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Skew Semi-Heyting Almost Distributive Lattices

Research Article

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Abstract: In this paper the concept of skew semi-Heyting almost distributive lattice is introduced. We characterize skew semi-Heyting almost distributive lattice interms of a congruence relation and we show that the quotient ADL of a skew semi-Heyting almost distributive lattice modulo the congruence relation is the maximal lattice image of the ADL.

Keywords: Almost distributive lattice (ADL), semi-Heyting almost distributive lattice(SHADL), semi-Heyting algebra, skew semi-Heyting algebra and skew semi-Heyting almost distributive lattice(skew SHADL).

1. Introduction

Modern theory of skew lattices is introduced by Jonathan Leech [6]. Leech [7, 8] showed that each right handed skew Boolean algebra can be embedded in to a generic skew Boolean algebra of partial functions from a given set to the co-domain $\{0, 1\}$. Heyting algebra is a relatively pseudo-complemented distributive lattice which arises from non-classical logic. It was first investigated by T. Skolem about 1920 [5] and introduced by G. Birkhoff [1]. While Boolean algebras provide algebraic models of classical logic, Heyting algebras provide algebraic models of intuitionistic logic. The notion of skew Heyting algebra was introduced by Karin Cvetko-vah [9]. In that paper it is proved that a Heyting algebras form a variety and that the maximal lattice image of a skew Heyting algebra is a generalized Heyting algebra. H.P. Sankappanavar [10] in 2007 define and investigate a new class of algebras, which is called Semi-Heyting Algebras, as an abstraction from Heyting algebras. He showed that semi-Heyting algebras share with Heyting algebras some strong properties, like these algebras are: pseudocomplemented, distributive, congruences on them are determined by filters and every interval in a semi-Heyting algebra is also pseudocomplemented.

The concept of an Almost Distributive Lattice was introduced by U.M.Swamy and G.C.Rao [11] as common abstraction to most of the existing ring theoretic generalization of a Boolean algebra and distributive lattices. G.C.Rao, Berhanu Assaye and M.V.Ratnamani in [3] introduced Heyting Almost Distributive Lattices as a generalization of Heyting algebra in the class of ADLs and they characterize an HADL in terms of the set of all of its principal ideals. Following this G.C.Rao, M.V.Ratnamani, K. P. Shum and Berhanu Assaye [4] introduce the concept of semi-Heyting almost distributive lattice as a generalization of a semi-Heyting algebra in the class of ADLs. In this paper we introduce and characterize two interrelated new concepts. Section two of our paper describes preliminary concepts which can be used in proving lemmas, theorems and corollaries in the sequel. In the third section we introduce and characterize the concept of skew semi-Heyting almost distributive lattices after we introduce and characterize the concepts of skew semi-Heyting Algebras.

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2. Preliminaries

In this section we give the necessary definitions and results on almost distributive lattices, Heyting algebras, semi Heyting algebras, skew lattices and skew Heyting algebras which will be used in the next section.

Definition 2.1 ([11]). *An algebra $(L, \vee, \wedge, 0)$ of type $(2, 2, 0)$ is called an Almost Distributive Lattice(ADL) with 0 if it satisfies the following axioms: for all $x, y, z \in L$*

(1) $x \vee 0 = x$

(2) $0 \wedge x = 0$

(3) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$

(4) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

(5) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

(6) $(x \vee y) \wedge y = y$.

Theorem 2.2 ([2]). *Let L be an ADL with 0. Then for any $w, x, y, z \in L$, we have the following.*

(1) $x \vee y = x \Leftrightarrow x \wedge y = y$

(2) $x \vee y = y \Leftrightarrow x \wedge y = x$

(3) $x \wedge y = y \wedge x = x$ *whenever* $x \leq y$

(4) $\wedge$ *is associative*

(5) $x \wedge y \wedge z = y \wedge x \wedge z$

(6) $(x \vee y) \wedge z = (y \vee x) \wedge z$

(7) $x \wedge y \leq y$ *and* $x \leq x \vee y$

(8) $x \wedge (y \wedge x) = y \wedge x$ *and* $x \vee (x \vee y) = x \vee y = (x \vee y) \vee y$

(9) $x \wedge x = x$ *and* $x \vee x = x$

(10) $x \wedge 0 = 0$ *and* $0 \vee x = x$

(11) $(w \vee (x \vee y)) \wedge z = ((w \vee x) \vee y) \wedge z$

(12) *If* $x \leq z$ *and* $y \leq z$*, then* $x \wedge y = y \wedge x$ *and* $x \vee y = y \vee x$.

An element m in an ADL L is said to be maximal if for each $a \in L, m \leq a$ implies that $a = m$.

Theorem 2.3 ([2]). *Let L be an ADL. For any $m \in L$ the following are equivalent.*

(1) m *is maximal element*

(2) $m \vee x = m$ *for all* $x \in L$

(3) $m \wedge x = x$ *for all* $x \in L$

Definition 2.4 ([10]). *An algebra $(L, \vee, \wedge, \rightarrow, 0, 1)$ of type $(2, 2, 2, 0, 0)$ is called a Heyting algebra if it satisfies the following axioms:*

(1) $(L, \vee, \wedge, 0, 1)$ *is a lattice with 0 and 1*

(2) $x \wedge (x \rightarrow y) = x \wedge y$

(3) $x \wedge (y \rightarrow z) = x \wedge ((x \wedge y) \rightarrow (x \wedge z))$

(4) $(x \wedge y) \rightarrow x = 1$

for all $x, y, z \in L$.

