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Multidimensional Fractional Integral Operators Involving General Class of Polynomial and Aleph (ℵ) Function

Research Article

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Abstract: In the Present Paper, we acquire a pair of Multidimensional Fractional Integral Operators whose kernels involving the product of Multivariable Polynomial and Aleph function. Firstly, in our operator of study we obtain images of two useful functions. Next, we authenticate two theorems given by Multidimensional generalized Stieltjes transform of fractional integral operators and conversely, lastly, for these operators, we present results concerning Mellin transform, Mellin Convolutions and inversion formulae for theses operators.

Keywords: Aleph function, Fractional integral operator, General class of Multivariable Polynomials, Mellin transform, Stieltjes transform.

1. Introduction

Various authors especially by Erdelyi [1, 3], Kalla [10], Saxena [8], Srivastava and Buschmann [6] etc., have been outlined and studied the Fractional integral operators. These operators play an important role in the theory of integral equations and problems concerning applying Mathematical Physics. The Multivariable polynomial $S_V^{U_1,\ldots,U_k}(x_1,...,x_k)$ adduced by Srivastava and Garg [7] is defined in the following way:

$$S_V^{U_1,\ldots,U_k}(x_1,...,x_k) = \sum_{R_1,\ldots,R_k=0}^{\sum_{j=1}^{k} U_j R_j \le V} (-V)_{\sum_{j=1}^{k} U_j R_j} \, A\left(V, R_1,...,R_k\right) \frac{x_1^{R_1}}{R_1!} \cdots \frac{x_k^{R_k}}{R_k!} \tag{1}$$

Where $V = 0, 1, 2, \ldots$ and $U_1, \ldots, U_k$ arbitrary positive integers and the coefficients are $A\left(V_1, R_1, R_2, ..., R_k\right)$ are arbitrary constants (real or complex).

1.1. Aleph (ℵ) Function

Sudland [2] Introduced the Aleph (ℵ) function, however the notation and complete definition is acquainted here in the following way in terms and the Mellin- Barnes type integrals

$$\begin{aligned} \aleph[z] &= \aleph^{M,N}_{P_i,Q_i;\tau_i;r}\left[z \Big|^{(a_j,A_j)_{1,N},\,[\tau_i(a_{ji},A_{ji})]_{N+1,P_i}}_{(b_j,B_j)_{1,M},\,[\tau_i(b_{ji},B_{ji})]_{M+1,Q_i}} \right] \\ &= \frac{1}{2\pi\omega} \int_L \Omega^{M,N}_{P_i,Q_i;\tau_i;r}(s)\, z^{-s} ds \end{aligned} \tag{2}$$

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For all $z \neq 0$ where $\omega = \sqrt{(-1)}$ and

$$\Omega^{M,N}_{P_i,Q_i,\tau_i;r}(s) = \frac{\prod\limits_{j=1}^{M} \Gamma(b_j + B_j s) \prod\limits_{j=1}^{N} \Gamma(1 - a_j - A_j s)}{\sum\limits_{i=1}^{r} \tau_i \prod\limits_{j=M+1}^{Q_i} \Gamma(1 - b_{ji} - B_{ji} s) \prod\limits_{j=N+1}^{P_i} \Gamma(a_{ji} + A_{ji} s)} \tag{3}$$

The integration path $L = L_{i\gamma\infty}$, $\gamma \in R$ extends from $\gamma - i\infty$ to $\gamma + i\infty$, and is such that the poles, assumed to be simple of $\Gamma(1-a_j-A_j s)$, $j = 1, \ldots, n$ do not coincide with the pole of $\Gamma(b_j + B_j s)$, $j = i, \ldots, M$ the parameter P_i, Q_i are non-negative integers satisfying: $0 \leq N \leq P_i$, $0 \leq M \leq Q_i$, $\tau_i > 0$, for $i = 1, \ldots, r$. $A_j, B_j, A_{ji}, B_{ji} > 0$ and $a_j, b_j, a_{ji}, b_{ji} \in C$ The empty product in (3) is interpreted as unity. The existence conditions for defining integral (2) are as following:

$$\phi_l > 0\,,\ |arg\,(z)| < \frac{\pi}{2}\phi_l\ ,\ l = 1, 2, ...r \tag{4}$$

$$\phi_l \geq 0\,,\ |arg\,(z)| < \frac{\pi}{2}\phi_l\ ,\ R\,(\xi_l) + 1 < 0 \tag{5}$$

Where

$$\phi_l = \sum_{j=1}^{N} A_j + \sum_{j=1}^{M} B_j - \tau_l \left(\sum_{j=N+1}^{P_l} A_{jl} + \sum_{j=M+1}^{Q_l} B_{jl} \right) \tag{6}$$

$$\xi_l = \sum_{j=1}^{M} b_j - \sum_{j=1}^{N} a_j + \tau_l \left(\sum_{j=N+1}^{Q_l} b_{jl} - \sum_{j=M+1}^{P_l} a_{jl} \right) + \frac{1}{2}\,(P_l - Q_l)\ ,\ l = 1, 2, ..., r \tag{7}$$

Detailed introduction of Aleph (ℵ) function is given in [2] and [4].

