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Cototal Domination Number of a Zero Divisor Graph

Research Article

K.Ananthi [1] and N.Selvi [2*]

1 Department of Mathematics, Bharathidasan University, Tiruchirapalli, Tamilnadu, India.

2 Department of Mathematics, ADM College for Women, Nagapattinam, Tamilnadu, India.

Abstract: Let R be a commutative ring and let Z(R) be its set of zero-divisors. We associate a graph $\Gamma(R)$ to R with vertices $Z(R)^* = Z(R) - \{0\}$, the set of non-zero zero divisors of R and for distinct $u, v \in Z(R)^*$, the vertices u and v are adjacent if and only if $uv = 0$. A dominating set D of G is a total dominating set if the induced subgraph of $\langle D\rangle$ contains no isolated vertices. The total domination number $\gamma_t(G)$ of G is the minimum cardinality of a total dominating set. A dominating set D of G is a cototal dominating set if every vertex $v \in V - D$ is not an isolated vertex in $\langle V - D\rangle$. The cotatal domination number $\gamma_{ct}(G)$ of G is the minimum cardinality of a cototal dominating set. In this paper, we evaluate the cototal domination number of $\Gamma(Z_n)$.

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1. Introduction

Let R be a commutative ring and let Z(R) be its set of zero-divisors. The zero-divisor graph of a ring is the graph(simple) whose vertex set is the set of non-zero zero-divisors, and an edge is drawn between two distinct vertices if their product is zero. Throughout this paper, we consider the commutative ring by R and zero divisor graph $\Gamma(R)$ by $\Gamma(Z_n)$. The idea of a zero-divisor graph of a commutative ring was introduced by I. Beck in [1, 2], where he was mainly interested in colorings. The zero divisor graph is very useful to find the algebraic structures and properties of rings. Domination is one of the most important area of graph theory with an extensive research activity. Domination came into existence from a chess problem in 1850s. Consider a chess board on which a queen can move any number of squares vertically, horizontally or diagonally. Many chess players were interested to find the minimum number of queens such that every square on the chess board either contains a queen or is attacked by a queen. This problem can be modelled on a graph by considering each square as a vertex and an edge connecting two vertices if and only if the corresponding squares are separated by any number of squares vertically, horizontally or diagonally. Now the minimum number of queens represents a dominating set which was latter found that the dominating set contains five queens. In 1862, the chess master C.V.de Jaenisch wrote "On the Application of Mathematical Analysis to Chess" in which he said the number of queens neccesary to attack each square on a n x n chess board.

Domination has several applications in other fields. Domination arrises in a facility locating problem so that a person who needs the facility (like hospiltal, fire station etc.,) can utilise it with in a minimum distance. This concept will also appears

* *E-mail: anakrish.maths@gmail.com*

in the problem involving electrical network, land surveying, communication network etc,. In 1998 T. W. Haynes, S. T. Hedetniemi and P. J. Slater wrote a book on domination " Fundamentals of domination in graphs" which listed 1222 papers in this area [3]. There are more than 75 variations of dominations [4, 5]. These variations are mainly formed by imposing additional conditions on $D, V(G) - DorV(G)$ but we focus mainly on the Cototal domination of a zero divisor graph.

2. Definitions

We present the definitions of some basic terms here and introduce other when necessary.

Definition 2.1. *A Dominating set is a set of vertices such that each vertex of V is either in D or as atleast one neighbour in D. The minimum cardinality of such a set is called the domination number of G denoted by $\gamma(G)$.*

Definition 2.2. *A dominating set D of G is a total dominating set if the induced subgraph of $\langle D\rangle$ contains no isolated vertices. The total domination number $\gamma_t(G)$ of G is the minimum cardinality of a total dominating set.*

Definition 2.3. *A dominating set D of G is a cototal dominating set if every vertex $v\varepsilon V - D$ is not an isolated vertex in $\langle V - D\rangle$. The cototal domination number $\gamma_{ct}(G)$ of G is the minimum cardinality of a cototal dominating set.*

3. Cototal Domination Number of a Zero Divisor Graph

In this section, we study the cototal domination number of some zero divisor graphs

Theorem 3.1. *For any prime $p > 2, \gamma_{ct}(\Gamma(Z_{2p})) = p$ if and only if $\Gamma(Z_{2p})$ is a star graph.*

Proof. The vertex set V of $\Gamma(Z_{2p})$ is $\{2, 4, 6, ..., 2(p-1), p\}$. Clearly the vertex p is adjacent to all the remaining vertices in $\Gamma(Z_{2p})$. Also $d(p) > d(q)$, where q is any vertex in $\Gamma(Z_{2p}) - p$. That is $d(p) = p - 1$ and remaining vertices in $\Gamma(Z_{2p})$ are non adjacent. Hence, $\Gamma(Z_{2p})$ is a star graph.

