x, y) - 1, y] \\ &= (x, y)^{-1}y^{-1}[x^{-1}y^{-1}[x, y], y] \end{aligned} \tag{1}$$

Now consider $[x^{-1}y^{-1}[x, y], y]$.

$$\begin{aligned} [x^{-1}y^{-1}[x, y], y] &= x^{-1}y^{-1}[[x, y], y] + [x^{-1}y^{-1}, y][x, y] \\ &= x^{-1}y^{-1}[x, y, y] + [x^{-1}y^{-1}, y][x, y] \\ &= [x^{-1}y^{-1}, y][x, y] \end{aligned} \tag{2}$$

Consider

$$\begin{aligned} 0 &= [[x^{-1}y^{-1}x, y], y] \\ &= [x^{-1}y^{-1}[x, y] + [x^{-1}y^{-1}, y]x, y] \\ &= [x^{-1}y^{-1}[x, y], y] + [[x^{-1}y^{-1}, y]x, y] \\ &= x^{-1}y^{-1}[x, y, y] + [x^{-1}y^{-1}, y][x, y] + [x^{-1}y^{-1}, y][x, y] + [[x^{-1}y^{-1}, y], y]x \\ &= x^{-1}y^{-1}[x, y, y] + 2[x^{-1}y^{-1}, y][x, y] + [x^{-1}y^{-1}, y, y]x \\ &= 2[x^{-1}y^{-1}, y][x, y] \end{aligned} \tag{3}$$

Now if Char$R \neq 2$, then from equation (3), $[x^{-1}y^{-1}, y][x, y] = 0$. Thus from equation (1), (2) and (3), we get $(x, y, y) - 1 = 0$ i.e. $(x, y, y) = 1$. Thus if characteristic of $R \neq 2$, then $U(R)$ is 2-Engel. □

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