2. Multidimensional Fractional Integral Operators

The following fractional integral operators are being studied in the recent Paper.

$$\begin{aligned} I_x\,[f\,(t_1, ..., t_s)] &= I^{\rho,\sigma;e,f;\eta,\lambda}_{x:U,V;Z}\,[f\,(t_1, ..., t_s)\,; x_1, ..., x_s] \\ &= \left(\prod_{j=1}^{s} x_j{}^{-\rho_j - \sigma_j} \right) \int_0^{x_1} \cdots \int_0^{x_s} \left[t_j^{\rho_j} (x_j - t_j)^{\sigma_j - 1} \right] S_V^{U_1,...,U_s} \left[E_1 \left(\frac{t_1}{x_1} \right)^{e_1} \left(1 - \frac{t_1}{x_1} \right)^{f_1}, ..., E_s \left(\frac{t_s}{x_s} \right)^{e_s} \left(1 - \frac{t_s}{x_s} \right)^{f_s} \right] \\ &\times \aleph^{M,N}_{P_i,Q_i,\tau_i,r} \left[z \prod_{j=1}^{s} \left(\frac{t_j}{x_j} \right)^{\eta_j} \left(1 - \frac{t_j}{x_j} \right)^{\lambda_j} \middle| \begin{matrix} (a_j, A_j)_{1,N}, [\tau_i (a_{ji}, A_{ji})]_{N+1,P_i;r} \\ (b_j, B_j)_{1,M}, [\tau_i (b_{ji}, B_{ji})]_{M+1,Q_i;r} \end{matrix} \right] \times f\,(t_1, ..., t_s)\, dt_1, ..., dt_s \end{aligned} \tag{8}$$

Where

(i). $\min Re\ \left(e_j, f_j, \eta_j, \lambda_j\right) \geq 0$, $(j = 1, ..., s)$ And all Parameters $e_j, f_j, \eta_j, \lambda_j$ are not zero simultaneously,

(ii). $\min\limits_{1 \leq k \leq M}\ Re\ \left[1 + \rho_j + U_j + \eta_j \frac{b_k}{B_k}\right] > 0,\ \min\limits_{1 \leq k \leq M}\ Re\ \left[\sigma_j + \lambda_j \frac{b_k}{B_k}\right] > 0$

$$\begin{aligned} J_x\,[f\,(t_1, ..., t_s)] &= J^{\rho,\sigma;e,f;\eta,\lambda}_{x:U,V;Z}\,[f\,(t_1, ..., t_s)\,; x_1, ..., x_s] \\ &= \left(\prod_{j=1}^{s} x_j{}^{\rho_j} \right) \int_{x_1}^{\infty} \cdots \int_{x_s}^{\infty} \left[t_j^{-\rho_j - \sigma_j} (t_j - x_j)^{\sigma_j - 1} \right] S_V^{U_1,...,U_s} \left[E_1 \left(\frac{x_1}{t_1} \right)^{e_1} \left(1 - \frac{x_1}{t_1} \right)^{f_1}, ..., E_s \left(\frac{x_s}{t_s} \right)^{e_s} \left(1 - \frac{x_s}{t_s} \right)^{f_s} \right] \\ &\times \aleph^{M,N}_{P_i,Q_i,\tau_i,r} \left[z \prod_{j=1}^{s} \left(\frac{x_j}{t_j} \right)^{\eta_j} \left(1 - \frac{x_j}{t_j} \right)^{\lambda_j} \middle| \begin{matrix} (a_j, A_j)_{1,N}, [\tau_i (a_{ji}, A_{ji})]_{N+1,P_i;r} \\ (b_j, B_j)_{1,M}, [\tau_i (b_{ji}, B_{ji})]_{M+1,Q_i;r} \end{matrix} \right] \times f\,(t_1, ..., t_s)\, dt_1, ..., dt_s \end{aligned} \tag{9}$$

Where

(i). $\min Re\ \left(e_j, f_j, \eta_j, \lambda_j\right) \geq 0$, $(j = 1, ..., s)$ And all Parameters $e_j, f_j, \eta_j, \lambda_j$ are not zero simultaneously,

(ii). $Re\ (W_j) = 0$, $\min\limits_{1\leq k\leq M} Re\left[1 + \rho_j + V_j + \eta_j \frac{b_k}{B_k}\right] > 0$, $\min\limits_{1\leq k\leq M} Re\left[\sigma_j + \lambda_j \frac{b_k}{B_k}\right] > 0$ Or $Re\ (W_j) > 0$, $\min\limits_{1\leq k\leq M} Re\left[\sigma_j + \lambda_j \frac{b_k}{B_k}\right] > 0$.