Case (i): First let us assume that $\Gamma(Z_{2p})$ is a star graph. Now let us show that $\gamma_{ct}(\Gamma(Z_{2p})) = p, where\ p > 2$ is any prime number. Since $\Gamma(Z_{2p})$ is a star graph a vertex p is adjacent to all the other vertices of V. Then the dominating set D contains the vertex p. Moreover the induced subgraph of $\langle V - D\rangle$ contains no isolated vertices. Then D will definitely contain all the pendent vertice. That is $|D|$ contains the sum of the vertex p and the number of pendent vertices. Therefore $|D| = 1 + p - 1 = p$. Then $\gamma_{ct}(\Gamma(Z_{2p})) = p$.

Case (ii): Suppose $\gamma_{ct}(\Gamma(Z_{2p})) = p$. To prove that $\Gamma(Z_{2p})$ is a star graph. Since a vertex p is adjacent to all the vertices in $\Gamma(Z_{2p})$ but no other vertices in $\Gamma(Z_{2p})$ are adjacent. Hence, $\Gamma(Z_{2p})$ is a star graph.

Using case(i) and case(ii) $\Gamma(Z_{2p})$ is a star graph iff $\gamma_{ct}(\Gamma(Z_{2p})) = p$. □

Theorem 3.2. *For any prime $p > 4, \gamma_{ct}(\Gamma(Z_{4p})) = p + 1$.*

Proof. The vertex set V of $\Gamma(Z_{4p})$ is $\{2, 4, 6, ..., 2(2p-1), p, 2p, 3p\}$. The vertex set V can be partitioned into three parts namely V_1 containing pendent vertices,V_2 containing multiples of 4 and V_3 containing multiples of p. Clearly $2p$ is adjacent to all the vertices in $V_1 and V_2$. Then the dominating set D will definitely contain $2p$ [6,7]. Moreover D contains all the pendent vertices Since the induced subgraph of $\langle V - D\rangle$ contains no isolated vertices. Now $|D|$ contains the sum of the vertex $2p$ and the number of pendent vertices. Therefore $|D| = 1 + p - 1 = p$. Also any element of the vertex set V_2 is adjacent to the vertices p and $3p$. As a result it arise to a complete bipartite graph $K_{2,2(p-1)}$. Then any one element of the vertex set V_2 dominates the vertex set V_3. Now the dominating set is added by one more element. That is $|D| = p + 1$. Moreover the induced subgraph of $\langle D\rangle$ has no isolated vertices. then $\gamma_{ct}(\Gamma(Z_{4p})) = p + 1$. □

Theorem 3.3. *For any prime $p > 6, \gamma_{ct}(\Gamma(Z_{6p})) = 2p + 1$.*

Proof. The vertex set V of $\Gamma(Z_{6p})$ is $\{p, 2p, 3p, ..., 5p, 2, 4, ..., 2(3p - 1), 3, 6, ..., 3(2p - 1)\}$. The vertex set V can be partitioned into four parts namely V_1 multiples of p,V_2 containing multiples of 6,V_3 containing multiples of 3 other than V_2 and V_4 containing multiples of 2 other than V_2 and V_3.Moreover V_4 are all pendent vertices of $\Gamma(Z_{6p})$. That is $|V_4| = 2(p-1)$. The vertex $3p$ is adjacent to all the vertices V_4 and V_2 also it is adjacent to $2p$ and and $4p$. Clearly the dominating set D contains $3p$. Then the induced subgraph $\langle V - D\rangle$ has isolated vertices since the vertex set V_4 are all pendent vertices. Now D contains the vertex set $3p$ as well as all the vertices of V_4. Then $|D|$ is sum of the vertex $3p$ and the number of pendent vertices. That is $|D| = 1 + 2(p - 1) = 2p - 1$. Also the vertex set V_3 are only adjacent to the vertices $2p$ and $4p$. Clearly either $2p$ or $4p$ dominates V_3. Similarly the vertex set V_2 are adjacent to the vertices p and $5p$. Then any one vertices of V_2 dominates the vertices p and $5p$. Now two new vertices are added to the dominating set D. That is $|D| = 2p-1+2 = 2p+1$. Moreover the induced subgraph $\langle V - D\rangle$ has no isolated vertices. Then $\gamma_{ct}(\Gamma(Z_{6p})) = 2p + 1$. □