Definition 2.5 ([3]). *Let $(L, \vee, \wedge, 0, m)$ be an ADL with 0 and a maximal element m. Suppose $\rightarrow$ is a binary operation on L satisfying the following conditions.*

(1) $x \rightarrow x = m$

(2) $(x \rightarrow y) \wedge y = y$

(3) $x \wedge (x \rightarrow y) = x \wedge y \wedge m$

(4) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$

(5) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$

for all $x, y, z \in L$. Then $(L, \vee, \wedge, \rightarrow, 0, m)$ is called a Heyting Almost Distributive Lattice(HADL).

Theorem 2.6 ([3]). *Let L be an ADL with 0 and a maximal element m, then the following are equivalent.*

(1) L is an HADL

(2) $[0, a]$ *is a Heyting algebra for all* $a \in L$

(3) $[0, m]$ *is a Heyting algebra.*

Lemma 2.7 ([3]). *Let L be a Heyting algebra, then an equivalence relation θ on L is a congruence relation if and only if for any $(a, b) \in \theta, d \in L$,*

(1) $(a \wedge d, b \wedge d) \in \theta$

(2) $(a \vee d, b \vee d) \in \theta$

(3) $(a \rightarrow d, b \rightarrow d) \in \theta$

(4) $(d \rightarrow a, d \rightarrow b) \in \theta$.

Definition 2.8 ([10]). *An algebra $(L, \vee, \wedge, \rightarrow, 0, 1)$ is a semi-Heyting algebra if the following conditions hold:*

(SH1) $(L, \vee, \wedge, 0, 1)$ *is a lattice with* 0 *and* 1

(SH2) $x \wedge (x \rightarrow y) = x \wedge y$

(SH3) $x \wedge (y \rightarrow z) = x \wedge ((x \wedge y) \rightarrow (x \wedge z))$

(SH4) $x \rightarrow x = 1$.

Theorem 2.9 ([10]). *Let L be a semi-Heyting algebra. Then for any $x, y, z \in L, L$ satisfies the following conditions:*

(a) $x \wedge (y \to z) = x \wedge ((x \wedge y) \to z)$

(b) $x \wedge (y \to z) = x \wedge (y \to (x \wedge z))$.

Definition 2.10 ([4]). *Let $(L, \vee, \wedge, 0, m)$ be an ADL with a maximal element m. Suppose there exists a binary operation $\to$ on L satisfying the following conditions:*

(1) $(x \to x) \wedge m = m$

(2) $x \wedge (x \to y) = x \wedge y \wedge m$

(3) $x \wedge (y \to z) = x \wedge ((x \wedge y) \to (x \wedge z))$

(4) $(x \to y) \wedge m = (x \wedge m) \to (y \wedge m)$

for all $x, y, z \in L$. Then $(L, \vee, \wedge, \to, 0, m)$ is called a semi- Heyting ADL(SHADL).

Theorem 2.11 ([4]). *Let L be an ADL with a maximal element m. Then the following are equivalent:*

(1) L is a SHADL

(2) $[0, a]$ is a semi-Heyting algebra for all $a \in L$

(3) $[0, m]$ is a semi-Heyting algebra.

Definition 2.12 ([6]). *A skew lattice is an algebra $\mathbf{L} = (L; \wedge, \vee)$ of type $(2, 2)$ such that $\wedge$ and $\vee$ are both idempotent and associative, and they satisfy the following absorption laws:*
$x \wedge (x \vee y) = x = x \vee (x \wedge y)$ and $(x \wedge y) \vee y = y = (x \vee y) \wedge y$ for all $x, y \in L$.

The natural partial order can be defined on a skew lattice $\mathbf{L}$ by stating that $x \leq y$ if and only if $x \vee y = y = y \vee x$, or equivalently $x \wedge y = x = y \wedge x$ for $x, y \in L$ (see [9]).

Definition 2.13 ([9]). *A skew lattice is called strongly distributive if for all $x, y, z \in L$ it satisfies the following identities: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$; and it is called co-strongly distributive if it satisfies the identities: $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$.*

Definition 2.14 ([8]). *A skew Lattice L is called normal if $w \wedge x \wedge y \wedge z = w \wedge y \wedge x \wedge z$ and it is called conormal if $w \vee x \vee y \vee z = w \vee y \vee x \vee z$ for all $w, x, y, z \in L$.*

Definition 2.15 ([9]). *An algebra $\mathbf{L} = (L; \vee, \wedge, \to, 1)$ of type $(2, 2, 2, 0)$ is said to be a skew Heyting algebra whenever the following conditions are satisfied:*

(1) $(L; \vee, \wedge, 1)$ is a co-strongly distributive skew lattice with top 1.

(2) For any $a \in L$, an operation $\to_a$ can be defined on $a\uparrow = \{a \vee x \vee a | x \in L\}$ such that $(a\uparrow, \vee, \wedge, \to_a, 1, a)$ is a Heyting algebra with top 1 and bottom a.

(3) An induced binary operation $\to$ from $\to_a$ is defined on L by $x \to y = (y \vee x \vee y) \to_y y$.

