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Analytical Solution for Three-Dimensional Transient Heat Conduction in a Multilayer Sphere

Research Article

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Abstract: An analytical solution is obtained for the problem of three dimensional heat conduction in sphere with multiple layers in the radial direction, spatially non-uniform but time independent volumetric heat sources are assumed in each layers, separation of variables method is used to obtain transient temperature distribution. The solution obtained is valid for any combination of homogeneous first and second kind boundary conditions in the angular and axial direction of the sphere and for the non-homogeneous third kind boundary condition in the radial direction. Proposed solution is also applicable to multiple layer with zero inner radius. An illustrative example problem for the three layer quarter-spherical region is solved.

Keywords: Three dimensional heat conduction, non-uniform but time independent volumetric heat sources, multilayer sphere.

1. Introduction

Composite materials are defined as materials consisting of two or more components with different properties and distinct boundaries between them, Thermal analysis of multilayer composite media is of great importance since it has been widely used in real physical and engineering systems. Multilayer material has benefit of combining various mechanical, physical and thermal properties of different substances. Multilayer materials are used in semicircular fibre insulated heaters, Multilayer insulation materials and nuclear fuel rods. Multilayer transient heat conduction finds applications in thermodynamics, fuel cells and electrochemical reactors. The layered sphere is utilized to investigate the thermal properties of composite media by assuming embedded spherical particles in the composite materials. Many researchers have solved the transient heat conduction problem in a composite medium. For instance, Salt [1] solved the transient heat conduction problem in a two dimensional composite slab using an orthogonal eigenfunction expansion technique de Monte [2, 3] applied the eigenfunction expansion method to obtain the transient temperature distribution for the heat conduction in a two-dimensional two-layer isotropic slab with homogenous boundary conditions. Lu et al. [4] and Lu and Viljanen [5] combined separation of variables and Laplace transforms to solve the transient conduction in the two-dimensional cylindrical and spherical media. Dalir and Nourazar [10] used the eigenfunction expansion method to solve the problem of three-dimensional transient heat conduction in a multilayer cylinder. Singh et al. [6, 7] and Jain et al. [8, 9] have studied 2D multilayer transient conduction problems in spherical and cylindrical coordinates. They have obtained analytical solutions for 2D multilayer transient heat conduction

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in spherical coordinates, in polar coordinates with multiple layers in the radial direction, and in a multilayer annulus. They have used the method of partial solutions to obtain the temperature distributions. In the method of partial solutions, the non homogeneous transient problem is split into two sub problems: a non homogeneous steady-state sub problem and a homogeneous transient sub problem. Then, the eigenfunction expansion method is used to solve the non homogeneous steady-state sub problem and the method of separation of variables is used to solve the homogeneous transient sub problem Thus, in the present paper, the non homogeneous transient problem is split into two sub problems: a non homogeneous steady-state sub problem and a homogeneous transient sub problem. Then, the eigenfunction expansion method is used to solve the non homogeneous steady-state sub problem and the method of separation of variables is used to solve the homogeneous transient sub problem in the 3D spherical coordinates for radial multilayer domain with spatially non uniform and time independent internal heat sources is obtained. Homogenous boundary conditions of the first or second kind can be applied on surfaces of θ = constant and ϕ = constant. However, boundary conditions of the third kind (convection) are used in the r-direction.

2. Mathematical Formulation

Consider an n-layer composite spherical slab with coordinates $r_0 = r = r_n$, $0 = \theta = \boldsymbol{\psi}$ and $0 = \phi = \boldsymbol{\chi}$. It is assume that all the layers are thermally isotropic and make a perfect thermal contact. At $t = 0$, the ith layer has a temperature $f_i(r, \theta, \emptyset)$. At $t > 0$, homogeneous boundary conditions of the first or second kind are set on the angular surfaces $\theta = 0$, $\theta = \boldsymbol{\psi}$ and on $\phi = 0$, $\phi = \boldsymbol{\chi}$. All these boundary conditions can be used for the inner ($i = 0$, $r = r_0$) and outer ($i = n$, $r = r_n$) radial surfaces. The time-independent heat sources $g_i(r, \theta, \emptyset)$ are actuated in each layer. The governing differential equation for the 3-D transient heat conduction in a multilayer sphere along with the boundary and initial conditions are as follows.

$$\frac{\partial^2 T_i}{\partial r^2} + \frac{2}{r}\frac{\partial T_i}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T_i}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial T_i}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T_i}{\partial \phi^2} + \frac{g_i(r,\theta,\phi)}{k_i} = \frac{1}{\alpha_i}\frac{\partial T_i}{\partial t} \tag{1}$$

$$T_i = T_i(r,\theta,\phi,t) r_0 \le r \le r_n r_{i-1} \le r \le r_i 1 \le i \le n 0 \le \theta \le \psi 0 \le \phi \le \chi t \ge 0$$

Boundary conditions

- Inner surface of first layer ($i = 1$)

$$A_{in}\frac{\partial T_1(r_0,\theta,\phi,t)}{\partial r} + B_{in}T_1(r_0,\theta,\phi,t) = 0 \tag{2}$$

- Outer surface of n$^{\text{th}}$ layer ($i = n$)

$$A_{out}\frac{\partial T_n(r_n,\theta,\phi,t)}{\partial r} + B_{out}T_n(r_n,\theta,\phi,t) = 0 \tag{3}$$

- $\theta = 0$ surface ($i = 1, 2, \ldots, n$)

$$T_i(r,\theta = 0,\phi,t) = 0 \quad \text{or} \quad \frac{\partial T_i(r,\theta = 0,\phi,t)}{\partial \theta} = 0 \tag{4}$$

- $\theta = \boldsymbol{\psi}$ surface ($i = 1, 2, \ldots, n$)

$$T_i(r,\theta = \psi,\phi,t) = 0 \quad \text{or} \quad \frac{\partial T_i(r,\theta = \psi,\phi,t)}{\partial \theta} = 0 \tag{5}$$

- $\phi = 0$ surface ($i = 1, 2, \ldots, n$)

$$T_i(r,\theta,\phi = 0,t) = 0 \quad \text{or} \quad \frac{\partial T_i(r,\theta,\phi = 0,t)}{\partial \phi} = 0 \tag{6}$$

