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On Total Product Cordial Labeling of Some Crown Graphs

Research Article

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Abstract: A total product cordial labeling of a graph G is a function $f : V \to \{0, 1\}$. For each xy, assign the label $f(x)f(y)$, f is called total product cordial labeling of G if it satisfies the condition that $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| \leq 1$ where $v_f(i)$ and $e_f(i)$ denote the set of vertices and edges which are labeled with $i = 0, 1$, respectively. A graph with a total product cordial labeling defined on it is called total product cordial. In this paper, we determined the total product cordial labeling of $P_m \circ C_n$, $P_m \circ P_n$, $C_m \circ P_n$, $P_m \circ F_n$, $P_m \circ W_n$ and $P_m \circ K_n$.

MSC: 05C78

Keywords: Graph Labeling, Total Product Cordial Labeling, Crown graphs.

1. Introduction

All graphs considered are finite, simple and undirected. The graph has vertex set $V = V(G)$ and edge set $E = E(G)$ and we let $e = |E|$ and $v = |V|$. A general reference for graph theoretic notions is in [5]. A graph labeling is an assignment of integers to the vertices or edges or both subject to certain condition(s). If the domain of the mapping is the set of vertices (or edges) then the labeling is called vertex labeling (or an edge labeling). Graph labeling is an active area of research in graph theory which has rigorous applications in coding theory, communication networks, optimal circuits layouts and graph decomposition problems. For a dynamic survey of various graph labeling problems along with an extensive bibliography we refer to Gallian [2]. The classic paper of β-valuations by Rosa in 1967 [3] laid the foundations for several graph labeling methods. For a simple graph of order $|V|$ and size $|E|$, Ibrahim Cahit [1] introduced a weaker version of β-valuation or graceful labeling in 1987 and called it cordial labeling. The following notions of product cordial labeling was introduced in 2004 [3]. For a simple graph $G = (V, E)$ and a function $f : V \to \{0, 1\}$, assign the label $f(x)f(y)$ for each edge xy. This function f is called a product cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ where $v_f(i)$ and $e_f(i)$ denote the number of vertices and edges labeled with $i = 0, 1$. Motivated by this definition, M. Sundaram, R. Ponraj and S. Somasundaram introduce a new type of graph labeling known as total product cordial labeling and investigate the total product cordial behavior of some standard graphs. A function f is called a total product cordial labeling of G if it satisfies the condition that $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| \leq 1$. A graph with a total product cordial labeling defined on it is called *total product cordial*. For any two graphs G and H of order p_1 and p_2, respectively, the graph obtained by taking one copy

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of G of order p_1 and p_1 copies of H and then connecting the ith vertex of G to every vertex of the ith copy of H (ith means first, second, third and so on) is called crown, denoted by $G \circ H$. The order and size of the crown $G \circ H$ is $p_1 + p_1 p_2$ and $q_1 + p_1 q_2 + p_1 p_2$, respectively, where q_1 and q_2 are the size of graphs G and H, respectively.

1.1. Preliminaries

This section presents some results of product cordial and total product cordial labeling on some graphs.

Theorem 1.1 ([6]). *The crown graph $P_n \circ C_m (n \geq 2, m \geq 3)$, is not product cordial except if n is even.*

Theorem 1.2 ([6]). *The crown graph $P_n \circ P_m (n, m \geq 2)$, is product cordial except if both n and m are odd.*

Theorem 1.3 ([6]). *The crown graph $P_n \circ K_m (n \geq 2, m \geq 4)$, is not product cordial except if n is even.*

Theorem 1.4 ([4]). *A product cordial graph G is total product cordial if either it is of even order or of odd order and even size.*

Corollary 1.5 ([4]). *Trees are total product cordial.*

Theorem 1.6 ([4]). *C_n is total product cordial if $n \neq 4$.*

Remark 1.1 ([4]). *The cycle C_4 is not total product cordial.*

Theorem 1.7 ([4]). *The complete graph K_n $(n \geq 4)$ is total product cordial if one of $2n^2 + 2n - 3$ or $2n^2 + 2n + 1$ or $2n^2 + 2n + 5$ is a square of an odd integer.*

Corollary 1.8 ([4]). *The crown $C_n \circ K_1$ is total product cordial.*

2. Main Section

Theorem 2.1. The crown graph $P_m \circ C_n$ is total product cordial graph for all $m \geq 1, \quad n \geq 3$.

Proof. Let $V(P_m) = \{v_1, v_2, \ldots, v_m\}$ be the vertex set of P_m and $V(C_n^i) = \{u_1^i, u_2^i, \ldots, u_n^i\}$ be the vertex set of the ith copy of $C_n, n \geq 3$. The order and size of the crown graph $P_m \circ C_n$ are $mn + m$ and $2mn + m - 1$, respectively. Consider the following cases:

Case 1: m is even and n is even or m is even and n is odd.

Observe that the order of the crown graph $P_m \circ C_n$ is $m + mn$. Since m is even, it follows that the order of the crown graph $P_m \circ C_n$ is even. By Theorem 1.1, the crown graph $P_m \circ C_n$ is product cordial, then by Theorem 1.4, the crown graph $P_m \circ C_n$ is total product cordial.

Case 2: m is odd and n is even.