Lemma 2.16 ([9]). *Let $(L, \vee, \wedge, \to, 1)$ be a skew Heyting algebra and let $x, y, a \in L$ be such that $x, y \in a\uparrow$ hold. Then $x \to y = x \to_a y$.*

3. Skew Semi-Heyting Almost Distributive Lattices

First we introduce the concept of skew semi-Heyting algebras, investigate some algebraic properties and characterize it as a skew Heyting algebra in terms of a binary operation on which an induced binary operation is defined on the underlying set. Through out this section L stands for a non empty set and we will use the following notations:

(i) For any $a \in L$ and a partial ordering $\leq$ on L, the set $\{x \in L | a \leq x \leq b\}$ is denoted by $[a, b]$,

(ii) If L contains 0 for any $a \in L$, $\rightarrow_a$ is the binary operation defined on $[0, a]$,

(iii) For any $b, c \in L$, ${}_b\!\rightarrow$ is the binary operation defined on $[b, c]$.

Definition 3.1. *An algebra $(L; \vee, \wedge, \rightarrow, 1)$ of type $(2, 2, 2, 0)$ is said to be a skew semi-Heyting algebra whenever the following conditions are satisfied:*

(1) $(L; \vee, \wedge, 1)$ is a co-strongly distributive skew lattice with top 1.

(2) For any $a \in L$, an operation ${}_a\!\rightarrow$ can be defined on $a\!\uparrow = \{x \in L | a \leq x\}$ such that $(a\!\uparrow, \vee, \wedge, {}_a\!\rightarrow, 1, a)$ is a semi-Heyting algebra with top 1 and bottom a.

(3) An induced binary operation $\rightarrow$ from ${}_a\!\rightarrow$ is defined on L by $x \rightarrow y = (y \vee x \vee y)\, {}_y\!\rightarrow y$.

A semi Heyting algebra is not necessarily a skew semi-Heyting algebra. The following example justifies this.

Example 3.2. *Let $L = \{0, x, 1\}$ be a chain with $0 < x < 1$ which is the lattice reduct of the two semi-Heyting algebras with the binary operation $\rightarrow$ defined on L by the tables given below. It can be routinely verified that the algebra $(L, \vee, \wedge, \rightarrow, 1)$ where the binary operation $\rightarrow$ defined by Table 1 is a skew semi-Heyting algebra. But $(L, \vee, \wedge, \rightarrow, 1)$ where the binary operation $\rightarrow$ defined by Table 2 is not a skew semi-Heyting algebra, because when we apply the definition of $\rightarrow$ in Table 2 we obtain that $0 \rightarrow x = (x \vee 0 \vee x)\, {}_x\!\rightarrow x \Rightarrow 0 \rightarrow x = x\, {}_x\!\rightarrow x \Rightarrow x = 1$ which is impossible.*

$\rightarrow$	0	x	1
0	1	1	1
x	0	1	1
1	0	x	1

Table 1. **(skew semi-Heyting algebra)**

$\rightarrow$	0	x	1
0	1	x	1
x	0	1	1
1	0	x	1

Table 2. **(Not a skew semi-Heyting algebra)**

Next we present some useful arithmetical properties of skew semi-Heyting algebras.

Theorem 3.3. *Let L be a skew semi-Heyting algebra and $x, y, z \in L$. Then the following hold:*

(a) $1 \rightarrow x = x$

(b) $x \rightarrow 1 = 1$

(c) $x \leq y \Rightarrow x \rightarrow y = 1$

(d) $y \leq x \rightarrow y$

(e) $(x \vee y) \rightarrow x = y \rightarrow x$

(f) $x \leq y \leq z \Rightarrow y \leq x \rightarrow z$

(g) $(x \wedge y) \rightarrow x = 1$

(h) $x \rightarrow (y \rightarrow x) = y \rightarrow (x \rightarrow y)$

Proof. (a) and (b) hold trivially.

(c) Suppose $x \leq y$. Then $x \rightarrow y = (y \vee x \vee y)_y\!\rightarrow y = 1$

(d) $y \wedge (x \rightarrow y) = y \wedge ((y \vee x \vee y)_y\!\rightarrow\! y) = y \wedge \{(y \wedge (y \vee x \vee y))_y\!\rightarrow\! y\} = y \wedge (y_y\!\rightarrow\! y) = y \wedge 1 = y$. Moreover, since $y \vee x \vee y$ and y belongs to $y\uparrow$, $((y \vee x \vee y)_y\!\rightarrow\! y) \wedge y = y$

(e) $(x \vee y) \rightarrow x = (x \vee (x \vee y) \vee x)_x\!\rightarrow\! x = (x \vee y \vee x)_x\!\rightarrow\! x = y \rightarrow x$.

(f) Let $x \leq y \leq z$. Then $y \wedge (x \rightarrow z) = y \wedge ((z \vee x \vee z)_z\!\rightarrow\! z) = y \wedge z \wedge ((z \vee x \vee z)_z\!\rightarrow\! z) = y \wedge z \wedge (z_z\!\rightarrow\! z) = y \wedge z = y$. Clearly, $(x \rightarrow z) \wedge y = y$

(g) $(x \wedge y) \rightarrow x = (x \vee (x \wedge y) \vee x)_x\!\rightarrow\! x = x\ {}_x\!\rightarrow\! x = 1$.