- $\phi = \chi$ surface $(i = 1, 2, \ldots, n)$

$$T_i(r,\theta,\phi=\chi,t) = 0 \quad \text{or} \quad \frac{\partial T_i(r,\theta,\phi=\chi,t)}{\partial \phi} = 0 \tag{7}$$

- Inner interface of the i$^{\text{th}}$ layer $(i = 2, 3, \ldots, n)$

$$T_i(r_{i-1},\theta,\phi,t) = T_{i-1}(r_{i-1},\theta,\phi,t) \tag{8}$$

$$k_i \frac{\partial T_i(r_{i-1},\theta,\phi,t)}{\partial r} = k_{i-1} \frac{\partial T_{i-1}(r_{i-1},\theta,\phi,t)}{\partial r} \tag{9}$$

- Outer interface of the i$^{\text{th}}$ layer $(i = 1, 2, \ldots, n-1)$

$$T_i(r_i,\theta,\phi,t) = T_{i+1}(r_i,\theta,\phi,t) \tag{10}$$

$$k_i \frac{\partial T_i(r_i,\theta,\phi,t)}{\partial r} = k_{i+1} \frac{\partial T_{i+1}(r_i,\theta,\phi,t)}{\partial r} \tag{11}$$

- Initial condition:

$$T_i(r,\theta,\phi,t=0) = f_i(r,\theta,\phi) \tag{12}$$

2.1. Solution methodology

In order to apply the separation of variable method , which is applicable to homogeneous problem has to be split into: (1) Homogeneous transient problem (2) Non-homogeneous steady state problem. This is accomplished by rewriting $T_i(r,\theta,\phi,t)$ as $\bar{T}_i(r,\theta,\phi,t) + T_{ss,i}(r,\theta,\phi)$, where $\bar{T}_i(r,\theta,\phi,t)$ is the "complementary transient" part and $T_{ss,i}(r,\theta,\phi)$ is the steady state part of the solution.

2.2. Homogeneous transient problem

$$\frac{1}{\alpha_i}\frac{\partial \bar{T}_i}{\partial t} = \frac{\partial^2 \bar{T}_i}{\partial r^2} + \frac{2}{r}\frac{\partial \bar{T}_i}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \bar{T}_i}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial \bar{T}_i}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 \bar{T}_i}{\partial \phi^2} \tag{13}$$

Where $\bar{T}_i = \bar{T}_i(r,\theta,\phi,t)$, $r_{i-1} \le r \le r_i$, $1 \le i \le n$.

2.3. Boundary conditions

- Inner surface of the first layer $(i = 0)$

$$A_{in}\frac{\partial \bar{T}_1(r_0,\theta,\phi,t)}{\partial r} + B_{in}\bar{T}_1(r_0,\theta,\phi,t) = 0 \tag{14}$$

- outer surface of the n$^{\text{th}}$ layer $(i = n)$

$$A_{out}\frac{\partial \bar{T}_n(r_n,\theta,\phi,t)}{\partial r} + B_{out}\bar{T}_n(r_n,\theta,\phi,t) = 0 \tag{15}$$

- $\theta = 0$ Surface at $i = 1, 2, \ldots, n$

$$\bar{T}_i(r,\theta=0,\phi,t) = 0 \quad \text{or} \quad \frac{\partial \bar{T}_i(r,\theta=0,\phi,t)}{\partial \theta} = 0 \tag{16}$$

- $\theta = \psi$ surface at $i = 1, 2, \ldots, n$

$$\bar{T}_i(r, \theta = \psi, \phi, t) = 0 \quad \text{or} \quad \frac{\partial \bar{T}_i(r, \theta = \psi, \phi, t)}{\partial \theta} = 0 \tag{17}$$

- $\phi = 0$ surface at $i = 1, 2, \ldots, n$

$$\bar{T}_i(r, \theta, \phi = 0, t) = 0 \quad \text{or} \quad \frac{\partial \bar{T}_i(r, \theta, \phi = 0, t)}{\partial \phi} = 0 \tag{18}$$

- $\phi = \chi$ surface at $i = 1, 2, \ldots, n$

$$\bar{T}_i(r, \theta, \phi = \chi, t) = 0 \quad \text{or} \quad \frac{\partial \bar{T}_i(r, \theta, \phi = \chi, t)}{\partial \phi} = 0 \tag{19}$$

- Inner interface of the i$^{\text{th}}$ layer ($i = 2, 3, \ldots, n$)

$$\bar{T}_i(r_{i-1}, \theta, \phi, t) = \bar{T}_{i-1}(r_{i-1}, \theta, \phi, t) \tag{20}$$

$$k_i \frac{\partial \bar{T}_i(r_{i-1}, \theta, \phi, t)}{\partial r} = k_{i-1} \frac{\partial \bar{T}_{i-1}(r_{i-1}, \theta, \phi, t)}{\partial r} \tag{21}$$

- Outer interface of the i$^{\text{th}}$ layer ($i = 1, 2, 3, \ldots, n-1$)

$$\bar{T}_i(r, \theta, \phi, t) = \bar{T}_{i+1}(r_i, \theta, \phi, t) \tag{22}$$

$$k_i \frac{\partial \bar{T}_i(r_i, \theta, \phi, t)}{\partial r} = k_{i+1} \frac{\partial \bar{T}_{i+1}(r_i, \theta, \phi, t)}{\partial r} \tag{23}$$

- Initial condition:

$$\bar{T}_i(r, \theta, \phi, t = 0) = f_i(r, \theta, \phi) - T_{ss,i}(r, \theta, \phi) \tag{24}$$

2.4. Non-homogeneous steady state problem

$$\frac{\partial^2 T_{ss,i}(r,\theta,\phi)}{\partial r^2} + \frac{2}{r}\frac{\partial T_{ss,i}(r,\theta,\phi)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T_{ss,i}(r,\theta,\phi)}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial T_{ss,i}(r,\theta,\phi)}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T_{ss,i}(r,\theta,\phi)}{\partial \phi^2} + \frac{g_i(r,\theta,\phi)}{k_i} = 0 \tag{25}$$