Define the function $f : V(P_m \circ C_n) \to \{0, 1\}$ by

$$f(v_i) = \begin{cases} 0, & 1 \leq i \leq \frac{m-1}{2} \\ 1, & \text{otherwise.} \end{cases}$$

$$f(u_j^i) = \begin{cases} 0, & 1 \leq i \leq \frac{m-1}{2}, 1 \leq j \leq n \quad \text{or} \\ & i = \frac{m+1}{2}, 1 \leq j \leq \frac{n}{2} \\ 1, & \text{otherwise.} \end{cases}$$

In view of the above labeling, we have $v_f(0) = \frac{m-1+mn}{2}$ and $v_f(1) = \frac{m+1+mn}{2}$. On the other hand, the edges of $P_m \circ C_n$ with labels zero are the following:

$$\begin{aligned} f(v_iv_{i+1}) &= 0,\ 1 \leq i \leq \frac{m-1}{2} \\ f(v_iu_j^i) &= 0,\ 1 \leq i \leq \frac{m-1}{2},\ 1 \leq j \leq n \quad \text{or} \\ i &= \frac{m+1}{2},\ 1 \leq j \leq \frac{n}{2} \\ f(u_j^iu_{j+1}^i) &= 0,\ 1 \leq i \leq \frac{m-1}{2},\ 1 \leq j \leq n-1 \quad \text{or} \\ i &= \frac{m+1}{2},\ 1 \leq j \leq \frac{n}{2} \\ f(u_1^iu_n^i) &= 0,\ 1 \leq i \leq \frac{m+1}{2}. \end{aligned}$$

In view of the above labeling, we have,

$$e_f(0) = \frac{2mn+m+1}{2} \tag{1}$$

and

$$e_f(1) = \frac{2mn+m-3}{2} \tag{2}$$

Hence, $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| = \left|\frac{3mn+2m-3mn-2m+2}{2}\right| = 1$. Thus, the crown graph $P_m \circ C_n$ is total product cordial if m is odd and n is even.

Case 3: m is odd and n is odd

Subcase 1: If $m = 1$ and $n \geq 3$, then the crown graph $P_1 \circ C_n \cong W_n$ which is total product cordial for all $n \geq 3$.

Subcase 2: If $m \geq 3$ and $n = 3$, define the function $f : V(P_m \circ C_3) \to \{0, 1\}$ by

$$f(v_i) = \begin{cases} 0, & 1 \leq i \leq \frac{m+1}{2} \\ 1, & \text{otherwise} \end{cases}$$

and for $j = 1, 2, 3$,

$$f(u_j^i) = \begin{cases} 0, & 1 \leq i \leq \frac{m-1}{2} \\ 1, & \text{otherwise.} \end{cases}$$

In view of the above labeling, we have

$$v_f(0) = 2m - 1 \tag{3}$$

and

$$v_f(1) = 2m + 1. \tag{4}$$

On the other hand, the edges of $P_m \circ C_3$ with labels zero are the following:

$$\begin{aligned} f(v_iv_{i+1}) &= 0,\ 1 \leq i \leq \frac{m+1}{2} \\ f(u_j^iu_{j+1}^i) &= 0,\ 1 \leq i \leq \frac{m-1}{2},\ j = 1, 2 \\ f(u_1^iu_3^i) &= 0,\ 1 \leq i \leq \frac{m-1}{2} \\ f(v_iu_j^i) &= 0,\ 1 \leq i \leq \frac{m+1}{2},\ j = 1, 2, 3. \end{aligned}$$

In view of the above labeling, we have

$$e_f(0) = \frac{7m+1}{2} \tag{5}$$

and

$$e_f(1) = \frac{7m-3}{2}. \tag{6}$$

Hence, $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| = \left|\frac{11m-1}{2} - \frac{11m-1}{2}\right| = 0$. Thus, the crown graph $P_m \circ C_3$ is total product cordial when m is odd, $m \geq 3$.

Subcase 3: If $m \geq 3$ and $n \geq 5$, define the function $f : V(P_m \circ C_n) \to \{0,1\}$ by

$$f(v_i) = \begin{cases} 0, & \frac{m+3}{2} \leq i \leq m \\ 1, & \text{otherwise} \end{cases}$$

$$f(u_j^i) = \begin{cases} 0, & \frac{m+3}{2} \leq i \leq m,\ 1 \leq j \leq n \quad \text{or} \\ & i = \frac{m+1}{2},\ j = 2,3,4,\ldots,\frac{n-1}{2},\frac{n+3}{2} \\ 1, & \text{otherwise.} \end{cases}$$

In view of the above labeling, we have

$$v_f(0) = \frac{m-2+mn}{2} \tag{7}$$

and

$$v_f(1) = \frac{m+2+mn}{2}. \tag{8}$$

On the other hand, the edges of $P_m \circ C_n$ with labels zero are the following:

$$\begin{aligned} f(v_iv_{i+1}) &= 0,\ \frac{m+1}{2} \leq i \leq m-1 \\ f(u_j^iu_{j+1}^i) &= 0,\ \frac{m+3}{2} \leq i \leq m\,,\ 1 \leq j \leq n-1 \quad \text{or} \\ &\quad i = \frac{m+1}{2}\,,\ 1 \leq j \leq \frac{n+3}{2} \\ f(u_1^iu_n^i) &= 0,\ \frac{m+3}{2} \leq i \leq m \\ f(v_iu_j^i) &= 0,\ \frac{m+3}{2} \leq i \leq m\,,\ 1 \leq j \leq n \quad \text{or} \\ &\quad i = \frac{m+1}{2}\,,\ j = 2,3,4,\ldots,\frac{n-1}{2},\frac{n+3}{2}. \end{aligned}$$

In view of the above labeling, we have

$$e_f(0) = \frac{m+1+2mn}{2} \tag{9}$$

and

$$e_f(1) = \frac{m-3+2mn}{2}. \tag{10}$$

Hence, $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| = \left|\frac{3mn+2m-1}{2} - \frac{3mn+2m-1}{2}\right| = 0$. Thus the crown graph $P_m \circ C_n$ is total product cordial if $m \geq 3$ and $n \geq 5$.