Throughout the paper we pretend that

$$f\left(t_1, ..., t_s\right) = \begin{cases} O\prod_{j=1}^{s}\left(\left|t_j\right|^{U_j}\right) & \max\left\{\left|t_j\right|\right\}\to 0 \\ O\prod_{j=1}^{s}\left(\left|t_j\right|^{-V_j} e^{-W_j\left|t_j\right|}\right) & \min\left\{\left|t_j\right|\right\}\to\infty \end{cases} \quad j = i, ..., s \tag{10}$$

Such a class of function will be denoted symbolically as $f\left(t_1, ..., t_s\right) \in A$. We also pretend that $\int ... \int_{\Lambda_s} \left|f\left(t_1, ..., t_s\right)\right| dt_1, ..., dt_s < \infty$ for every bounded s-dimensional region Λ_s excluding the origin.

3. Some Useful Images

Now in our operators of study we find the images of some useful functions.

(a)

$$\begin{aligned} I_x\left[\prod_{j=1}^{s} t_j^{\gamma_j}\left(h_j + t_j\right)^{-\delta_j}\right] &= \sum_{n=0}^{\infty}\frac{1}{\Gamma(n+1)}\prod_{j=1}^{s}\left(-\frac{x_j}{h_j}\right)^n\left(\frac{x_j^{\gamma_j}}{h_j^{\delta_j}}\right) \\ &\times \sum_{R_1,...,R_s=0}^{\sum_{j=1}^{s}U_jR_j\leq V}(-V)_{\sum_{j=1}^{s}U_jR_j} A\left(V, R_1, ..., R_s\right)\frac{E_1^{R_1}}{R_1!}...\frac{E_s^{R_s}}{R_s!} \\ &\times \left(1 + \frac{x_j}{h_j}\right)^{\sigma_j + f_jR_j - \delta_j} \aleph_{P_i+3s, Q_i+2s, \tau_i, r}^{M, N+3s}\left[z\prod_{j=1}^{s}\left(1 + \frac{x_j}{h_j}\right)^{\lambda_j}\Bigg|_{B^*}^{A^*}\right] \end{aligned} \tag{11}$$

$$\begin{aligned} A^* &= \left(a_j, A_j\right)_{1,N}, \left(-\rho_j - \gamma_j - e_jR_j; \eta_j\right)_{1,s}, \left(1 - \sigma_j - f_jR_j - n; \lambda_j\right)_{1,s} \\ &\left(-\sigma_j - \rho_j - \gamma_j - \left(e_j + f_j\right)R_j + \delta_j - n; \lambda_j + \eta_j\right)_{1,s}, \left[\tau_i\left(a_{ji}, A_{ji}\right)\right]_{N+1,P_i;r} \\ B^* &= \left(b_j, B_j\right)_{1,M}, \left(-\sigma_j - \rho_j - \gamma_j - \left(e_j + f_j\right)R_j + \delta_j; \lambda_j + \eta_j\right)_{1,s} \\ &\left(-\sigma_j - \rho_j - \gamma_j - \left(e_j + f_j\right)R_j - n; \lambda_j + \eta_j\right)_{1,s}, \left[\tau_i(b_{ji}, B_{ji})\right]_{M+1,Q_i;r} \end{aligned}$$

Provided that $\min Re\left(e_j, f_j, \eta_j, \lambda_j\right) \geq 0$, $(j = 1, ..., s)$ and all Parameters $e_j, f_j, \eta_j, \lambda_j$ are not zero simultaneously,

$$\min_{1\leq k\leq M} Re\left[1 + \rho_j + \gamma_j + \eta_j\frac{b_k}{B_k}\right] > 0 \text{ and } \min_{1\leq k\leq M} Re\left[\sigma_j + \lambda_j\frac{b_k}{B_k}\right] > 0$$