- Inner surface of first layer ($i = 1$)

$$A_{in} \frac{\partial T_{ss,1}(r_0, \theta, \phi)}{\partial r} + B_{in} T_{ss,1}(r_0, \theta, \phi) = 0 \tag{26}$$

- Outer surface of i$^{\text{th}}$ layer ($i = n$)

$$A_{out} \frac{\partial T_{ss,n}(r_n, \theta, \phi)}{\partial r} + B_{out} T_{ss,n}(r_n, \theta, \phi) = 0 \tag{27}$$

- $\theta = 0$ Surface at $i = 1, 2, \ldots, n$

$$T_{ss,i}(r, \theta = 0, \phi) = 0 \quad \text{or} \quad \frac{\partial T_{ss,i}(r, \theta = 0, \phi)}{\partial \theta} = 0 \tag{28}$$

- $\theta = \boldsymbol{\psi}$ surface at $i = 1, 2, \ldots, n$

$$T_{ss,i}(r, \theta = \psi, \phi) = 0 \quad \text{or} \quad \frac{\partial T_{ss,i}(r, \theta = \psi, \phi)}{\partial \theta} = 0 \tag{29}$$

- $\phi = 0$ surface at $i = 1, 2, \ldots, n$

$$T_{ss,i}(r, \theta, \phi = 0) = 0 \quad \text{or} \quad \frac{\partial T_{ss,i}(r, \theta, \phi = 0)}{\partial \phi} = 0 \tag{30}$$

- $\phi = \boldsymbol{\chi}$ surface at $i = 1, 2, \ldots, n$

$$T_{ss,i}(r, \theta, \phi = \chi) = 0 \quad \text{or} \quad \frac{\partial T_{ss,i}(r, \theta, \phi = \chi)}{\partial \phi} = 0 \tag{31}$$

- Inner interface of $\mathrm{i^{th}}$ layer $(i = 2, 3, \ldots, n)$

$$T_{ss,i}(r_{i-1}, \theta, \phi) = T_{ss,i-1}(r_{i-1}, \theta, \phi) \tag{32}$$

$$k_i \frac{\partial T_{ss,i}(r_{i-1}, \theta, \phi)}{\partial r} = k_{i-1} \frac{\partial T_{ss,i-1}(r_{i-1}, \theta, \phi)}{\partial r} \tag{33}$$

- Outer interface of $\mathrm{n^{th}}$ layer $(i = 1, 2, \ldots, n-1)$

$$T_{ss,i}(r_i, \theta, \phi) = T_{ss,i+1}(r_i, \theta, \phi) \tag{34}$$

$$k_i \frac{\partial T_{ss,i}(r_i, \theta, \phi)}{\partial r} = k_{i+1} \frac{\partial T_{ss,i+1}(r_i, \theta, \phi)}{\partial r} \tag{35}$$

3. Solution to the Homogeneous Transient Problem

Using the Separation of variable method :

$$\bar{T}_i(r, \theta, \phi, t) = R_i(r)\theta_i(\theta)\phi_i(\phi)T_i(t) \tag{36}$$

$$\frac{1}{\alpha_i}\frac{\Gamma_i^{'}}{\Gamma_i} = \frac{R_i^{''}}{R_i} + \frac{2}{r}\frac{R_i^{'}}{R_i} + \frac{1}{r^2}\frac{\theta_i^{''}}{\theta_i} + \frac{\cot\theta}{r^2}\frac{\theta_i^{'}}{\theta_i} + \frac{1}{r^2\sin^2\theta}\frac{\phi_i^{''}}{\phi_i} = -\lambda_i^2$$

$$\Gamma_i^{'} + \alpha_i\lambda_i^2\Gamma_i = 0$$

$$\Gamma_i = c_1 e^{-\alpha_i\lambda_i^2 t} \tag{37}$$

$$\frac{R_i^{''}}{R_i} + \frac{2}{r}\frac{R_i^{'}}{R_i} + \frac{1}{r^2}\frac{\theta_i^{''}}{\theta_i} + \frac{\cot\theta}{r^2}\frac{\theta_i^{'}}{\theta_i} = -\frac{1}{r^2\sin^2\theta}\frac{\phi_i^{''}}{\phi_i} - \lambda_i^2$$

$$\sin^2\theta\left[r^2\frac{R_i^{''}}{R_i} + 2r\frac{R_i^{'}}{R_i} + \frac{\theta_i^{''}}{\theta_i} + \cot\theta\frac{\theta_i^{'}}{\theta_i} + \lambda_i^2 r^2\right] = -\frac{\phi_i^{''}}{\phi_i} = \nu^2$$

$$\phi_i^{''} + \nu_{ip}^2\phi_i = 0$$

$$\phi_{ip} = c_2\sin\nu_{ip}\phi + c_3\cos\nu_{ip}\phi \tag{38}$$

By using the boundary condition on ϕ, we obtain as

$$\phi_{ip}(\phi) = \sin \nu_{ip}\phi \tag{39}$$

$$r^2\frac{R_i^{''}}{R_i} + 2r\frac{R_i^{'}}{R_i} + \lambda_i^2 r^2 = -\frac{\theta_i^{''}}{\theta_i} - \cot\theta\frac{\theta_i^{'}}{\theta_i} + \frac{\nu^2}{\sin^2\theta} = \beta^2 \tag{40}$$

$$r^2 R_i^{''} + 2rR_i^{'} + (\lambda_{iml}^2 r^2 - \beta_m^2)R_i = 0$$

Substituting $R_i(r) = r^{\frac{1}{2}}V_i$, $\beta_m^2 = m(m+1)$

$$R_{iml}(r) = \frac{1}{\sqrt{r}}\left[c_{iml}J_{m+0.5}(\lambda_{iml}r) + d_{iml}Y_{m+0.5}(\lambda_{iml}r)\right] \tag{41}$$