(h) Apply (c) and (d). □

Lemma 3.4. *Let L be a skew semi-Heyting algebra. Then for any $a \in L$, $a\uparrow$ is a distributive lattice.*

Proof. Suppose L be a skew semi-Heyting algebra. Let $a \in L$. Then $a\uparrow$ is a semi-Heyting algebra. Hence $a\uparrow$ is a distributive lattice. □

Next we give a characterization of skew semi-Heyting algebra.

Theorem 3.5. *Let L be a skew semi-Heyting algebra and $a \in L$ such that $x, y, z \in a\uparrow$. Then the following are equivalent:*

(a) L *is a skew Heyting algebra*

(b) $x \leq y \Rightarrow x\ {}_a\!\rightarrow y = 1$

(c) $x \leq y \Rightarrow y\ {}_a\!\rightarrow z \leq x\ {}_a\!\rightarrow z$

(d) $(x \vee y)_a\!\rightarrow\! z = (x\ {}_a\!\rightarrow z) \wedge (y\ {}_a\!\rightarrow z)$.

Proof. Suppose L be a skew Heyting algebra. Let $x, y, z \in a\uparrow$. Clearly $a\uparrow$ is a skew Heyting algebra. If $x \leq y$, then $x\ {}_a\!\rightarrow y = (y \vee x \vee y)_y\!\rightarrow\! y = y\ {}_y\!\rightarrow y = 1$. Thus (a)⇒(b) holds. Using the fact that $z\uparrow$ is a Heyting algebra and L is regular, whenever $x \leq y$ we have

$$\begin{aligned}(y\ {}_a\!\rightarrow z) \wedge (x\ {}_a\!\rightarrow z) &= ((z \vee y \vee z)_z\!\rightarrow\! z) \wedge ((z \vee x \vee z)_z\!\rightarrow\! z)\\ &= \{(z \vee y \vee z) \vee (z \vee x \vee z)\}_z\!\rightarrow\! z\\ &= (z \vee x \vee y \vee z)_z\!\rightarrow\! z\\ &= (z \vee y \vee z)_z\!\rightarrow\! z\\ &= y\ {}_a\!\rightarrow \mathrm{z}\,.\end{aligned}$$

This shows that (a)⇒(c). Like wise we have (a)⇒(d). To prove the converse, we show that for any $a \in L$, the semi-Heyting algebra $a\uparrow$ is a Heyting algebra. For this it is sufficient to show that each of the conditions (b), (c) and (d) implies that $(x \wedge y)_a\!\rightarrow\! x = 1$. Suppose (b) holds. Let $a \in L$ such that $x, y \in a\uparrow$. Then $x \wedge y = y \wedge x$ and $(y \wedge x) \vee x = x \Rightarrow y \wedge x \leq x \Rightarrow x \wedge y \leq x$. From the given assumption we obtain that $(x \wedge y)_a\!\rightarrow\! x = 1$. Now assume that (c) holds. Clearly $x, y \in$

$a\uparrow$ implies that $x \wedge y \leq x$. Then by (c), we have $x \;_a\!\!\to z \leq (x \wedge y)_a\!\!\to z$ for any $z \in L$. Taking $z = x$ we get $x \;_a\!\!\to x \leq (x \wedge y)_a\!\!\to x$. This in turn implies that $1 \leq (x \wedge y)_a\!\!\to x$. Hence $(x \wedge y)_a\!\!\to x = 1$. Finally let (d) holds. Then for any $x, y \in a\uparrow, 1 = x_a\!\!\to x = (x \vee (x \wedge y))_a\!\!\to x = (x \;_a\!\!\to x) \wedge ((x \wedge y)_a\!\!\to x) = 1 \wedge ((x \wedge y)_a\!\!\to x)$. Therefore $(x \wedge y)_a\!\!\to x = 1$. □

In the following theorem we give another characterization of a skew semi-Heyting algebra.

Theorem 3.6. *Let L be a skew semi-Heyting algebra.Then L is a skew Heyting algebra if and only if for all $b \in L$, the binary operation $_b\!\!\to$ on $b\uparrow$ is the same as the induced binary operation $\to$ on L.*

Proof. Assume L is skew Heyting algebra. Thus for any $b \in L$, $b\uparrow$ is a Heyting algebra so that it is a skew Heyting algebra. Since the binary operation $\to$ on a skew Heyting algebra is unique, $x \;_b\!\!\to y = x \to y$ for any $x, y \in b\uparrow$. Conversely suppose $x \;_b\!\!\to y = x \to y$ for any $x, y \in b\uparrow$. Since $b\uparrow$ is semi-Heyting algebra we need to prove that $(x \wedge y)_b\!\!\to x = 1$. Using (SH4) for the semi-Heyting algebra $x\uparrow$ we get ,$(x \wedge y)_b\!\!\to x = (x \wedge y)\to x = (x \vee (x \wedge y) \vee x)_x\!\!\to x = x \;_x\!\!\to x = 1$. Hence $b\uparrow$ is a Heyting algebra and therefore L is a skew Heyting algebra. □

Consider the skew semi-Heyting algebra L given by Example 3.1 whose induced binary operation $\to$ is defined by Table 1. In this example the induced binary operation $\to$ on L is the same as the binary operation $_b\!\!\to$ defined on $b\uparrow$ for each $b \in L$. Therefore this skew semi-Heyting algebra is a skew Heyting algebra. In the next theorem we give an axiomatization for a skew semi-Heyting algebra.