Theorem 3.4. *For any prime $p > 8, \gamma_{ct}(\Gamma(Z_{8p})) = 2p$.*

Proof. The vertex set V of $\Gamma(Z_{8p})$ is $\{2, 4, 6, ..., 2(4p - 1), p, 2p, ..., 7p\}$. The vertex set V can be partitioned into four parts namely V_1 containing multiples of 8, V_2 containing multiples of 4 other than V_1, V_3 containing multiples of 2 other than V_1 and V_2 and V_4 containing multiples of p. Clearly $4p$ is adjacent to V_1, V_2 and V_3 as well as $2p$ and $6p$. Therefore the dominating set D contains $4p$. Moreover $\Gamma(Z_{8p})$ has $2(p - 1)$ pendent vertices. Then definitely D must contain $2(p - 1$ pendent vertices since the induced subgraph of $\langle V - D\rangle$ has no isolated vertices. Then $|D|$ is sum of the vertex $4p$ and the number of pendent vertices. That is $|D| = 1 + 2(p - 1) = 2p - 1$. Moreover the vertex set $\{p, 3p, 5p, 7p\}$ are adjacent to all the vertices of V_1. Then any one vertices of V_1 dominates $\{p, 3p, 5p, 7p\}$. Now the dominating set D is increased by one more vertices of V_1. That is $|D| = 2p - 1 + 1 = 2p$. Then $\gamma_{ct}(\Gamma(Z_{8p})) = 2p$. □

Theorem 3.5. *For any prime $p > 2, \gamma_{ct}(\Gamma(Z_{p^2})) = 1$.*

Proof. The vertex set V of $\Gamma(Z_{p^2})$ is $\{p, 2p, 3p, ..., p(p - 1)\}$. Any two vertices of V are adjacent. Clearly $\Gamma(Z_{p^2})$ is a complete graph K_{p-1}. Then any one vertices of V will dominate the other vertices. Moreover the induced subgraph of $\langle V - D\rangle$ has no isolated vertices. That is $|D| = 1$. Then $\gamma_{ct}(\Gamma(Z_{p^2})) = 1$. □

Theorem 3.6. *For any prime $p > 2, \gamma_{ct}(\Gamma(Z_{2p^2})) = p^2 - p + 2$.*

Proof. The vertex set V of $\Gamma(Z_{2p^2})$ is $\{2, 4, 6, ..., 2(p^2 - 1), p, 2p, 3p, ..., p(2p - 1)\}$. The vertex set V can be partitioned into three parts namely V_1 containing pendent vertices, V_2 containing odd multiples of p and V_3 containing even multiples of p. The vertex p^2 of V_2 is adjacent to all the vertices of V_1 and V_3. Then the dominating set D will definitely contain p^2. Moreover D contains all the pendent vertices since the induced subgraph of $\langle V - D\rangle$ must contain no isolated vertices. Now $|D|$ is sum of the vertex p^2 and the number of pendent vertices. That is $|D| = 1 + p(p - 1)$. Also the remaining vertices of V_2 other than p^2 is adjacent to all the vertices of V_3 which forms a complete bipartite graph. Therefore any one vertices in V_3 dominates the vertices of V_2. Therefore $|D| = 1 + p(p - 1) + 1 = p^2 - p + 2$. Moreover the induced subgraph of $\langle V - D\rangle$ has no isolated vertices. Then $\gamma_{ct}(\Gamma(Z_{2p^2})) = p^2 - p + 2$. □

Theorem 3.7. *For any prime $p > 3, \gamma_{ct}(\Gamma(Z_{3p^2})) = 2$.*

Proof. The vertex set V of $\Gamma(Z_{3p^2})$ is $\{p, 2p, ..., p(3p-1), 3, 6, ..., 3(P^2-1)\}$. The vertex set V can be partitioned into four parts namely V_1 containing multiples of 3, V_2 containing the vertices $\{p^2, 2p^2\}$, V_3 containing the vertices $\{3p, 6p, ..., 3(p-1)\}$ and V_4 containing multiples of p other than V_2 and V_3. Any vertices of V_2 are adjacent to all the vertices of V_1. Then any one vertices of V_2 will be in the dominating set D. Similarly we can find any vertices of V_3 adjacent to all the vertices of V_4. Therefore any one vertices of V_3 will also belong to the dominating set D. Also $\Gamma(Z_{3p^2})$ is a connected graph as the vertex $3p$ of the vertex set V_3 is adjacent to all the vertices of V_2. Moreover the induced subgraph of $\langle V - D\rangle$ has no isolated vertices. Therefore the dominating set D contains any one vertices of V_2 any any one vertices of V_3. That is $|D| = 1 + 1 = 2$. Then $\gamma_{ct}(\Gamma(Z_{3p^2})) = 2$. □