(b)

$$\begin{aligned} J_x\left[\prod_{j=1}^{s} t_j^{\gamma_j}\left(h_j + t_j\right)^{-\delta_j}\right] &= \sum_{n=0}^{\infty}\frac{1}{\Gamma(n+1)}\prod_{j=1}^{s}\left(-\frac{h_j}{x_j}\right)^n\left(x_j^{\gamma_j - \delta_j}\right) \\ &\times \sum_{R_1,...,R_s=0}^{\sum_{j=1}^{s}U_jR_j\leq V}(-V)_{\sum_{j=1}^{s}U_jR_j} A\left(V, R_1, ..., R_s\right)\frac{E_1^{R_1}}{R_1!}...\frac{E_s^{R_s}}{R_s!} \\ &\times \left(1 + \frac{h_j}{x_j}\right)^{\sigma_j + f_jR_j - \delta_j} \aleph_{P_i+3s, Q_i+2s, \tau_i, r}^{M, N+3s}\left[z\prod_{i=1}^{s}\left(1 + \frac{h_j}{x_j}\right)^{\lambda_j}\Bigg|_{D^*}^{C^*}\right] \end{aligned} \tag{12}$$

$$C^{*}=(a_j,A_j)_{1,N},(1-\rho_j+\gamma_j-e_jR_j-\delta_j;\eta_j)_{1,s},(1-\sigma_j-f_jR_j-n;\lambda_j)_{1,s}$$
$$(1-\sigma_j-\rho_j+\gamma_j-(e_j+f_j)R_j-n;\lambda_j+\eta_j)_{1,s},[\tau_i(a_{ji},A_{ji})]_{N+1,P_i;r}$$
$$D^{*}=(b_j,B_j)_{1,M},(1-\sigma_j-\rho_j+\gamma_j-(e_j+f_j)R_j;\lambda_j+\eta_j)_{1,s}$$
$$(1-\sigma_j-\rho_j+\gamma_j-(e_j+f_j)R_j-n;\lambda_j+\eta_j)_{1,s},[\tau_i(b_{ji},B_{ji})]_{M+1,Q_i;r}$$

Provided that $\min Re\left(e_j,f_j,\eta_j,\lambda_j\right)\geq 0$, $(j=1,...,s)$ and all Parameters e_j,f_j,η_j,λ_j are not zero simultaneously,

$$\min_{1\leq k\leq M} Re\left[\rho_j-\gamma_j+\delta_j+\eta_j\frac{b_k}{B_k}\right]>0 \text{ and } \min_{1\leq k\leq M} Re\left[\sigma_j+\lambda_j\frac{b_k}{B_k}\right]>0$$

Proof. To prove (11) with help of equation (8), we express the I-operator involved in its left hand side in the integral form. Next, by using (1), we express generalized multivariable Polynomial $S_V^{U_1,\dots,U_s}(x_1,...,x_s)$ which is occurring there in the series. Then, by changing the order of the series and t_j-integral and expressing the Aleph function in term of mellin Barnes type contour integrals with help of (2). Now changing the order of $\xi$and t_j-integrals$(j=1,2,...,s)$, (which is permissible under the given conditions) finally, solving the t_j-integrals with help of known result [5]. We get

$$I_x\left[\prod_{j=1}^{s}t_j^{\gamma_j}(h_j+t_j)^{-\delta_j}\right]=\sum_{R_1,...,R_s}^{\sum_{j=1}^{s}U_jR_j\leq V}(-V)_{\sum_{j=1}^{s}U_jR_j}A(V,R_1,...,R_s)\frac{E_1^{R_1}}{R_1!}...\frac{E_s^{R_s}}{R_s!}$$
$$\prod_{j=1}^{s}\left(x_j^{\gamma_j-\delta_j}\right)\left(\frac{h_j}{x_j}\right)^{-\delta_j}\frac{1}{2\pi i}\int_L\varphi(\xi)\ z^{-\xi}\ \beta(\sigma_j+f_jR_j-\lambda_j\xi,\rho_j+\gamma_j+e_jR_j-\eta_j\xi+1)$$
$${}_2F_1\left[{}^{\delta_j,\rho_j+\gamma_j+e_jR_j-\eta_j\xi+1}_{\sigma_j+\rho_j+\gamma_j+(e_j+f_j)R_j-(\lambda_j+\eta_j)\xi+1};-\frac{x_j}{h_j}\right]d\xi \tag{13}$$

Where

$$\left|\arg\left(\frac{x_j}{h_j}\right)\right|<\pi, Re\left(\sigma_j+f_jR_j+\lambda_j\xi\right)>0, Re\left(\rho_j+\gamma_j+e_jR_j+\eta_j\xi+1\right)>0$$

Now reinterpreting the result which is obtained in terms of the Aleph function, after a little simplification we can easily reach at the wanting result. Similar we can Proof to Result (b) as (a). □

4. Multidimensional Generalized Stieltjes Transform and Fractional Integral Operators

The Multidimensional Generalized Stieltjes Transform of the function $\phi(t_1,...,t_s)$ is express as

$$S_{w_1,...,w_s}(\phi)\ (g_1,...,g_s)\ =\int_0^\infty...\int_0^\infty\phi\ (t_1,...,t_s)\prod_{j=1}^{s}\left\{(t_j+g_j)^{-w_j}\right\}dt_1...dt_s \tag{14}$$

Provided that the integral exists. The following theorem gives the Multidimensional Generalized Stieltjes transform of the generalized fractional operators given by (8) and (9).