Application of the interface conditions (20)-(23) and boundary conditions (14)-(15) to the transverse eigenfunction $R_{iml}(\lambda_{iml}r)$, The matrix (2n x 2n) are as follows:

$$\begin{bmatrix} a_{1in} & a_{2in} & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ x_{11} & x_{12} & x_{13} & x_{14} & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ y_{11} & y_{12} & y_{13} & y_{14} & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & x_{i1} & x_{i2} & x_{i3} & x_{i4} & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & y_{i1} & y_{i2} & y_{i3} & y_{i4} & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & x_{i-1,1} & x_{i-1,2} & x_{i-1,3} & x_{i-1,4} \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & y_{i-1,1} & y_{i-1,2} & y_{i-1,3} & y_{i-1,4} \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & a_{1out} & a_{2out} \end{bmatrix} \begin{bmatrix} c_{1ml} \\ d_{1ml} \\ \dots \\ \dots \\ c_{iml} \\ d_{iml} \\ \dots \\ \dots \\ c_{nml} \\ d_{nml} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dots \\ \dots \\ 0 \\ 0 \\ \dots \\ \dots \\ 0 \\ 0 \end{bmatrix} \tag{42}$$

$$a_{1in} = \frac{1}{\sqrt{r_0}}\left[A_{in}J_{m+0.5}^{'}(\lambda_{1ml}r_0) - \frac{A_{in}}{2r_0}J_{m+0.5}(\lambda_{1ml}r_0) + B_{in}J_{m+0.5}(\lambda_{1ml}r_0)\right]$$

$$a_{2in} = \frac{1}{\sqrt{r_0}}\left[A_{in}Y_{m+0.5}^{'}(\lambda_{1ml}r_0) - \frac{A_{in}}{2r_0}Y_{m+0.5}(\lambda_{1ml}r_0) + B_{in}Y_{m+0.5}(\lambda_{1ml}r_0)\right]$$

$$x_{i1} = \frac{1}{\sqrt{r_1}}J_{m+0.5}(\lambda_{1ml}r_1) \qquad y_{i1} = \frac{1}{\sqrt{r_1}}\left[k_1J_{m+0.5}^{'}(\lambda_{1ml}r_1) - \frac{k_1}{2r_1}J_{m+0.5}(\lambda_{1ml}r_1)\right]$$

$$x_{i2} = \frac{1}{\sqrt{r_1}}Y_{m+0.5}(\lambda_{1ml}r_1) \qquad y_{i2} = \frac{1}{\sqrt{r_1}}\left[k_1Y_{m+0.5}^{'}(\lambda_{1ml}r_1) - \frac{k_1}{2r_1}Y_{m+0.5}(\lambda_{1ml}r_1)\right]$$

$$x_{i3} = -\frac{1}{\sqrt{r_1}}J_{m+0.5}(\lambda_{2ml}r_1) \qquad y_{i3} = \frac{1}{\sqrt{r_1}}\left[-k_2J_{m+0.5}^{'}(\lambda_{2ml}r_1) + \frac{k_2}{2r_1}J_{m+0.5}(\lambda_{2ml}r_1)\right]$$

$$x_{i4} = -\frac{1}{\sqrt{r_1}}Y_{m+0.5}(\lambda_{2ml}r_1) \qquad y_{i4} = \frac{1}{\sqrt{r_1}}\left[-k_2Y_{m+0.5}^{'}(\lambda_{2ml}r_1) + \frac{k_2}{2r_1}Y_{m+0.5}(\lambda_{2ml}r_1)\right]$$

$$a_{1out} = \frac{1}{\sqrt{r_n}}\left[A_{out}J_{m+0.5}^{'}(\lambda_{nml}r_n) - \frac{A_{out}}{2r_n}J_{m+0.5}(\lambda_{nml}r_n) + B_{out}J_{m+0.5}(\lambda_{nml}r_n)\right]$$

$$a_{2out} = \frac{1}{\sqrt{r_n}}\left[A_{out}Y_{m+0.5}^{'}(\lambda_{nml}r_n) - \frac{A_{out}}{2r_n}Y_{m+0.5}(\lambda_{nml}r_n) + B_{out}Y_{m+0.5}(\lambda_{nml}r_n)\right]$$

For heat flux to be continuous at the layer interfaces for all values of t, $\alpha_i\lambda_{iml}^2 = \alpha_1\lambda_{1ml}^2$, $i = 1, 2, \dots, n$. In the above matrix equation, λ_{iml} $(i \neq 1)$ may be written in terms of λ_{1ml} using the above equation. Subsequently, transverse Eigen condition can be obtained by setting the determinant of the $(n\times n)$ coefficient matrix equal to zero. And after that eigenvalue determined the constants c_{iml} and d_{iml} by solving (42). From equation (40)

$$\theta_i^{''} + (\cot\theta)\theta^{'} + \left(\beta^2 - \frac{\nu^2}{\sin^2\theta}\right)\theta_i = 0$$

Put $\mu = \cos\theta$, $1-\mu^2 = \sin^2\theta$.

$$\sin^2\theta\frac{d^2\theta_i}{d\mu^2} - \cos\theta\frac{d\theta_i}{d\mu} - \cos\theta\frac{d\theta_i}{d\mu} + \left(\beta^2 - \frac{\nu^2}{\sin^2\theta}\right)\theta_i = 0$$
$$(1-\mu^2)\frac{d^2\theta_i}{d\mu^2} - 2\mu\frac{d\theta_i}{d\mu} + \left(\beta^2 - \frac{\nu^2}{\sin^2\theta}\right)\theta_i = 0$$

If $\beta_m^2 = m(m+1)$ is the associated Legendre equation, its solution is written as follows

$$\theta_{il}(\mu) = c_6 P_{il}^{\nu_{ip}}(\mu) + c_7 Q_{il}^{\nu_{ip}}(\mu) \tag{43}$$

But $\mu = \cos\theta$

$$\theta_{il}(\theta) = c_6 P_{il}^{\nu_{ip}}(\cos\theta) + c_7 Q_{il}^{\nu_{ip}}(\cos\theta) \tag{44}$$