Theorem 3.8. *For any positive integer $n > 2, \gamma_{ct}(\Gamma(Z_{2^n})) = 2^{n-2} + 1$.*

Proof. The vertex set V of $\Gamma(Z_{2^n})$ is $\{2, 4, 6, ..., 2(2^{n-1} - 1)\}$. The vertex 2^{n-2} is adjacent to all the remaining vertices of V. Then we have $deg(2^{n-2}) = 2^{n-1} - 2$. Then the vertex must definitely lie on the dominating set D. Moreover there are 2^{n-2} pendent vertices in $\Gamma(Z_{2^n})$. Now the induced subgraph of D contains 2^{n-2} isolated vertices. Therefore 2^{n-2} isolated vertices will also lie on the dominating set D. Now the induced subgraph of $\langle V - D\rangle$ has no isolated vertices. Therefore $|D|$ is sum of the vertex 2^{n-2} and the number of pendent vertices. That is $|D| = 1 + 2^{n-2}$. Then $\gamma_{ct}(\Gamma(Z_{2^n})) = 2^{n-2} + 1$. □

Theorem 3.9. *For any positive integer $n > 3, \gamma_{ct}(\Gamma(Z_{3^n})) = 1$.*

Proof. The vertex set V of $\Gamma(Z_{3^n})$ is $\{3, 6, ..., 3(3^{n-1} - 1)\}$. The vertex can be partitioned into parts two parts namely V_1 contains the vertex set $\{3^{n-1}, 2.3^{n-1}\}$ and V_2 containing the vertex set V other than V_1. Clearly any vertices of V_1 are adjacen to all the vertices of V_2 which implies $\Gamma(Z_{3^n})$ is a complete bipartite graph $K_{2,3^{n-1}-3}$. Moreover the vertex 3^{n-1} and the vertex 2.3^{n-1} are adjacent to each other. Then the dominating set D contains any one of the vertices of V_1 since the induced subgraph $\langle V - D\rangle$ has no isolated vertices. Therefore the dominating set D contains any one vertices of V_1. That is $\gamma_{ct}(\Gamma(Z_{3^n})) = 1$. □

Theorem 3.10. *For any two distinct prime p and q such that $q > p$ and both p and q are > 2, then $\gamma_{ct}(\Gamma(Z_{pq})) = 2$.*

Proof. The vertex set V of $\Gamma(Z_{pq})$ is $\{p, 2p, ...p(q - 1), q, 2q, ...q(p - 1)\}$. The vertex set V can be partitioned into two parts namely V_1 containing multiples of p , V_2 containing multiples of q. Clearly all the vertices of V_1 are adjacent to all the vertices of V_2 which is a complete bipartite graph $K_{p-1,q-1}$. Then any one of the vertices of V_1 dominates all the vertices of V_2 and similarly any one of the vertices of V_2 dominates all the vertices of V_1. Clearly D contains only two vertices each from V_1 and V_2. That is $|D| = 2$. Moreover the induced subgraph $\langle V - D\rangle$ has no isolated vertices. Then $\gamma_{ct}(\Gamma(Z_{pq})) = 2$. □

4. Conclusion

In this paper, we study the cototal domination number of some zero divisor graphs namely, $2p, 4p, 6p, 8p$ and also for $p^2, 2p^2, 3p^2$ and $2^n, 3^n, pq$. Graphs are the most ubiquitous models of both natural and human made structures. In computer science, zero divisor graphs are used to represent networks of communication, network flow, clique problems.

References

[1] D.F.Anderson and P.S.Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra, 217(2)(1999), 434-447.

[2] I.Beck, *Coloring of Commutative Rings*, J. Algebra, 116(1988), 208-226.

[3] T.N.Haynes, S.T.Hedetniemi and P.J.Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc., Newyork, (1998).

[4] V.R.Kulli, B.Janakiram and Radha R.Iyer, *The cototal domination number of a graph*, Journal of Discrete Mathematical Sciences and Cryptography, 2(2-3)(1999), 179-184.

[5] E.Sampathkumar and H.B.Walikar, *The connected Domination Number of a Graph*, Journal of Math. Phy. Science, 13(6)(1979).

[6] J.Ravi Sankar and S.Meena, *Changing and unchanging the Domination Number of a commutative ring*, International Journal of Algebra, 6(27)(2012), 1343-1352.

[7] J.Ravi Sankar and S.Meena, *Connected Domination Number of a commutative ring*, International Journal of Mathematical Research, 5(1)(2012), 5-11.