Theorem 4.1. *Let $\phi(t_1,...,t_s)\in A$, then*

(1).

$$S_{w_1,...,w_s}(I_t\phi)(h_1,...,h_s)=\int_0^\infty...\int_0^\infty\phi(x_1,...,x_s)\,\psi_1(x_1,...,x_s;h_1,...,h_s)\,dx_1...dx_s \tag{15}$$

Where

$$\psi_1(x_1,...,x_s;h_1,...,h_s) = J_x\left[\prod_{j=1}^{s}(h_j+t_j)^{-w_j}\right]$$
$$= \sum_{R_1,...,R_s=0}^{\sum_{j=1}^{s}U_jR_j\le V}(-V)_{\sum_{j=1}^{s}U_jR_j}A(V,R_1,...,R_s)\frac{E_1^{R_1}}{R_1!}...\frac{E_s^{R_s}}{R_s!}\sum_{n=0}^{\infty}\frac{1}{\Gamma(n+1)}\prod_{j=1}^{s}\left(-\frac{h_j}{x_j}\right)^n$$
$$\times\left(x_j^{-w_j}\right)\left(1+\frac{h_j}{x_j}\right)^{\sigma_j+f_jR_j-w_j}\aleph^{M,N+3s}_{P_i+3s,Q_i+2s,\tau_i,r}\left[z\prod_{i=1}^{s}\left(1+\frac{h_j}{x_j}\right)^{\lambda_j}\Bigg|^{E^*}_{F^*}\right] \tag{16}$$

$$E^* = (a_j,A_j)_{1,N},(1-\rho_j-e_jR_j-w_j;\eta_j)_{1,s},(1-\sigma_j-f_jR_j-n;\lambda_j)_{1,s}$$
$$(1-\sigma_j-\rho_j-(e_j+f_j)R_j-n;\lambda_j+\eta_j)_{1,s},[\tau_i(a_{ji},A_{ji})]_{N+1,P_i;r}$$
$$F^* = (b_j,B_j)_{1,M},(1-\sigma_j-\rho_j-(e_j+f_j)R_j;\lambda_j+\eta_j)_{1,s}$$
$$(1-\sigma_j-\rho_j-(e_j+f_j)R_j-n;\lambda_j+\eta_j)_{1,s},[\tau_i(b_{ji},B_{ji})]_{M+1,Q_i;r}$$

Provided that $\min Re(e_j,f_j,\eta_j,\lambda_j)\ \ge\ 0$ $,(j=1,...,s)$, *all Parameters* e_j,f_j,η_j,λ_j *are not zero simultaneously, min* $Re[\rho_j+\eta_j+w_j]>0$ *and* $\min\ Re[\sigma_j+\lambda_j]>0$.

(2).

$$S_{w_1,...,w_s}(J_t\phi)(h_1,...,h_s) = \int_0^\infty...\int_0^\infty\phi(x_1,...,x_s)\psi_2(x_1,...,x_s;h_1,...,h_s)dx_1...dx_s \tag{17}$$

Where

$$\psi_2(x_1,...,x_s;h_1,...,h_s) = I_x\left[\prod_{j=1}^{s}(h_j+t_j)^{-w_j}\right]$$
$$= \sum_{R_1,...,R_s=0}^{\sum_{j=1}^{s}U_jR_j\le V}(-V)_{\sum_{j=1}^{s}U_jR_j}A(V,R_1,...,R_s)\frac{E_1^{R_1}}{R_1!}...\frac{E_s^{R_s}}{R_s!}\sum_{n=0}^{\infty}\frac{1}{\Gamma(n+1)}\prod_{j=1}^{s}\left(-\frac{x_j}{h_j}\right)^n$$
$$\times\left(\frac{1}{h_j^{w_j}}\right)\left(1+\frac{x_j}{h_j}\right)^{\sigma_j+f_jR_j-w_j}\aleph^{M,N+3s}_{P_i+3s,Q_i+2s,\tau_i,r}\left[z\prod_{j=1}^{s}\left(1+\frac{x_j}{h_j}\right)^{\lambda_j}\Bigg|^{G^*}_{H^*}\right] \tag{18}$$