If $\theta = 0$ then $\theta_{il}(\theta) = 0$. Therefore, $Q_{il}^{\nu_{ip}}(\cos 0) = Q_{il}^{\nu_{ip}}(1) = \infty \Rightarrow c_7 = 0$. Hence $c_6 \neq 0$. Therefore

$$\theta_{il}(\theta) = P_{il}^{\nu_{ip}}(\cos\theta) \tag{45}$$

Orthogonality condition for the r-direction eigenfunctions as

$$\sum_{i=1}^{n}\frac{k_i}{\alpha_i}\int_{r_{i-1}}^{r_i} r_i^2 R_{iml}(\lambda_{iml}r)R_{inl}(\lambda_{inl}r)dr = 0 \;\text{ if }\; m\neq n \tag{46}$$
$$\sum_{i=1}^{n}\frac{k_i}{\alpha_i}\int_{r_{i-1}}^{r_i} r_i^2 R_{iml}(\lambda_{iml}r)R_{inl}(\lambda_{inl}r)dr = N_{rml} \;\text{ if }\; m = n \tag{47}$$

Orthogonal condition for the θ-direction

$$\int_0^{\psi}\theta_{il}(\theta)\theta_{is}(\theta) = 0 \;\text{ if }\; l\neq s \tag{48}$$
$$\int_0^{\psi}\theta_{il}(\theta)\theta_{is}(\theta) = N_{\theta_l} \;\text{ if }\; n = s \tag{49}$$

Orthogonal condition for the ϕ-direction

$$\int_0^{\chi}\phi_{ip}(\nu_{ip}\phi)\phi_{iq}(\nu_{iq}\phi) = 0 \;\text{ if }\; p\neq q \tag{50}$$
$$\int_0^{\chi}\phi_{ip}(\nu_{ip}\phi)\phi_{iq}(\nu_{iq}\phi) = N_{\phi_p} \;\text{ if }\; p = q \tag{51}$$

A general solution for the homogeneous transient problem may be considered as:

$$\bar{T}(r,\theta,\phi,t) = \sum_{p=1}^{\infty}\sum_{m=1}^{\infty}\sum_{l=1}^{\infty} D_{imlp}e^{-\alpha_i\lambda_{imlp}^2 t}R_{iml}(\lambda_{iml}r)P_{il}^{\nu_p}(\cos\theta)\sin\nu_{ip}\phi \tag{52}$$

3.1. Determination of coefficient D_{lmn}

Coefficient of D_{imlp} in equation (52) may be obtained by applying the initial condition and then making use of the orthogonality conditions in the r, θ, ϕ directions as follows

$$D_{ipmn} = \frac{1}{N_{\theta l} N_{rml} N_{\phi p}} \sum_{i=1}^{n} \frac{k_i}{\alpha_i} \int_0^{\chi} \int_0^{\psi} \int_{r_{i-1}}^{r_i} r^2 R_{iml}(\lambda_{iml} r) \theta_{il}(\theta) \phi_{ip}(\nu_{ip}\phi) \bar{T}_i(r,\theta,\phi,t=0) dr d\theta d\phi \tag{53}$$

3.2. Solution of inhomogeneous steady state problem

The inhomogeneous steady state problem is solved using eigenfunction expansion method.

$$T_{ss,i}(r,\theta,\phi) = \sum_{m=1}^{\infty}\sum_{p=1}^{\infty}\sum_{l=1}^{\infty} \hat{T}_{iml}(r)\theta_{il}(\theta)\phi_{ip}(\phi) \tag{54}$$

$$g_i(r,\theta,\phi) = \sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\sum_{p=1}^{\infty} g_{iml}(r)\theta_{il}(\theta)\phi_{ip}(\phi) \tag{55}$$

Substituting Equation (54) and Equation (55) in Equation (25)

$$\frac{\hat{T}''_{iml}(r)}{\hat{T}_{iml}(r)} + \frac{2}{r}\frac{\hat{T}'_{iml}(r)}{\hat{T}_{iml}(r)} + \frac{1}{r^2}\frac{\theta''_{il}(\theta)}{\theta_{il}(\theta)} + \frac{\cot\theta}{r^2}\frac{\theta'_{il}(\theta)}{\theta_{il}(\theta)} + \frac{1}{r^2\sin^2\theta}\frac{\phi''_{ip}(\phi)}{\phi_{ip}(\phi)} + \frac{1}{k_i}\frac{g_{iml}(r)}{\hat{T}_{iml}(r)} = 0$$

$$\hat{T}''_{iml}(r) + \frac{2}{r}\hat{T}'_{iml}(r) + \frac{1}{r^2}(-\beta_{im}^2)\hat{T}_{iml}(r) + \frac{1}{k_i}g_{iml}(r) = 0$$

$$\left[r^2D^2 + 2rD - \beta_{im}^2\right]\hat{T}_{iml}(r) = -\frac{r^2}{k_i}g_{iml}(r) \tag{56}$$

Where $D = \frac{d}{dr}$. Above equation is a Cauchy's homogeneous linear equation and its solution is "Complimentary function (C.F.) + particular integral (P.I.)".

$$C.F. = a_{ss,i} r^{-1+\sqrt{1+4\beta_{im}^2}/2} + b_{ss,i} r^{-1-\sqrt{1+4\beta_{im}^2}/2} \tag{57}$$

And is particular integral that can be obtained by application of method of variation of parameters

$$P.I. = \frac{r^{\frac{-1+\sqrt{1+4\beta_{im}^2}}{2}}}{k_i\sqrt{1+4\beta_{im}^2}} \int r^{\frac{5-\sqrt{1+4\beta_{im}^2}}{2}} g_{iml}(r)dr + \frac{r^{\frac{-1-\sqrt{1+4\beta_{im}^2}}{2}}}{k_i\sqrt{1+4\beta_{im}^2}} \int r^{\frac{5+\sqrt{1+4\beta_{im}^2}}{2}} g_{iml}(r)dr$$

$$P.I. = \frac{1}{k_i\sqrt{1+4\beta_{im}^2}}\left[r^{\frac{-1+\sqrt{1+4\beta_{im}^2}}{2}} \int r^{\frac{5-\sqrt{1+4\beta_{im}^2}}{2}} g_{iml}(r)dr + r^{\frac{-1-\sqrt{1+4\beta_{im}^2}}{2}} \int r^{\frac{5+\sqrt{1+4\beta_{im}^2}}{2}} g_{iml}(r)dr\right] \tag{58}$$