Considering all the cases above, then we can say that the crown graph $P_m \circ C_n$ is total product cordial for $m \geq 1$ and $n \geq 3$. □

Theorem 2.2. The crown graph $P_m \circ P_n$ is total product cordial graph for all $m, n \geq 1$.

Proof. Let $V(P_m) = \{v_1, v_2, \ldots, v_m\}$ be the vertex set of P_m and $V(P_n^i) = \{u_1^i, u_2^i, \ldots, u_n^i\}$ be the vertex set of the ith copy of P_n. The order and size of the crown graph $P_m \circ P_n$ are $m + mn$ and $2mn - 1$, respectively. To prove the theorem, we will consider the following cases:

Case 1: m is even and n is even or m is even and n is odd.

Observe that the order of the crown graph $P_m \circ P_n$ is $m + mn$. Since m is even, it follows that the order of the crown graph $P_m \circ P_n$ is even. By Theorem 1.2, the crown graph $P_m \circ P_n$ is product cordial, then by Theorem 1.4, the crown graph $P_m \circ P_n$ is total product cordial.

To prove the remaining cases, observe that the crown graph $P_m \circ P_n$ is obtained from the crown graph $P_m \circ C_n$ by deleting the edges $u_1^i u_n^i$. Since only edges $u_1^i u_n^i$ are deleted, $V(P_m \circ P_n) = V(P_m \circ C_n)$. Hence, we will use the functions defined on Theorem 2.1 to prove the following cases.

Case 2: m is odd and n is even.

We will label the vertices and edges of $P_m \circ C_n$ using the function defined on Theorem 2.1, Case 2 by omitting the edges $u_1^i u_n^i$. Accordingly, since there are no vertices deleted, it follows that the number of vertices labeled with 0 and 1 would be the same. That is, $v_f(0) = \frac{m-1+mn}{2}$ and $v_f(1) = \frac{m+1+mn}{2}$.

Now, counting all the deleted edges $u_1^i u_n^i$ labeled with 0 and 1, we will obtain $\frac{m+1}{2}$ and $\frac{m-1}{2}$. Subtracting this deleted edges from the computed number of edges having labels 0 and 1 in Equations 1 and 2, it follows that $e_f(0) = mn$ and $e_f(1) = mn - 1$.

Hence, $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| = \left|\frac{3mn+m-1}{2} - \frac{3mn+m-1}{2}\right| = 0$. Thus, the crown graph $P_m \circ P_n$ is total product cordial if m is odd and n is even.

Case 3: m is odd and n is odd.

Subcase 1: If $m = 1$ and $n = 1$, then the crown graph $P_1 \circ P_1 \cong P_2$ which is total product cordial by Corollary 1.5.

Subcase 2: If $m = 1$ and $n \geq 3$, then the crown graph $P_1 \circ P_n \cong F_n$ which is total product cordial.

Subcase 3: If $m \geq 3$ and $n = 3$, we will label the vertices and edges of $P_m \circ P_3$ using the function defined on Theorem 2.1, Case 3, Subcase 2. Accordingly, since there are no vertices deleted, then the number of vertices labeled with 0 and 1 would be the same. That is, $v_f(0) = 2m - 1$ and $v_f(1) = 2m + 1$.

Now, counting all the deleted edges $u_1^i u_3^i$ labeled with 0 and 1, we will obtain $\frac{m-1}{2}$ and $\frac{m+1}{2}$, respectively. Subtracting this deleted edges from the computed number of edges having labels 0 and 1 in Equation 5 and 6, it follows that $e_f(0) = 3m+1$ and $e_f(1) = 3m - 2$.

Hence, $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| = |5m - 5m + 1| = 1$. Thus, the crown graph $P_m \circ P_3$ is total total product cordial for $m \geq 3$.

Subcase 4: If $m \geq 3$ and $n \geq 5$, we will label the vertices and edges of $P_m \circ P_n$ using the function defined on Theorem 2.1, Case 4, Subcase 3 by omitting the edges $u_1^i u_n^i$. Accordingly, since there are no vertices deleted, then the number of vertices labeled with 0 and 1 would be the same. That is $v_f(0) = \frac{m-2+mn}{2}$ and $v_f(1) = \frac{m+2+mn}{2}$.

Now counting all the deleted edges $u_1^i u_n^i$ labeled with 0 and 1, we will obtain $\frac{m-1}{2}$ and $\frac{m+1}{2}$, respectively. Subtracting this deleted edges from the computed number of edges having labels 0 and 1 in Equation 9 and 10, it follows that $e_f(0) = mn+1$ and $e_f(1) = mn - 2$.