$$G^* = (a_j,A_j)_{1,N},(-\rho_j-e_jR_j;\eta_j)_{1,s},(1-\sigma_j-f_jR_j-n;\lambda_j)_{1,s}$$
$$(-\sigma_j-\rho_j-(e_j+f_j)R_j+w_j-n;\lambda_j+\eta_j)_{1,s},[\tau_i(a_{ji},A_{ji})]_{N+1,P_i;r}$$
$$H^* = (b_j,B_j)_{1,M},(-\sigma_j-\rho_j-(e_j+f_j)R_j+w_j;\lambda_j+\eta_j)_{1,s}$$
$$(-\sigma_j-\rho_j-(e_j+f_j)R_j-n;\lambda_j+\eta_j)_{1,s},[\tau_i(b_{ji},B_{ji})]_{M+1,Q_i;r}$$

Provided that $\min Re(e_j,f_j,\eta_j,\lambda_j)\ \ge\ 0$ $,(j=1,...,s)$, *all Parameters* e_j,f_j,η_j,λ_j *are not zero simultaneously, min* $Re[1+\rho_j+\eta_j]>0$ *and* $\min\ Re[\sigma_j+\lambda_j]>0$.

It is pretended that the integrals on the RHS of equation (15) *and* (17) *exist.*

Proof. To Prove first of Theorem 4.1, with help of (8) and (14). We express the left hand side of (15), then the order of t_j and x_j-integrals must be interchanged (which is the Permissible the conditions stated with the theorem), finally evaluating the inner t_j-integrals with the help of result (13) (Taking $\gamma_j=0$ therein), we reach at wanting result after a little simplification. Similarly the result (17) of the Theorem 4.1 can be established on using (14). Now, the following theorem gives the fractional integrals of generalized Stieltjes transform given by (14). □

Theorem 4.2. *Let* $\phi(t_1,...,t_s) \in A$, $\min Re\ \left(e_j, f_j, \eta_j, \lambda_j\right) \geq 0$, $(j = 1,...,s)$ *all Parameters* $e_j, f_j, \eta_j, \lambda_j$ *are not zero simultaneously,* $\min\ Re\ [\sigma_j + \lambda_j] > 0$. *Then*

(1). For $\min\ Re\ [1 + \rho_j + \eta_j] > 0\ (j = 1,...,s)$

$$I_y\left[S_{w_1,...,w_s}\ \phi\ (t_1,...,t_s)\ ; (x_1,...,x_s)\right] = \int_0^\infty ... \int_0^\infty \phi(t_1,...,t_s)\,\psi_2(t_1,...,t_s;x_1,...,x_s)\,dt_1...dt_s \tag{19}$$

(2). For $\min\ Re\ [\rho_j + \eta_j + w_j] > 0\ (j = 1,...,s)$

$$J_y\left[S_{w_1,...,w_s}\ \phi\ (t_1,...,t_s)\ ; (x_1,...,x_s)\right] = \int_0^\infty ... \int_0^\infty \phi(t_1,...,t_s)\,\psi_1(t_1,...,t_s;x_1,...,x_s)\,dt_1...dt_s \tag{20}$$

Where $\psi_1(t_1,...,t_s;x_1,...,x_s)$ *and* $\psi_2(t_1,...,t_s;x_1,...,x_s)$ *are as given in* (16) *and* (18) *respectively, provided that the integrals in the Right hand side of the equations* (19) *and* (20) *abide.*

Proof. Results (19) and (20) of Theorem 4.2 can be gained on the similar lines to proof of Theorem 4.1. Also we can easily obtain the one dimensional analogues of the Theorem 4.1 and 4.2. □

5. Mellin Transforms

The Multidimensional Generalized Mellin transform of the function $f(t_1,...,t_s) \in A$ is defined by [9] given below:

$$M\ [\ f\ (t_1,...,t_s)\ ;\theta_1,...,\theta_s] = \int_0^\infty ... \int_0^\infty \prod_{j=1}^{s} t^{\theta_j - 1}\ f(t_1,...,t_s)\,dt_1...dt_s \tag{21}$$

Now we shall establish the results given as follows provided that the integral exists.