$$\hat{T}_{iml}(r) = a_{ss,i} r^{\frac{-1+\sqrt{1+4\beta_{im}^2}}{2}} + b_{ss,i} r^{\frac{-1-\sqrt{1+4\beta_{im}^2}}{2}} + \frac{1}{k_i\sqrt{1+4\beta_{im}^2}}\left[r^{\frac{-1+\sqrt{1+4\beta_{im}^2}}{2}} \int r^{\frac{5-\sqrt{1+4\beta_{im}^2}}{2}} g_{iml}(r)dr + r^{\frac{-1-\sqrt{1+4\beta_{im}^2}}{2}} \int r^{\frac{5+\sqrt{1+4\beta_{im}^2}}{2}} g_{iml}(r)dr\right] \tag{59}$$

by using the boundary and interface conditions determined the constants $a_{ss,i}$ and $b_{ss,i}$. Where

$$g_{iml}(r) = \frac{1}{N_{\theta l} N_{\phi p}} \int_0^{\psi}\int_0^{\chi} g_{iml}(r,\theta,\phi)\theta_{il}(\theta)\phi_{ip}(\phi) d\theta . d\phi. \tag{60}$$

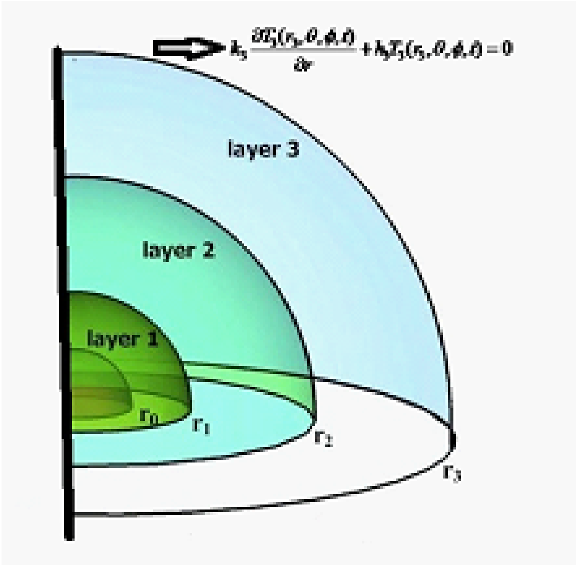

Figure 1. **3D hollow quarter sphere**

3.3. Case Study Problem

We consider a three layer quarter sphere with co-ordinates $0 \le r \le r_3$, $0 \le \theta \le \frac{\pi}{2}$ and $0 \le \phi \le \pi$. At the initial instant of time $(t = 0)$ the sphere has a uniform temperature distribution. At $t > 0$ the temperature of the surfaces $\theta = 0$, $\theta = \frac{\pi}{2}$, $\phi = 0$ and $\phi = \pi$ are uniform and equal to zero temperatures. Thermal convection occurs from the outer radial surface $(r = r_3)$ at zero temperature. These boundary conditions are defined by the relation $A_{in} = 1$, $B_{in} = 0$, $A_{out} = k_3$, $B_{out} = h_3$. A heat source $g_i(r, \theta, \phi)$ for i=1,2,3 is activated in each layer at $t = 0$.

The governing differential equation for the three dimension transient heat conduction in the indicated three layer quarter-spherical region is as follows:

$$\frac{\partial^2 T_i}{\partial r^2} + \frac{2}{r}\frac{\partial T_i}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T_i}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial T_i}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T_i}{\partial^2 \phi^2} + \frac{g_i(r, \theta, \phi)}{k_i} = \frac{1}{\alpha_i}\frac{\partial T_i}{\partial t}$$

$$T_i = T_i(r, \theta, \phi, t),\ r_{i-1} \le r \le r_i,\ 1 \le i \le 3,\ r_0 \le r \le r_3,\ 0 \le \theta \le \frac{\pi}{2},\ 0 \le \phi \le \pi$$

Boundary condition for inner surface

$$\frac{\partial T_1(r_0, \theta, \phi, t)}{\partial r} = 0 \tag{61}$$

Boundary condition for outer surface

$$k_3 \frac{\partial T_3(r_3, \theta, \phi, t)}{\partial r} + h_3 T_3(r_3, \theta, \phi, t) = 0 \tag{62}$$

Boundary condition for θ and ϕ direction i=1,2,3.

$$T_i(r, 0, \phi, t) = 0 \tag{63}$$

$$T_i(r, \frac{\pi}{2}, \phi, t) = 0 \tag{64}$$

$$T_i(r, \theta, 0, t) = 0 \tag{65}$$

$$T_i(r, \theta, \pi, t) = 0 \tag{66}$$

For inner interface surface of the ith layer (i = 2,3)

$$T_i(r_{i-1},\theta,\phi,t) = T_{i-1}(r_{i-1},\theta,\phi,t)$$

$$k_i\frac{\partial T_i(r_{i-1},\theta,\phi,t)}{\partial r} = k_{i-1}\frac{\partial T_{i-1}(r_{i-1},\theta,\phi,t)}{\partial r} \tag{67}$$

For outer interface surface of the ith layer (i = 1,2)

$$T_i(r_i,\theta,\phi,t) = T_{i+1}(r_i,\theta,\phi,t)$$

$$k_i\frac{\partial T_i(r_i,\theta,\phi,t)}{\partial r} = k_{i+1}\frac{\partial T_{i+1}(r_i,\theta,\phi,t)}{\partial r} \tag{68}$$

The initial condition has the form

$$T_i(r,\theta,\phi,0) = 1 \quad i = 1,2,3. \tag{69}$$

Solution Method by rewriting $T_i(r,\theta,\phi,t)$ as $\bar{T}_i(r,\theta,\phi,t)+T_{ss,i}(r,\theta,\phi)$, where $\bar{T}_i(r,\theta,z,t)$ is the "complementary transient" part and $T_{ss,i}(r,\theta,z)$ is the steady state part of the solution.