Theorem 3.7. *Let $(L, \vee, \wedge, \to, 0, 1)$ be an algebra of type $(2, 2, 2, 0, 0)$ such that $(L, \vee, \wedge, 0, 1)$ is a co-strongly distributive skew lattice and let $b \in L$. Then $(L, \vee, \wedge, \to, 1)$ is a skew semi-Heyting algebra if and only if the following conditions hold*

(a) $x \;_b\!\!\to x = 1$ *for all* $x \in b\uparrow$

(b) $x \wedge (x \;_b\!\!\to y) = x \wedge y$ *for all* $x, y \in b\uparrow$

(c) $x \wedge (y \;_b\!\!\to z) = x \wedge ((x \wedge y)_b\!\!\to z)$ *for all* $x, y, z \in b\uparrow$

(d) $x \wedge (y \;_b\!\!\to z) = x \wedge (y \;_b\!\!\to (x \wedge z))$ *for all* $x, y, z \in b\uparrow$

(e) $y \leq (x \to y)$ *for all* $x, y \in L$

(f) $x \to y = (y \vee x \vee y)_y\!\!\to y$ *for all* $x, y \in L$.

Proof. Assume that L is a skew semi-Heyting algebra. Then for any $b \in L$, $b\uparrow$ is a semi-Heyting algebra and hence by Theorem 2.4 (c) and (d) hold. But (a) and (b) hold directly from the definition of semi-Heyting algebra. Condition (e) and (f) is direct from the assumption. Since L is co-strongly distributive skew lattice, for any $b \in L$, $b\uparrow$ is a lattice. So to prove the converse it is enough to prove that $b\uparrow$ is a semi-Heyting algebra and for this we show that $x\wedge(y \;_b\!\!\to z) = x\wedge((x\wedge y)_b\!\!\to(x\wedge z))$. Then from (c) we have $x \wedge (y \;_b\!\!\to z) = x \wedge ((x \wedge y)_b\!\!\to z)$, and by (d) we get $x \wedge ((x \wedge y)_b\!\!\to z) = x \wedge ((x \wedge y)_b\!\!\to (x \wedge z)$. Hence $x \wedge (y \;_b\!\!\to z) = x \wedge ((x \wedge y)_b\!\!\to (x \wedge z))$. Now for $b \in L$ and $x, y, z \in b\uparrow$, set $x_b\!\!\to y = x \to y$. Clearly (e) implies that $x \to y \in y\uparrow \subseteq b\uparrow$. Thus the restriction $_b\!\!\to$ of $\to$ to $b\uparrow$ is well defined. Since $b\uparrow$ is commutative (a), (b) and, (c) and (d) for $\to$ simplify respectively to (SH4), (SH2) and (SH3) for $_b\!\!\to$ making $_b\!\!\to$ is the binary operation on $b\uparrow$. This shows that for each $b \in L$, $(b\uparrow, \vee, \wedge,_b\to, b, 1)$ is a semi-Heyting algebra. Therefore using (f) it is possible to define an induced binary operation $\to$ on L by $x \to y = (y \vee x \vee y)_y\!\!\to y$ that makes L is a skew semi-Heyting algebra. □

Theorem 3.8. *Let $(L, \vee, \wedge, \to, 1)$ be a skew semi-Heyting algebra. Then for any $x, y \in L$, the algebra $([x, y], \vee, \wedge,_x\to, y)$ is a skew semi-Heyting algebra.*

Lemma 3.9. *Let the algebra $(L, \vee, \wedge, \rightarrow, 1)$ be a skew semi-Heyting algebra with bottom 0. Then the algebra $(L, \vee, \wedge, \rightarrow, 0, 1)$ is a Heyting algebra.*

Proof. Let $a, b \in L$. Then $(a \wedge b) \rightarrow a = (a \vee (a \wedge b) \vee a)_a \rightarrow a = a\ {}_a\rightarrow a = 1$. Thus $(L, \vee, \wedge, \rightarrow, 0, 1)$ becomes a Heyting algebra. □

Next using the concepts of skew semi-Heyting algebras we introduce and characterize skew semi- Heyting almost distributive lattices(skew SHADLs).

Definition 3.10. *Let $(L; \vee, \wedge, 0, m)$ be an ADL with 0 and a maximal element m. Then an algebra $(L; \vee, \wedge, \rightarrow, 0, m)$ of type (2,2,2,0,0) is said to be a skew SHADL whenever $([0, a], \vee, \wedge, \rightarrow_a, a)$ is a skew semi-Heyting algebra for each $a \in L$.*

Example 3.11. *Take an element a in a semi-Heyting algebra L and define a binary operation $\rightarrow_a$ on $[0, a]$ by $x \rightarrow_a y = (y \vee x \vee y)_y \rightarrow y$ we obtain that $([0, a], \vee, \wedge, \rightarrow_a, a)$ is a skew semi-Heyting algebra. This is due to the fact that L is co-strongly distributive skew lattice and for any $a \in L, [0, a]$ is a semi-Heyting algebra so that $[b, a]$ is a semi-Hyeting algebra for all $b \in [0, a]$. Therefore we can deduce that L together with the binary operation $\rightarrow_a$ defined on $[0, a]$ for each $a \in L$ as above is a skew SHADL.*