Result 5.1. *If* $M\left[I_x\left\{f(t_1,...,t_s);\theta_1,...,\theta_s\right\}\right]$ *and the conditions of the Existence of the operator* $I^{\rho,\sigma;e,f;\eta,\lambda}_{x:U,V;Z}\left[f(t_1,...,t_s)\right]$ *abide, then*

$$M\left[I_x\left\{f(t_1,...,t_s)\ ;\ \theta_1,...,\theta_s\right\}\right] = M\left[f(t_1,...,t_s)\ ;\ \theta_1,...,\theta_s\right]\ \Lambda\ (\theta_1,...,\theta_s) \tag{22}$$

Where

$$\Lambda(\theta_1,...,\ \theta_s) = \sum_{R_1,...,R_s=0}^{\sum_{j=1}^{s} U_j R_j \leq V} (-V)_{\sum_{j=1}^{s} U_j R_j} \times\ A\ (V\ ,\ R_1,...,\ R_s)\ \frac{E_1^{R_1}}{R_1\,!}...\frac{E_s^{R_s}}{R_s\,!} \aleph^{M\ ,N+3\,s}_{P_i+3s\ ,Q_i+2s\ ,\tau_i,r}\left[\ z\ \middle|\ \begin{matrix} I^* \\ J^* \end{matrix}\right] \tag{23}$$

$$I^* = (a_j, A_j)_{1,N}, (-\rho_j + \theta_j - e_j R_j; \eta_j)_{1,s}, (1 - \sigma_j - f_j R_j; \lambda_j)_{1,s}, [\tau_i\,(a_{ji}, A_{ji})]_{N+1,P_i;r}$$

$$J^* = (b_j, B_j)_{1,M}, (-\sigma_j - \rho_j + \theta_j - (e_j + f_j)\,R_j; \lambda_j + \eta_j)_{1,s}, [\tau_i(b_{ji}, B_{ji})]_{M+1,Q_i;r}$$

Result 5.2. *If* $M\left[J_x\left\{f(t_1,...,t_s);\theta_1,...,\theta_s\right\}\right]$ *and the conditions of the Existence of the operator* $J^{\rho,\sigma;e,f;\eta,\lambda}_{x:U,V;Z}\left[f(t_1,...,t_s)\right]$ *exists, then*

$$M\left[J_x\left\{f(t_1,...,t_s);\theta_1,...,\theta_s\right\}\right]\ =\ M\left[f(t_1,...,t_s);\theta_1,...,\theta_s\right]\ \Lambda\ (1-\theta_1,...,1-\theta_s) \tag{24}$$

Where $\Lambda(1-\theta_1,...,1-\theta_s)$ *can be obtaining by replacing* θ_s *by* $1-\theta_s$ *in* (22).

Proof. To prove the Result 5.1, we write the Multidimensional Mellin transform of the I-operator with help of equation (21), and then we change the order of t_j and x_j-integrals. Next, we arrive the desired result (22) after simplification with the help of (13) and (21). The Proof of Result 5.2 can be proceeding on the lines similar to those indicated above. □

6. Inversion Formulas

On using inversion theorems for the multidimensional Melline transform (21), given by Srivastava and Panda [9], the following inversion formula for the fractional integral operators defined by (8) and (9) can be gained as follows

Result 6.1.

$$f\left(t_1,...,t_s\right)=\frac{1}{\left(2\pi i\right)^s}\int\limits_{c_1-i\infty}^{c_1+i\infty}...\int\limits_{c_s-i\infty}^{c_s+i\infty}\frac{\prod\limits_{j=1}^{s}t^{-\theta_j}}{\Lambda\left(\theta_1,...,\theta_s\right)}M\left[I_x\left\{f\left(t_1,...,t_s\right);\theta_1,...,\theta_s\right\}\right]d\theta_1...d\theta_s \tag{25}$$

Where $\Lambda\left(\theta_1,...,\theta_s\right)$ *is given by* (22).

Result 6.2.

$$f\left(t_1,...,t_s\right)=\frac{1}{\left(2\pi i\right)^s}\int\limits_{c_1-i\infty}^{c_1+i\infty}...\int\limits_{c_s-i\infty}^{c_s+i\infty}\frac{\prod\limits_{j=1}^{s}t^{-\theta_j}}{\Lambda\left(1-\theta_1,...,1-\theta_s\right)}M\left[J_x\left\{f\left(t_1,...,t_s\right);\theta_1,...,\theta_s\right\}\right]d\theta_1...d\theta_s \tag{26}$$

Where $\Lambda\left(1-\theta_1,...,1-\theta_s\right)$ *can be obtaining by replacing* θ_s, *by* $1-\theta_s$ *in* (22).

The accurate validity conditions for the inversion formula (25) and (26) can be deduced from the existence condition of the fractional integral operators defined by (8) and (9) and their multidimensional Mellin transform stared earlier.