Hence, $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| = \left|\frac{3mn+m}{2} - \frac{3mn+m-2}{2}\right| = 1$. Thus the crown graph $P_m \circ P_n$ is total product cordial if $m \geq 3$ and $n \geq 5$.

Considering all the cases above, then we can say that the crown graph $P_m \circ P_n$ is total product cordial for $m, n \geq 1$. □

Theorem 2.3. The crown graph $C_m \circ P_n$ is total product cordial graph for all $m \geq 3$, $n \geq 1$.

Proof. Let $V(C_m)=\{v_1,v_2,v_3,\dots,v_m\}$ be the vertex set of C_m, $m\geq 3$ and $V(P_n^i)=\{u_1^i,u_2^i,\dots,u_n^i\}$ be the vertex set of the ith copy of P_n, $n\geq 1$. The order and size of the crown graph $C_m\circ P_n$ are $mn+m$ and $2mn$, respectively. Consider the following cases:

Case 1: m is even and $n\geq 1$.

Subcase 1: If m is even and $n=1$, then the crown graph $C_m\circ P_1\cong C_m\circ K_1$ which is total product cordial by Corollary 1.8.

Subcase 2: If m is even and $n\geq 2$, then define the function $f:V(C_m\circ P_n)\to\{0,1\}$ by

$$f(v_i)=\begin{cases}0, & \frac{m}{2}+1\leq i\leq m\\ 1, & \text{otherwise}\end{cases}$$

$$f(u_j^i)=\begin{cases}0, & \frac{m}{2}+2\leq i\leq m,\ 1\leq j\leq n \quad \text{or}\\ & i=\frac{m}{2}+1,\ 2\leq j\leq n\\ 1, & \text{otherwise.}\end{cases}$$

In view of the above labeling, we have $v_f(0)=\frac{mn+m-2}{2}$ and $v_f(1)=\frac{mn+m+2}{2}$. On the other hand, the edges of $C_m\circ P_n$ with labels zero are the following:

$$\begin{aligned}
f(v_iv_{i+1})&=0, && \frac{m}{2}\leq i\leq m-1\\
f(v_mv_1)&=0\\
f(u_j^iu_{j+1}^i)&=0, && \frac{m}{2}+1\leq i\leq m\,,\ 1\leq j\leq n-1\\
f(v_iu_j^i)&=0, && \frac{m}{2}+1\leq i\leq m\,,\ 1\leq j\leq n.
\end{aligned}$$

In view of the above labeling, we have $e_f(0)=mn+1$ and $e_f(1)=mn-1$.

Hence, $|v_f(0)+e_f(0)-v_f(1)-e_f(1)|=\left|\frac{3mn+m}{2}-\frac{3mn+m}{2}\right|=0$. Thus, the crown graph $C_m\circ P_n$ is total product cordial if m is even and $n\geq 2$.

Case 2: m is odd and n is odd.

Define the function $f:V(C_m\circ P_n)$ by

$$f(v_i)=\begin{cases}0, & \frac{m+3}{2}\leq i\leq m\\ 1, & \text{otherwise}\end{cases}$$

$$f(u_j^i)=\begin{cases}0, & \frac{m+3}{2}\leq i\leq m,\ 1\leq j\leq n \quad \text{or}\\ & i=\frac{m+1}{2},\ j=\frac{n+1}{2},\frac{n+5}{2},\frac{n+5}{2}+1,\frac{n+5}{2}+2,\dots,n\\ 1, & \text{otherwise.}\end{cases}$$

In view of the above labeling, we have $v_f(0)=\frac{mn+m-2}{2}$ and $v_f(1)=\frac{mn+m+2}{2}$. On the other hand, the edges of $C_m\circ P_n$

with labels zero are the following:

$$
\begin{aligned}
f(v_iv_{i+1}) &= 0, && \frac{m+1}{2} \leq i \leq m-1\\
f(v_mv_1) &= 0 &&\\
f(u_j^iu_{j+1}^i) &= 0, && \frac{m+3}{2} \leq i \leq m\,,\ 1 \leq j \leq n-1 \quad \text{or}\\
& && i = \frac{m+1}{2}\,,\ \frac{n-1}{2} \leq j \leq n-1\\
f(v_iu_j^i) &= 0, && \frac{m+3}{2} \leq i \leq m\,,\ 1 \leq j \leq n \quad \text{or}\\
& && i = \frac{m+1}{2}\,,\ j = \frac{n+1}{2}, \frac{n+5}{2}, \frac{n+5}{2}+1, \frac{n+5}{2}+2, \ldots, n.
\end{aligned}
$$

In view of the above labeling, we have $e_f(0) = mn+1$ and $e_f(1) = mn-1$.

Hence, $|v_f(0)+e_f(0)-v_f(1)-e_f(1)| = \left|\frac{3mn+m}{2} - \frac{3mn+m}{2}\right| = 0$. Thus the crown graph $C_m \circ P_n$ is total product cordial if m and n is odd.

Case 3: m is odd and n is even.