Lemma 3.12. *Let L be a skew semi-Heyting algebra with 0. Then L is a skew SHADL.*

Proof. From lemma 3.7 we obtain that L is a Heyting algebra so that for all $a \in L, [0, a]$ is semi-Heyting algebra. Now take $x \rightarrow_a y = x \rightarrow y$ for all $a \in L$ and $x, y \in [0, a]$. Hence we obtain that $x \rightarrow_a y = (y \vee x \vee y)_y \rightarrow y$. This implies that $[0, a]$ is a skew semi-Heyting algebra. Therefore L is a skew SHADL. □

The following lemma is one of the consequences of lemma 3.7.

Lemma 3.13. *Let L be a skew SHADL. Then L is a semi-Heyting almost distributive lattice.*

Proof. Suppose L be a skew SHADL. Then for all $a \in L, [0, a]$ is skew semi-Heyting algebra. Thus from the consequence of lemma 3.7 we obtain that $[0, a]$ is semi-Heyting algebra. Therefore L is semi-Heyting almost distributive lattice. □

The next theorem gives another justification to our Definition 3.8 of skew SHADL.

Theorem 3.14. *Let $(L; \vee, \wedge, 0, m)$ be an ADL with 0 and a maximal element m. Then $(L; \vee, \wedge, \rightarrow, 0, m)$ is a skew SHADL if and only if*

(i) for any $b \in L, ([b, m]; \vee, \wedge, {}_b\rightarrow, b, m)$ is a SHADL

(ii) a binary operation $\rightarrow$ on L can be defined by $x \rightarrow y = ((x \vee y) \wedge m)_{(y \wedge m)} \rightarrow (y \wedge m)$.

Proof. Assume L be a skew SHADL. This implies that for any $a \in L, [0, a]$ is a skew semi-Heyting algebra. If we take $a = m$, then we get $[0, m]$ is a skew semi-Heyting algebra. Cosequently, from the definition of skew semi-Heyting algebra we have seen that; for any $b \in [0, m], ([b, m]; \vee, \wedge, {}_b\rightarrow, b, m)$ is a semi-Heyting algebra so that for any $a \in [b, m], [b, a]$ is a semi-Heyting algebra. Then by theorem 2.11 $([b, m]; \vee, \wedge, {}_b\rightarrow, b, m)$ is a SHADL. This proves (i). Since $[0, m]$ is a skew semi-Heyting algebra, the induced operation $\rightarrow_m$ on $[0, m]$ from ${}_b\rightarrow$ on $[b, m]$, is given by $x \rightarrow_m y = (y \vee x \vee y)\ {}_y\rightarrow y$. From the fact that $[0, m]$ is a semi-Heyting algebra, it is possible to define a binary operation $\rightarrow$ on L by $x \rightarrow y = (x \wedge m) \rightarrow_m (y \wedge m)$ so that $(L; \vee, \wedge, \rightarrow, 0, m)$ is a SHADL. But $(x \wedge m) \rightarrow_m (y \wedge m) = ((y \wedge m) \vee (x \wedge m) \vee (y \wedge m))_{(y \wedge m)} \rightarrow (y \wedge m) = ((x \wedge m) \vee (y \wedge m))_{(y \wedge m)} \rightarrow (y \wedge m) = ((x \vee y) \wedge m)_{(y \wedge m)} \rightarrow (y \wedge m)$, and hence $x \rightarrow y = ((x \vee y) \wedge m)_{(y \wedge m)} \rightarrow (y \wedge m)$. Conversely, suppose conditions (1) and (2) hold. Let $a \in L$. Then $[0, a]$ is a co-strongly distributive skew lattice. By (1) for any $b \in [0,a], [b,m]$ is a SHADL. Thus by theorem 2.11 $([b,a]; \vee, \wedge, _b\!\rightarrow, b, a)$ is a semi-Heyting algebra. Now define $\rightarrow_a$ on $[0,a]$ by $x \rightarrow_a y = x \rightarrow y$. Using (ii) we obtain that $x \rightarrow_a y = ((x \vee y) \wedge a)_{(y \wedge a)}\!\rightarrow\!(y \wedge a)$. But $((x \vee y) \wedge a)_{(y\wedge a)}\!\rightarrow\!(y \wedge a) = ((y \vee x \vee y) \wedge a)_{(y\wedge a)}\!\rightarrow\!(y \wedge a) = (y \vee x \vee y)\ _y\!\rightarrow y$. Therefore $([0,a]; \vee, \wedge, \rightarrow_a, 0, a)$ is a skew semi-Heyting algebra and hence L is a skew SHADL. □

Corollary 3.15. *Let L be a skew SHADL. Then for any $a \in L, [a,m]$ is a semi-Heyting algebra.*

The following lemma is analogous with the statement, any interval on a semi-Heyting algebra is again a semi-Heyting algebra.