7. Mellin Convolutions

The Multidimensional Mellin convolutions of two functions $f\left(t_1,...,t_s\right)$ and $g\left(t_1,...,t_s\right)$ will be defined by

$$\left(f*g\right)\left(t_1,...,t_s\right)=\left(g*f\right)\left(t_1,...,t_s\right)=\int\limits_0^{\infty}...\int\limits_0^{\infty}\prod_{j=1}^{s}x_j^{-1}f\left(\frac{t_1}{x_1},...,\frac{t_s}{x_s}\right)g\left(x_1,...,x_s\right)dx_1...dx_s \tag{27}$$

Provided the Multiple integral exists. If $f\left(t_1,...,t_s\right)\in A$, then the fractional integral operators which is defined by (8) and (9) can be expressed as Multidimensional Mellin convolutions in the following manner.

Result 7.1.

$$I_{x:U,V;Z}^{\rho,\sigma;e,f;\eta,\lambda}g\left(t_1,...,t_s\right)=\left(I_{\rho,\sigma;e,f;\eta,\lambda,x:U,V;Z}*g\right)\left(x_1,...,x_s\right) \tag{28}$$

Where

$$I_{\rho,\sigma;e,f;\eta,\lambda,x:U,V;Z}=\left(\prod_{j=1}^{s}x_j{}^{-\rho_j-\sigma_j}\left(x_j-1\right)^{\sigma_j-1}U\left(x_j-1\right)\right)$$
$$\times S_V^{U_1,...,U_s}\left[\prod_{j=1}^{s}E_j(x_j)^{-e_j-f_j}(x_j-1)^{f_j}\right]\times\aleph_{P_i,Q_i,\tau_i,r}^{M,N}\left[z\prod_{j=1}^{s}(x_j)^{-\eta_j-\lambda_j}(x_j-1)^{\lambda_j}\right] \tag{29}$$

Result 7.2.

$$J_{x:U,V;Z}^{\rho,\sigma;e,f;\eta,\lambda}g\left(t_1,...,t_s\right)=\left(J_{\rho,\sigma;e,f;\eta,\lambda,x:U,V;Z}*g\right)\left(x_1,...,x_s\right) \tag{30}$$

Where

$$J_{\rho,\sigma;e,f;\eta,\lambda,x:U,V;Z}=\left(\prod_{j=1}^{s}x_j{}^{\rho_j}\left(1-x_j\right)^{\sigma_j-1}U\left(1-x_j\right)\right)$$
$$\times S_V^{U_1,...,U_s}\left[\prod_{j=1}^{s}E_j(x_j)^{-e_j-f_j}(x_j-1)^{f_j}\right]\times\aleph_{P_i,Q_i,\tau_i,r}^{M,N}\left[z\prod_{j=1}^{s}(x_j)^{\eta_j}(1-x_j)^{\lambda_j}\right] \tag{31}$$

Where $U\left(x\right)$ *being the Heaviside's unit function.*

Proof. To Prove Result 7.1, with help of the Heaviside's unit function. we write the I-operator defined by (8) in the following form.

$$I_{x:U,V;Z}^{\rho,\sigma;e,f;\eta,\lambda} g\left(t_1,...,t_s\right) = \int_0^\infty \int_0^\infty \left(\prod_{j=1}^s t_j{}^{-1}\right) \left[\prod_{j=1}^s \left\{\left(\frac{x_j}{t_j}\right)^{-\rho_j-\sigma_j}\left(\frac{x_j}{t_j}-1\right)^{\sigma_j-1} U\left(\frac{x_j}{t_j}-1\right)\right\}\right]$$
$$\times S_V^{U_1,...,U_s}\left[E_1\left(\frac{x_1}{t_1}\right)^{-e_1-f_1}\left(\frac{x_1}{t_1}-1\right)^{f_1},..., E_s\left(\frac{x_s}{t_s}\right)^{-e_s-f_s}\left(\frac{x_s}{t_s}-1\right)^{f_s}\right]$$
$$\times \aleph_{P_i,Q_i,\tau_i,r}^{M,N}\left[z\prod_{j=1}^s\left(\frac{x_j}{t_j}\right)^{-\eta_j-\lambda_j}\left(\frac{x_j}{t_j}-1\right)^{\lambda_j}\left|\begin{matrix}\left(a_j,A_j\right)_{1,N},\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{N+1,P_i;r}\\ \left(b_j,B_j\right)_{1,M},\left[\tau_i\left(b_{ji},B_{ji}\right)\right]_{M+1,Q_i;r}\end{matrix}\right.\right]$$
$$\times \left(t_1,...,t_s\right) dt_1,...,dt_s \tag{32}$$

Now in the above equation we use the equation (29) and the definition of the Mellin convolutions given by (27) in the above equation, we easily reach at the wanting result. The Proof of the Result 7.2 can be developed on the same basis. □

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