Define the function $f : V(C_m \circ P_n)$ by

$$
f(v_i) = \begin{cases} 0, & \frac{m+3}{2} \leq i \leq m\\ 1, & \text{otherwise} \end{cases}
$$

$$
f(u_j^i) = \begin{cases} 0, & \frac{m+3}{2} \leq i \leq m,\ 1 \leq j \leq n \quad \text{or}\\ & i = \frac{m+1}{2},\ \frac{n}{2}+1 \leq j \leq n\\ 1, & \text{otherwise.} \end{cases}
$$

In view of the above labeling, we have $v_f(0) = \frac{mn+m-1}{2}$ and $v_f(1) = \frac{mn+m+1}{2}$. On the other hand, the edges of $C_m \circ P_n$ with labels zero are the following:

$$
\begin{aligned}
f(v_iv_{i+1}) &= 0, && \frac{m+1}{2} \leq i \leq m-1\\
f(v_mv_1) &= 0 &&\\
f(u_j^iu_{j+1}^i) &= 0, && \frac{m+3}{2} \leq i \leq m\,,\ 1 \leq j \leq n-1 \quad \text{or}\\
& && i = \frac{m+1}{2}\,,\ \frac{n}{2} \leq j \leq n-1\\
f(v_iu_j^i) &= 0, && \frac{m+3}{2} \leq i \leq m\,,\ 1 \leq j \leq n \quad \text{or}\\
& && i = \frac{m+1}{2}\,,\ \frac{n}{2}+1 \leq j \leq n.
\end{aligned}
$$

In view of the above labeling, we have $e_f(0) = mn+1$ and $e_f(1) = mn-1$.

Hence, $|v_f(0)+e_f(0)-v_f(1)-e_f(1)| = \left|\frac{3mn+m+1}{2} - \frac{3mn+m-1}{2}\right| = 1$. Thus, the crown graph $C_m \circ P_n$ is total product cordial if m is odd and n is even.

Considering the cases above, we can say that the crown graph $C_m \circ P_n$ is total product cordial for all $m \geq 3$,$n \geq 1$. □

Theorem 2.4. The crown graph $P_m \circ F_n$ is total product cordial graph for all $m \geq 2$, $n \geq 1$.

Proof. Let $V(P_m) = \{v_1, v_2, v_3, \ldots, v_m\}$ be the vertex set of P_m and $V(F_n^i) = \{u_1^i, u_2^i, \ldots, u_{n+1}^i\}$ be the vertex set of the ith copy of F_n. The order and size of the crown graph $P_m \circ F_n$ are $2m+mn$ and $3mn+m-1$, respectively. Consider the following cases:

Case 1: m is even and $n \geq 1$.

Subcase 1: If m is even and $n = 1$, then the crown graph $P_m \circ F_1 \cong P_m \circ P_2$ which is total product cordial by Theorem 2.2.

Subcase 2: If m is even and $n \geq 2$, define the function $f : V(P_m \circ F_n) \to \{0, 1\}$ by

$$f(v_i) = \begin{cases} 0, & \frac{m}{2} + 1 \leq i \leq m \\ 1, & \text{otherwise} \end{cases}$$

$$f(u_j^i) = \begin{cases} 0, & \frac{m}{2} + 1 \leq i \leq m,\ 1 \leq j \leq n+1 \\ 1, & \text{otherwise.} \end{cases}$$

In view of the above labeling, we have $v_f(0) = \frac{mn+2m}{2}$ and $v_f(1) = \frac{mn+2m}{2}$. On the other hand, the edges of $P_m \circ F_n$ with labels zero are the following:

$$\begin{aligned} f(v_i v_{i+1}) &= 0, & &\frac{m}{2} \leq i \leq m-1 \\ f(u_j^i u_{j+1}^i) &= 0, & &\frac{m}{2} + 1 \leq i \leq m\,,\ 1 \leq j \leq n \\ f(u_j^i u_{n+1}^i) &= 0, & &\frac{m}{2} + 1 \leq i \leq m\,,\ 1 \leq j \leq n-1 \\ f(v_i u_j^i) &= 0, & &\frac{m}{2} + 1 \leq i \leq m\,,\ 1 \leq j \leq n+1. \end{aligned}$$

In view of the above labeling, we have $e_f(0) = \frac{3mn+m}{2}$ and $e_f(1) = \frac{3mn+m-2}{2}$.
Hence, $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| = \left|\frac{4mn+3m}{2} - \frac{4mn+3m-2}{2}\right| = 1$. Thus, the crown graph $P_m \circ F_n$ is total product cordial if m is even, $n \geq 1$.

Case 2: m is odd, $(m \geq 3)$ and n is odd, $(n \geq 1)$.

Subcase 1: If m is odd, $(m \geq 3)$ and $n = 1$, the the crown graph $P_m \circ F_1 \cong P_m \circ P_2$, which is total product cordial by Theorem 2.2.