4. Solution to the Homogeneous Transient Problem

With the use of the separation of variable method the associated eigenvalue problem is solved in the r, θ & ϕ- directions of equation (39), (43) and (44) are as follows respectively:

$$\phi_{ip} = c_2\sin\nu_{ip}\phi + c_3\cos\nu_{ip}\phi$$
$$R_{iml}(r) = \frac{1}{\sqrt{r}}\left[c_{iml}J_{m+0.5}(\lambda_{iml}r) + d_{iml}Y_{m+0.5}(\lambda_{iml}r)\right]$$
$$\theta_{il}(\theta) = c_6P_{il}^{\nu_{ip}}(\cos\theta) + c_7Q_{il}^{\nu_{ip}}(\cos\theta)$$

The Eigen functions $R_{iml}(r)$, $\theta_{il}(\theta)$ and $\phi_{ip}(\phi)$ in the r, θ, ϕ-directions are determined with the use of relevant boundary conditions in each direction. In this case for the ϕ-direction with the use of boundary condition in equation (65) and (66), we obtain

$\phi_{ip}(0) = 0 \Rightarrow c_3 = 0$

$\phi_{ip}(\pi) = 0 \Rightarrow c_2\sin\nu_{ip}\pi = 0$ here $c_2 \neq 0$

$\sin\nu_{ip}\pi = 0 = \sin p\pi$, $\nu_{ip} = p$

$$\phi_{ip}(\phi) = \sin p\phi \tag{70}$$

With the use of boundary condition in equations (63) and (64) in the θ-direction we obtain. If $\theta = 0$ then $\theta_{il}(\theta) = 0$. Therefore, $Q_{il}^{\nu_{ip}}(\cos 0) = Q_{il}^{\nu_{ip}}(1) = \infty \Rightarrow c_7 = 0$. Hence $c_6 \neq 0$. Therefore

$$\theta_{il}(\theta) = P_{il}^{\nu_{ip}}(\cos\theta)$$

Using the boundary condition equation (64)

$$P_{il}^{p}\left[\cos\left({}^{\pi}\!/_{2}\right)\right] = 0 \Rightarrow P_{il}^{p}(0) = 0$$

Where $P_{il}^{p}(0) = 0$ is only satisfied when " l" are odd integers; that is $l = 1, 3, 5, \ldots$. Thus the θ- direction eigenvalues and eigenfunction are as follows

$$\therefore\ \theta_{il}(\theta) = P_{il}^{p}(\cos\theta),\ \ \nu_{ip} = p \tag{71}$$

By using the boundary conditions and interface conditions in the r-direction

$$\begin{bmatrix} a_{1in} & a_{2in} & 0 & 0 & 0 & 0 \\ x_{11} & x_{12} & x_{13} & x_{14} & 0 & 0 \\ y_{11} & y_{12} & y_{13} & y_{14} & 0 & 0 \\ 0 & 0 & x_{21} & x_{22} & x_{23} & x_{24} \\ 0 & 0 & y_{21} & y_{22} & y_{23} & y_{24} \\ 0 & 0 & 0 & 0 & a_{1out} & a_{2out} \end{bmatrix} \begin{bmatrix} c_{1ml} \\ d_{1ml} \\ c_{2ml} \\ d_{2ml} \\ c_{3ml} \\ d_{3ml} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{72}$$

$$a_{1in} = \frac{1}{\sqrt{r_0}}\left[J'_{m+0.5}(\lambda_{1ml}r_0) - \frac{1}{2r_0}J_{m+0.5}(\lambda_{1ml}r_0)\right]$$

$$a_{2in} = \frac{1}{\sqrt{r_0}}\left[Y'_{m+0.5}(\lambda_{1ml}r_0) - \frac{1}{2r_0}Y_{m+0.5}(\lambda_{1ml}r_0)\right]$$

$$x_{11} = \frac{J_{m+0.5}(\lambda_{1ml}r_1)}{\sqrt{r_1}},\ \ x_{12} = \frac{Y_{m+0.5}(\lambda_{1ml}r_1)}{\sqrt{r_1}},\ \ x_{13} = -\frac{J_{m+0.5}(\lambda_{2ml}r_1)}{\sqrt{r_1}},\ \ x_{14} = -\frac{Y_{m+0.5}(\lambda_{2ml}r_1)}{\sqrt{r_1}}$$

$$x_{21} = \frac{J_{m+0.5}(\lambda_{2ml}r_2)}{\sqrt{r_2}},\ \ x_{22} = \frac{Y_{m+0.5}(\lambda_{2ml}r_2)}{\sqrt{r_2}},\ \ x_{23} = -\frac{J_{m+0.5}(\lambda_{3ml}r_2)}{\sqrt{r_2}},\ \ x_{24} = -\frac{Y_{m+0.5}(\lambda_{3ml}r_2)}{\sqrt{r_2}}$$

$$y_{11} = \frac{1}{\sqrt{r_1}}\left[k_1J'_{m+0.5}(\lambda_{1ml}r) - \frac{k_1}{2r_1}J_{m+0.5}(\lambda_{1ml}r_1)\right],\ \ y_{12} = \frac{1}{\sqrt{r_1}}\left[k_1Y'_{m+0.5}(\lambda_{1ml}r) - \frac{k_1}{2r_1}Y_{m+0.5}(\lambda_{1ml}r_1)\right]$$

$$y_{13} = \frac{1}{\sqrt{r_1}}\left[k_2J'_{m+0.5}(\lambda_{2ml}r) - \frac{k_2}{2r_1}J_{m+0.5}(\lambda_{2ml}r_1)\right],\ \ y_{14} = \frac{1}{\sqrt{r_1}}\left[k_2Y'_{m+0.5}(\lambda_{2ml}r) - \frac{k_2}{2r_1}Y_{m+0.5}(\lambda_{2ml}r_1)\right]$$

$$a_{1out} = \frac{1}{\sqrt{r_3}}\left[k_3J'_{m+0.5}(\lambda_{3ml}r_3) - \frac{k_3}{2r_3}J_{m+0.5}(\lambda_{3ml}r_3) + h_3J_{m+0.5}(\lambda_{3ml}r_3)\right]$$

$$a_{2out} = \frac{1}{\sqrt{r_3}}\left[k_3Y'_{m+0.5}(\lambda_{3ml}r_3) - \frac{k_3}{2r_3}Y_{m+0.5}(\lambda_{3ml}r_3) + h_3Y_{m+0.5}(\lambda_{3ml}r_3)\right]$$