Lemma 3.16. *Let L be a skew SHADL. Then for any $b \in L, [b,m]$ is a skew SHADL.*

Proof. Suppose L be a skew SHADL. Then from corollary 3.12 we have $[b,m]$ is a semi-Heyting algebra for any $b \in L$. Clearly $[b,m]$ is an ADL with its zero is b. Following this $[b,c]$ is a semi-Heyting algebra for any $c \in [b,m]$. Now for all $x,y \in [b,c]$ define $x\ _b\!\rightarrow y = x \rightarrow_m y$ which yields that $x\ _b\!\rightarrow y = (y \vee x \vee y)_y\!\rightarrow\!y$ so that $[b,c]$ is a skew semi-Heyting algebra with top element c. Therefore $[b,m]$ is a skew SHADL. □

Corollary 3.17. *Let L be a skew SHADL. If $x,y \in L$ such that $x \leq y$ and $a,b \in [y,m]$, then $a\ _x\!\rightarrow b = a\ _y\!\rightarrow b$.*

Proof. Let $x,y \in L$ such that $x \leq y$. Then by lemma 3.7 $[x,m]$ and $[y,m]$ are Heyting algebras with $[y,m] \subseteq [x,m]$. If $a,b \in [y,m]$, then $a\ _y\!\rightarrow b \in [y,m]$ and hence $a\ _y\!\rightarrow b \in [x,m]$. Since $a,b \in [x,m]$, $a\ _x\!\rightarrow b$ also belongs to $[x,m]$. The maximal element characterization of $a\ _x\!\rightarrow b$ and $a\ _y\!\rightarrow b$ on the Heyting algebra $[x,m]$ forces the two elements are equal. □

The following lemma gives a justification for lemma 3.10.

Lemma 3.18. *The binary operation $\rightarrow$ on a skew SHADL L, defined by $x \rightarrow y = ((x \vee y) \wedge m)_{(y\wedge m)}\!\rightarrow\!(y \wedge m)$ satisfies all the axioms of SHADLs.*

Proof. Let L be a skew SHADL and $x,y,z \in L$. Applying lemma 3.7 and lemma 3.13 consecutively we obtain that for any $y \in L, [y,m]$ is a Heyting algebra. Then (1), (2) and (4) hold directly. Thus we need to prove (3).

$$\begin{aligned}
(2)\ \ (3)\ \ x \wedge \{(x \wedge y) \rightarrow (x \wedge z)\} &= x \wedge \{\{((x \wedge y) \vee (x \wedge z)) \wedge m\}_{(x\wedge z\wedge m)}\!\rightarrow\!(x \wedge z \wedge m)\} \\
&= x \wedge \{(x \wedge m \wedge (y \vee z) \wedge m)_{(x\wedge m\wedge z\wedge m)}\!\rightarrow\!(x \wedge m \wedge z \wedge m)\} \\
&= x \wedge (x \wedge m) \wedge \{(y \vee z) \wedge m)_{(x\wedge m\wedge z\wedge m)}\!\rightarrow\!(z \wedge m)\} \\
&= x \wedge (x \wedge m) \wedge \{(y \vee z) \wedge m)_{(z\wedge m)}\!\rightarrow\!(z \wedge m)\} \\
&= x \wedge (y \rightarrow z)
\end{aligned}$$

□

To prove the next theorem first we observe the following lemma which can be verified routinely.

Lemma 3.19. *Let L be a skew SHADL and $x,y,z \in L$ such that $x \wedge m = y \wedge m$. Then the following statements hold:*

(1) $x \rightarrow y = m$

(2) $x \rightarrow z = y \rightarrow z$ *and* $z \rightarrow x = z \rightarrow y$

Theorem 3.20. *Let L be a skew SHADL and θ defined by $\theta = \{(x,y) \in L \times L | x \wedge y = y$ and $y \wedge x = x\}$ is a relation on L. Then the following conditions hold:*

(1) θ is a congruence relation on L

(2) L/θ is the maximal lattice image of L.