Subcase 2: If m is odd, $(m \geq 3)$ and $n = 3$, define the function $f : V(P_m \circ F_3)$ by

$$f(v_i) = \begin{cases} 0, & \frac{m+3}{2} \leq i \leq m \\ 1, & \text{otherwise} \end{cases}$$

$$f(u_j^i) = \begin{cases} 0, & \frac{m+5}{2} \leq i \leq m,\ 1 \leq j \leq 4 \quad \text{or} \\ & i = \frac{m+3}{2},\ j = 2, 3, 4 \quad \text{or} \\ & i = \frac{m+1}{2},\ j = 3, 4 \\ 1, & \text{otherwise.} \end{cases}$$

In view of the labeling above, we have $v_f(0) = \frac{5m-3}{2}$ and $v_f(1) = \frac{5m+3}{2}$. On the other hand, the edges of $P_m \circ F_3$ with labels zero are the following:

$$\begin{aligned}
f(v_iv_{i+1}) &= 0, && \frac{m+1}{2} \le i \le m-1\\
f(u_j^iu_{j+1}^i) &= 0, && \frac{m+5}{2} \le i \le m\,,\ j=1,2,3 \quad \text{or}\\
& && i=\frac{m+1}{2}\,,\ j=2,3 \quad \text{or}\\
& && i=\frac{m+3}{2}\,,\ j=1,2,3\\
f(u_j^iu_4^i) &= 0, && \frac{m+1}{2} \le i \le m\,,\ j=1,2\\
f(v_iu_j^i) &= 0, && \frac{m+3}{2} \le i \le m\,,\ j=1,2,3,4 \quad \text{or}\\
& && i=\frac{m+1}{2}\ j=3,4.
\end{aligned}$$

In view of the above labeling, we have $e_f(0)=5m+1$ and $e_f(1)=5m-2$.

Hence, $|v_f(0)+e_f(0)-v_f(1)-e_f(1)| = \left|\frac{15m-1}{2} - \frac{15m-1}{2}\right| = 0$. Thus, the crown graph $P_m \circ F_n$ is total product cordial if m is odd, $m \ge 3$ and $n=3$.

Subcase 3: If m is odd, $m \ge 3$ and $n \ge 5$, define the function $f : V(P_m \circ F_n) \to \{0,1\}$ by

$$f(v_i) = \begin{cases} 0, & \frac{m+3}{2} \le i \le m \\ 1, & \text{otherwise} \end{cases}$$

$$f(u_j^i) = \begin{cases} 0, & \frac{m+5}{2} \le i \le m,\ 1 \le j \le n+1 \quad \text{or} \\ & i=\frac{m+1}{2},\ \frac{n+1}{2} \le j \le n \quad \text{or} \\ & i=\frac{m+3}{2},\ 1 \le j \le n \\ 1, & \text{otherwise.} \end{cases}$$

In view of the above labeling, we have $v_f(0)=\frac{mn+2m-3}{2}$ and $v_f(1)=\frac{mn+2m+3}{2}$. On the other hand, the edges of $P_m \circ F_n$ with labels zero are the following:

$$\begin{aligned}
f(v_iv_{i+1}) &= 0, && \frac{m+1}{2} \le i \le m-1\\
f(u_j^iu_{j+1}^i) &= 0, && \frac{m+3}{2} \le i \le m\,,\ 1 \le j \le n \quad \text{or}\\
& && i=\frac{m+1}{2}\,,\ \frac{n-1}{2} \le j \le n\\
f(u_j^iu_{n+1}^i) &= 0, && \frac{m+3}{2} \le i \le m\,,\ 1 \le j \le n-1 \quad \text{or}\\
& && i=\frac{m+1}{2}\,,\ \frac{n+1}{2} \le j \le n-1\\
f(v_iu_j^i) &= 0, && \frac{m+3}{2} \le i \le m\,,\ 1 \le j \le n+1 \quad \text{or}\\
& && i=\frac{m+1}{2}\,,\ \frac{n+1}{2} \le j \le n.
\end{aligned}$$

In view of the above labeling, we have $e_f(0)=\frac{3mn+m+2}{2}$ and $e_f(1)=\frac{3mn+m-4}{2}$.

Hence, $|v_f(0)+e_f(0)-v_f(1)-e_f(1)| = \left|\frac{4mn+3m-1}{2} - \frac{4mn+3m-1}{2}\right| = 0$. Thus, the crown graph $P_m \circ F_n$ is total product cordial if m is odd, $m \ge 3$ and n is odd, $n \ge 5$.

Case 3: m is odd, $(m \ge 3)$ and n is even.

Subcase 1: If m is odd, $(m \ge 3)$ and $n=2$, then the crown graph $P_m \circ F_2 \cong P_m \circ C_3$ which is total product cordial by Theorem 2.1.

Subcase 2: If m is odd, $(m \geq 3)$ and $n \geq 4$, define the function $f : V(P_m \circ F_n) \to \{0, 1\}$ by

$$f(v_i) = \begin{cases} 0, & \frac{m+1}{2} \leq i \leq m-1 \\ 1, & \text{otherwise} \end{cases}$$

$$f(u_j^i) = \begin{cases} 0, & \frac{m+1}{2} \leq i \leq m-1,\ 1 \leq j \leq n+1 \quad \text{or} \\ & i = m,\ 1 \leq j \leq \frac{n}{2} \\ 1, & \text{otherwise.} \end{cases}$$

In view of the above labeling, we have $v_f(0) = \frac{mn+2m-2}{2}$ and $v_f(1) = \frac{mn+2m+2}{2}$. On the other hand, the edges of $P_m \circ F_n$ with labels zero are the following:

$$f(v_i v_{i+1}) = 0, \qquad \frac{m-1}{2} \leq i \leq m-1$$

$$\begin{aligned} f(u_j^i u_{j+1}^i) = 0, &\qquad \frac{m+1}{2} \leq i \leq m-1\,,\ 1 \leq j \leq n \quad \text{or} \\ &\qquad i = m\,,\ 1 \leq j \leq \frac{n}{2} \\ f(u_j^i u_{n+1}^i) = 0, &\qquad \frac{m+1}{2} \leq i \leq m-1\,,\ 1 \leq j \leq n-1 \quad \text{or} \\ &\qquad i = m\,,\ 1 \leq j \leq \frac{n}{2} \\ f(v_i u_j^i) = 0, &\qquad \frac{m+1}{2} \leq i \leq m-1\,,\ 1 \leq j \leq n+1 \quad \text{or} \\ &\qquad i = m\,,\ 1 \leq j \leq \frac{n}{2}. \end{aligned}$$