The heat flux continuity conditions at the interfaces imply the following:

$$\lambda_{iml} = \lambda_{1ml}\sqrt{\frac{\alpha_1}{\alpha_i}}$$

In the above matrix equation, λ_{iml} $(i \neq 1)$ may be written in terms of λ_{1ml} using the above equation. Subsequently, transverse eigencondition can be obtained by setting the determinant of the (6×6) coefficient matrix equal to zero. And after that eigenvalue determined the constants c_{iml} and d_{iml} by solving Equation (72). We solved in the form of the triple-series expansion.

$$\bar{T}_i(r,\theta,z,t) = \sum_{p=1}^{\infty}\sum_{m=1}^{\infty}\sum_{l=1}^{\infty} D_{imlp}e^{-\alpha_i\lambda_{imlp}^2 t}R_{iml}(\lambda_{iml}r)P_{il}^{p}(\cos\theta)\sin p\phi \tag{73}$$

Coefficient of D_{imlp} may be obtained by applying the initial condition equation (69) and then making use of the orthogonality conditions in the r, θ, ϕ directions as follows:

$$D_{imlp} = \frac{\sum_{i=1}^{3}\frac{k_i}{\alpha_i}\int_{r_{i-1}}^{r_i}\int_0^{\frac{\pi}{2}}\int_0^{\pi} r^2R_{iml}(\lambda_{iml}r)P_{il}^{p}(\cos\theta)\sin p\phi.drd\theta d\phi}{\sum_{i=1}^{3}\frac{k_i}{\alpha_i}\left(\int_{r_{i-1}}^{r_i} r^2R_{iml}(\lambda_{iml}r)dr\right)\left(\int_0^{\frac{\pi}{2}}\left(P_{il}^{p}(\cos\theta)\right)^2d\theta\right)\left(\int_0^{\pi}\sin^2(p\phi)d\phi\right)}$$

$$D_{imlp} = \frac{\left(\int_{r_{i-1}}^{r_i} r^2 R_{iml}(\lambda_{iml} r)dr\right)\left(\int_0^{\frac{\pi}{2}} \left(P_{il}^p(\cos\theta)\right) d\theta\right)\left(\int_0^{\pi} \sin(p\phi)d\phi\right)}{\left(\int_{r_{i-1}}^{r_i} r^2 R_{iml}^2(\lambda_{iml} r)dr\right)\left(\int_0^{\frac{\pi}{2}} \left(P_{il}^p(\cos\theta)\right)^2 d\theta\right)\left(\int_0^{\pi} \sin^2(p\phi)d\phi\right)}$$

$$D_{imlp} = \frac{\left(\int_{r_{i-1}}^{r_i} r^2 R_{iml}(\lambda_{iml} r)dr\right)\left(\frac{\sqrt{\pi}.\left|\overline{\frac{2l+1}{2}}\right.}{2^{l+1}.\left(\left|\overline{l+1}\right.\right)^2}\right)\left(\frac{1}{p}\left(1-(-1)^p\right)\right)}{\left(\int_{r_{i-1}}^{r_i} r^2 R_{iml}^2(\lambda_{iml} r)dr\right)\left(\frac{\left|\overline{2l+1}\right.\sqrt{\pi}.\left|\overline{\frac{2l+1}{2}}\right.}{2^{2l+1}\left(\left|\overline{l+1}\right.\right)^3}\right)\frac{\pi}{2}}$$

$$D_{imlp} = \frac{\left(\int_{r_{i-1}}^{r_i} r^2 R_{iml}(\lambda_{iml} r)dr\right)\left(\frac{2^l\Gamma(l+1)}{\Gamma(2l+1)}\right)\left(\frac{1}{p}\left(1-(-1)^p\right)\right)}{\left(\int_{r_{i-1}}^{r_i} r^2 R_{iml}^2(\lambda_{iml} r)dr\right)\frac{\pi}{2}} \tag{74}$$

5. Solution of Inhomogeneous Steady State Problem

And in steady state solution for this particular problem can easily be obtained as,

$$T_{ss,i}(r,\theta,\phi) = \sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\sum_{p=1}^{\infty} \hat{T}_{iml}(r) P_{il}^p(\cos\theta)\sin p\phi$$

And $g_i(r,\theta,\phi) = \sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\sum_{p=1}^{\infty} g_{iml}(r) P_{il}^p(\cos\theta)\sin p\phi$

$$g_{iml}(r) = \frac{g_i(r,\theta,\phi)\left(\int_0^{\frac{\pi}{2}} \left(P_{il}^p(\cos\theta)\right) d\theta\right)\left(\int_0^{\pi} \sin(p\phi)d\phi\right)}{\left(\int_0^{\frac{\pi}{2}} \left(P_{il}^p(\cos\theta)\right)^2 d\theta\right)\left(\int_0^{\pi} \sin^2(p\phi)d\phi\right)}$$

$$g_{iml}(r) = \frac{g_i(r,\theta,\phi)\left(\frac{2^l\Gamma(l+1)}{\Gamma(2l+1)}\right)\left(\frac{1}{p}\left(1-(-1)^p\right)\right)}{\frac{\pi}{2}} \tag{75}$$

$$g_{iml}(r) = b_{il} g_i(r,\theta,\phi) \tag{76}$$

Where

$$b_{il} = \frac{\left(\frac{2^l\Gamma(l+1)}{\Gamma(2l+1)}\right)\left(\frac{1}{p}\left(1-(-1)^p\right)\right)}{\frac{\pi}{2}}$$

$$\hat{T}_{iml}(r) = a_{ss,i} r^{\frac{-1+\sqrt{1+4\beta_{in}^2}}{2}} + b_{ss,i} r^{\frac{-1-\sqrt{1+4\beta_{in}^2}}{2}} + \frac{1}{k_i\sqrt{1+4\beta_{in}^2}}\left[r^{\frac{-1+\sqrt{1+4\beta_{in}^2}}{2}}\int r^{\frac{5-\sqrt{1+4\beta_{in}^2}}{2}} g_{iml}(r)dr + r^{\frac{-1-\sqrt{1+4\beta_{in}^2}}{2}}\int r^{\frac{5+\sqrt{1+4\beta_{in}^2}}{2}} g_{iml}(r)dr\right]$$

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