Proof. (1) Suppose L be a skew SHADL. Let $a,x,y,z \in L$. It is easly seen that θ is reflexive and symmetric. Assume that $x\theta y$ and $y\theta z$. Then $x \wedge y = y, y \wedge x = x, y \wedge z = z$ and $z \wedge y = y$. As $z \wedge x = z \wedge y \wedge x = y \wedge x = x$ and $x \wedge z = y \wedge x \wedge z = x \wedge y \wedge z = y \wedge z = z$, it follows that $x\theta z$. Consequently θ is transitive and hence it is an equivalence relation. To show that θ is a congruence relation it is sufficient to show that θ satisfies lemma 2.7. Given that $x\theta y$ and $a \in L$. Then $(y \wedge a) \wedge (x \wedge a) = y \wedge x \wedge a = x \wedge a$ and $(x \wedge a) \wedge (y \wedge a) = x \wedge y \wedge a = y \wedge a$, and hence $(x \wedge a)\theta(y \wedge a)$. Also $(x\vee a)\wedge(y\vee a) = ((x\vee a)\wedge y)\vee((x\vee a)\wedge a) = ((a\vee x)\wedge y)\vee a = ((a\wedge y)\vee(x\wedge y))\vee a = y\vee a$ and $(y\vee a)\wedge(x\vee a) = ((y\vee a)\wedge x)\vee((y\vee a)\wedge a) = ((a\vee y)\wedge x)\vee a = ((a\wedge x)\vee(y\wedge x))\vee a = x\vee a$. Hence $(x\vee a)\theta(y\vee a)$. Finally we show that $(x \to a)\theta(y \to a)$ and $(a \to x)\theta(a \to y)$ hold. From the property of ADLs given by theorem 2.2 and the given conditions $x \wedge y = y$ and $y \wedge x = x$, one can simply observe that $x \wedge m = y \wedge x \wedge m = x \wedge y \wedge m = y \wedge m$. Indeed, (2) of lemma 3.16 assures that $(x \to a)\theta(y \to a)$ and $(a \to x)\theta(a \to y)$. Hence θ is a congruence relation. To prove (2) for any $x, y \in L$ define $[x]\theta \wedge [y]\theta = [x \wedge y]\theta$ and $[x]\theta \vee [y]\theta = [x \vee y]\theta$. Then for any $a \in L$, $a \in [x \wedge y]\theta \Leftrightarrow a\theta(x \wedge y) \Leftrightarrow a\theta(y \wedge x) \Leftrightarrow a \in [y \wedge x]\theta \Leftrightarrow a \in [x]\theta \wedge [y]\theta$. This shows that $[x]\theta \wedge [y]\theta = [y]\theta \wedge [x]\theta$. Similarly $[x]\theta \vee [y]\theta = [y]\theta \vee [x]\theta$ holds. Hence L/θ is a lattice. Now Let β be a congruence relation such that L/β is a lattice and $x\theta y$. Then $(x \vee y)\beta(y \vee x) \Rightarrow (x \wedge (x \vee y))\beta(x \wedge (y \vee x)) \Rightarrow x\beta((x \wedge y) \vee x) \Rightarrow x\beta(y \vee x)$ and by transitivity we obtain that $x\beta(x \vee y)$. Similarly $(x \vee y)\beta(y \vee x) \Rightarrow (y \wedge (x \vee y))\beta(y \wedge (y \vee x)) \Leftrightarrow (y \wedge x)\beta y \Leftrightarrow (x \vee y)\beta y$. Which shows that $x\beta y$. Thus $\theta \leq \beta$ and $L/\beta \subseteq L/\theta$. Therefore L/θ is a maximal lattice image of L. □

The next theorem is another characterization of a skew SHADL.

Theorem 3.21. *Let L be an ADL with 0 and a maximal elemment m. Then L is a skew SHADL if and only if for any $a \in L$ and $x, y, z \in [0, a]$ the following hold:*

(1) $y \leq x \to_a y$

(2) $(y \to_a z) \wedge x = x$ if and only if $z \wedge x \wedge y = x \wedge y$.

Proof. Suppose L be a skew SHADL. Indeed for any $a \in L$, $[0, a]$ is a skew semi-Heyting algebra . Thus from lemma 3.7 $[0, a]$ is a Heyting algebra for any $a \in L$. Hence $y \leq x \to_a y$. Now, assume that $(y \to_a z) \wedge x = x$. Then $x \wedge y = x \wedge x \wedge y = x\wedge(y \to_a z)\wedge x\wedge y = x\wedge y\wedge(y\vee z)\wedge a\wedge\{((y\vee z)\wedge a)_{(z\wedge a)}{\to}(z\wedge a)\}\wedge x\wedge y = z\wedge x\wedge y$. On the other hand given that $z\wedge x\wedge y = x\wedge y$, then $y \to_a (x \wedge y) = y \to_a (z \wedge x \wedge y)$. Since $[0, a]$ is a Heyting algebra we obtain that $(y \to_a x) \wedge a = (y \to_a z) \wedge (y \to_a x) \wedge a$. Thus $(y \to_a z) \wedge x = (y \to_a z) \wedge (y \to_a x) \wedge x = (y \to_a z) \wedge (y \to_a x) \wedge a \wedge x = (y \to_a x) \wedge a \wedge x = a \wedge (y \to_a x) \wedge x = a \wedge x = x$. Conversely assume that (1) and (2) hold. Now for any $b \in [0, a]$, take $c, d, e \in [b, a]$. By (1) $c \leq e\ {}_b{\to}\ c$ such that $e\ {}_b{\to}\ c \in [b, a]$, and then by (2) we get $d \leq e\ {}_b{\to}\ c \Leftrightarrow d = d \wedge (e\ {}_b{\to}\ c) \Leftrightarrow d = d \wedge (e\ {}_b{\to}\ c) \wedge a \Leftrightarrow d \wedge e = d \wedge e \wedge c \Leftrightarrow d \wedge e \leq c$. Hence $[b, a]$ is a Heyting algebra(and thus a semi-Heyting algebra). Now, define an induced binary operation $\to_a$ on $[0, a]$ by $y \to_a z = (z \vee y \vee z)_z{\to}z$ so that $[0, a]$ is a skew semi-Heyting algebra. This proves the converse of the theorem. □

Lemma 3.22. *On a skew SHADL H, $((x \vee y) \wedge m)_{(y\wedge m)}{\to}(y \wedge m) = m$ if and only if $y \wedge x = x$.*

Proof. Assume $((x\vee y)\wedge m)_{(y\wedge m)}{\to}(y\wedge m) = m$. Then $x = x\wedge m\wedge x = x\wedge(x\vee y)\wedge m\wedge\{((x\vee y)\wedge m)_{(y\wedge m)}{\to}(y\wedge m)\}\wedge x = y\wedge x$. Hence $y \wedge x = x$. The converse is straight forward □

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