In view of the above labeling, we have $e_f(0) = \frac{3mn+m+1}{2}$ and $e_f(1) = \frac{3mn+m-3}{2}$.
Hence, $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| = \left|\frac{4mn+3m-1}{2} - \frac{4mn+3m-1}{2}\right| = 0$. Thus, the crown graph $P_m \circ F_n$ is total product cordial if m is odd, $m \geq 3$ and n is even, $n \geq 4$.
Considering the cases above, we can say that the crown graph $P_m \circ F_n$ is total product cordial if $m \geq 2$, $n \geq 1$. ☐

Theorem 2.5. The crown graph $P_m \circ W_n$ is total product cordial graph for all $m \geq 2$, $n \geq 3$.

Proof. Let $V(P_m) = \{v_1, v_2, v_3, \ldots, v_m\}$ be the vertex set of P_m and $V(W_n^i) = \{u_1^i, u_2^i, \ldots, u_{n+1}^i\}$ be the vertex set of the ith copy of W_n, $n \geq 3$. The order and size of the crown graph $P_m \circ W_n$ are $2m + mn$ and $3mn + 2m - 1$, respectively. Consider the following cases:

Case 1: m is even and $n \geq 3$. Define the function $f : V(P_m \circ W_n) \to \{0, 1\}$ by

$$f(v_i) = \begin{cases} 0, & \frac{m}{2} + 1 \leq i \leq m \\ 1, & \text{otherwise} \end{cases}$$

$$f(u_j^i) = \begin{cases} 0, & \frac{m}{2} + 1 \leq i \leq m,\ 1 \leq j \leq n+1 \\ 1, & \text{otherwise.} \end{cases}$$

In view of the above labeling, we have $v_f(0) = \frac{mn+2m}{2}$ and $v_f(1) = \frac{mn+2m}{2}$. On the other hand, the edge labels of $P_m \circ W_n$ are the following:

$$
\begin{aligned}
f(v_i v_{i+1}) &= 0, && \frac{m}{2} \le i \le m-1 \\
f(u_j^i u_{j+1}^i) &= 0, && \frac{m}{2}+1 \le i \le m\,,\ 1 \le j \le n \\
f(u_j^i u_{n+1}^i) &= 0, && \frac{m}{2}+1 \le i \le m\,,\ 1 \le j \le n-1 \\
f(u_1^i u_n^i) &= 0, && \frac{m}{2}+1 \le i \le m \\
f(v_i u_j^i) &= 0, && \frac{m}{2}+1 \le i \le m\,,\ 1 \le j \le n+1.
\end{aligned}
$$

In view of the above labeling, we have $e_f(0) = \frac{3mn+2m}{2}$ and $e_f(1) = \frac{3mn+2m-2}{2}$.
Hence, $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| = |2mn + 2m - (2mn + 2m - 1)| = 1$. Thus, the crown graph $P_m \circ W_n$ is total product cordial if m is even, $n \ge 3$.

Case 2: m is odd and n is odd, $(m, n \ge 3)$

Subcase 1: If m is odd, $(m \ge 3)$ and $n = 3$, then we will label the vertices based on the function defined on Theorem 2.4, Case 2, Subcase 2. Accordingly, the number of vertices labeled with 0 and 1 are, $\frac{5m-3}{2}$ and $\frac{5m+3}{2}$, respectively. For the number of edges labeled with 0 and 1, it follows that $e_f(0) = \frac{11m+3}{2}$ and $e_f(1) = \frac{11m-5}{2}$.
Hence, $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| = |8m - (8m - 1)| = 1$. Thus, the crown graph $P_m \circ W_n$ is total product cordial if m is odd, $m \ge 3$, $n = 3$.

Subcase 2: If m is odd, $(m \ge 3)$, n is odd, $(n \ge 5)$, then we will label the vertices of $P_m \circ W_n$ using the function defined on Theorem 2.4, Case 2, Subcase 3. Accordingly, the number of vertices labeled with 0 and 1 is, $\frac{mn+2m-3}{2}$ and $\frac{mn+2m+3}{2}$, respectively. For the number of edges labeled with 0 and 1, it follows that $e_f(0) = \frac{3mn+2m+3}{2}$ and $e_f(1) = \frac{3mn+2m-5}{2}$.
Hence, $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| = |2mn + 2m - (2mn + 2m - 1)| = 1$. Thus, the crown graph $P_m \circ W_n$ is total product cordial if m is odd, $m \ge 3$ and n is odd, $n \ge 5$.

Case 3: If m is odd, $(m \ge 3)$ and n is even, $(n \ge 4)$. We will label the vertices of $P_m \circ W_n$ using the function defined on Theorem 2.4, Case 3, Subcase 2. Accordingly, the number of vertices labeled with 0 and 1 is, $\frac{mn+2m-2}{2}$ and $\frac{mn+2m+2}{2}$, repectively. For the edges labeled with 0 and 1, it follows that $e_f(0) = \frac{3mn+2m+2}{2}$ and $e_f(1) = \frac{3mn+2m-4}{2}$.
Hence, $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| = |2mn + 2m - (2mn + 2m - 1)| = 1$. Thus, the crown graph $P_m \circ W_n$ is total product cordial if m is odd, $m \ge 3$ and n is even, $n \ge 4$.

Considering the cases above, we can say that the crown graph $P_m \circ W_n$ is total product cordial for all $m \ge 2$ and $n \ge 3$. □

Theorem 2.6. The crown graph $P_m \circ K_n$ is total product cordial graph if m is even and $n \ge 3$ or m is odd and if one of $2(n+1)^2 + 2(n+1) - 3$ or $2(n+1)^2 + 2(n+1) + 1$ or $2(n+1)^2 + 2(n+1) + 5$ is a square of an odd integer.

Proof.

Case 1: m is even and n is odd or m is even and n is even

The order and size of the crown graph $P_m \circ K_n$ are $m + mn$ and $\frac{mn^2+mn+2m-2}{2}$, respectively. Now, if m is even and n is odd, $(n \ge 3)$ or m is even and n is even, $(n \ge 4)$, the order of the crown graph $P_m \circ K_n$ is even. Since $P_m \circ K_n$ is a product cordial graph by Theorem 1.3, then by Theorem 1.4, the crown graph $P_m \circ K_n$ is total product cordial.

Case 2: $m = 1$ and if one of $2(n+1)^2 + 2(n+1) - 3$ or $2(n+1)^2 + 2(n+1) + 1$ or $2(n+1)^2 + 2(n+1) + 5$ is a square of an odd integer.

The crown graph $P_1 \circ K_n \cong K_{n+1}$ which is total product cordial by Theorem 1.7.

Case 3: $m \geq 3$ and $2(n+1)^2 + 2(n+1) - 3 = (2r+1)^2$ for some $r \in \mathbb{Z}^+$. Define $f : V(P_m \circ K_n) \to \{0,1\}$ by:

$$f(v_i) = \begin{cases} 0, & \frac{m+3}{2} \leq i \leq m \\ 1, & \text{otherwise} \end{cases}$$

$$f(u_j^i) = \begin{cases} 0, & \frac{m+3}{2} \leq i \leq m,\ 1 \leq j \leq n \quad \text{or} \\ & i = \frac{m+1}{2},\ r \leq j \leq n \\ 1, & \text{otherwise.} \end{cases}$$

In view of the above labeling, we have $v_f(0) = \frac{m+mn+1+n-2r}{2}$ and $v_f(1) = \frac{m+mn-1-n+2r}{2}$. On the other hand, the edge labels for $f(v_iv_{i+1})$ is defined as:

$$f(v_iv_{i+1}) = 0, \qquad \frac{m+1}{2} \leq i \leq m-1.$$

For the edges on every ith copy of K_n where $i = \frac{m+1}{2}$, we have,

$$e_f(0) = \frac{n(n+1)}{2} - \frac{r(r-1)}{2}.$$

For the edges on every ith copy of K_n where $\frac{m+3}{2} \leq i \leq m$, we have,

$$e_f(0) = \left(\frac{m}{2} - \frac{1}{2}\right)\left(\frac{n(n+1)}{2}\right).$$

Hence, we have $e_f(0) = \frac{mn^2+mn+n^2+n-2r^2+2r+2m-2}{4}$ and $e_f(1) = \frac{mn^2+mn+2m-n^2}{4} + \frac{-n+2r^2-2r-2}{4}$. Now $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| = \left|\frac{6n+2n^2+4-2n^2-4n-2-2n-2+4}{4}\right| = 1$. Thus, the crown graph $P_m \circ K_n$ is total product cordial if m is odd, $m \geq 3$ and if $2(n+1)^2 + 2(n+1) - 3$ is a square of an odd integer.

Case 4: $m \geq 3$ and $2(n+1)^2 + 2(n+1) + 1 = (2r+1)^2$ for some $r \in \mathbb{Z}^+$.

Similarly, we have $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| = \left|\frac{6n+2n^2+4-2n^2-4n-2-2n-2}{4}\right| = 0$. Thus, the crown graph $P_m \circ K_n$ is total product cordial if m is odd, $m \geq 3$ and if $2(n+1)^2 + 2(n+1) + 1$ is a square of an odd integer.

Case 5: $m \geq 3$ and $2(n+1)^2 + 2(n+1) + 5 = (2r+1)^2$ for some $r \in \mathbb{Z}^+$.

Similarly, we have $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| = \left|\frac{6n+2n^2+4-2n^2-4n-2-2n-2-4}{4}\right| = |-1| = 1$. Thus, the crown graph $P_m \circ K_n$ is total product cordial if m is odd, $m \geq 3$ and if $2(n+1)^2 + 2(n+1) + 5$ is a square of an odd integer.

Considering the cases above, we can say that the crown graph $P_m \circ K_n$ is total product cordial if m is even and $n \geq 3$ or m is odd and if one of $2(n+1)^2 + 2(n+1) - 3$ or $2(n+1)^2 + 2(n+1) + 1$ or $2(n+1)^2 + 2(n+1) + 5$ is a square of an odd integer. $\square$

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