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Quotient Heyting Algebras Via Fuzzy Congruence Relations

Research Article

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Abstract: This paper aims to introduce fuzzy congruence relations over Heyting algebras (HA) and give constructions of quotient Heyting algebras induced by fuzzy congruence relations on HA. The Fuzzy First, Second and Third Isomorphism Theorems of HA are established.

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Keywords: Heyting algebra, Fuzzy Heyting algebra, Fuzzy Congruence relation, Quotiont HA.

1. Introduction

Fuzzy set theory, proposed by L.A. Zadeh [6], has been extensively applied to many scientific fields. Following the discovery of fuzzy sets, much attention has been paid to generalize the basic concepts of classical algebra in a fuzzy framework, and thus developing a theory of fuzzy algebras. In recent years, much interest is shown to generalize algebraic structures of groups, rings, modules, etc. The notion of fuzzy ideals of a ring R was put forward and the operations on fuzzy ideals was discussed by several researchers [1].

V. Murali [8] studied fuzzy congruence relations on universal algebras. Fuzzy isomorphism theorems of soft rings were shown by X.P. Liu [9], [10]. General algebraic structure, such as group and ring of congruence relations and ideals to depict the algebraic structure has played a very important role. The various constructions of quotient groups and quotient rings by fuzzy ideals was introduced by Y.L. Liu [7]. Moreover, N. Kuroki has been shown that there exists a one-to-one mapping from all fuzzy normal subgroups and all fuzzy congruence relations of groups. Naturally, the study of the definition and properties about fuzzy congruence relations on rings is a meaningful work. Heyting Algebra is a relatively pseudocomplemented distributive lattice. It arises from non classical logic and was first investigated by Skolem T. [11]. It is named as Heyting Algebra after the Dutch Mathematician Arend Heyting [11].

In this paper, we introduce the notion of fuzzy congruence relations on HA's and introduce the notion of quotient HA's by fuzzy congruence relations on HA's and give the Fuzzy First, Second and Third Isomorphism Theorems of HA based on fuzzy congruence relation. Moreover, we give some properties between fuzzy ideals and fuzzy congruence relations on HA's.

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2. preliminaries

From the properties of fuzzy set theory, we know that a fuzzy set defined on a set as follows: let H be a non-empty set, then $\mu : H \to [0,1]$ is called a fuzzy set of H. In this paper,H is always a Heyting algebra(HA).

Definition 2.1. *An algebra* $(H, \vee, \wedge \to, 0, 1)$ *is called a Heyting algebra if it satisfies the following*

(1). $(H, \vee, \wedge, 0, 1)$ *is a bounded distributive lattice*

(2). $a \to a = 1$

(3). $b \leq a \to b$

(4). $a \wedge (a \to b) = a \wedge b$

(5). $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$

(6). $(a \vee b) \to c = (a \to c) \wedge (b \to c)$ *,for all a,b,c* $\in H$

Theorem 2.2. *If* $(H, \vee, \wedge, \to, 0, 1)$ *is a Heyting Algebra and a,b* $\in$ *H,then* $a \to b$ *is the largest element c of H such that* $a \wedge c \leq b$.

Definition 2.3. *Let X be a set. A function A:X* $\times$ *X* $\to [0,1]$ *is called a fuzzy relation in X. The fuzzy relation A in X is reflexive iff*$A(x,x) = 1$*,for all*$x \in X$*.The fuzzy relation A in X is anti symmetric iff* $A(x,y) > 0$ *and* $A(y,x) > 0$ $\Rightarrow x = y$*.The fuzzy relation A in X is transitive iff* $A(x,z) \geq Sup_{y \in X}(min(A(x,y), A(y,z)))$*.A fuzzy relation A is fuzzy partial order relation if A is reflexive,symmetric and transitive. A fuzzy partial order relation A is fuzzy total order relation iff* $A(x,y) > 0$ *or* $A(y,x) > 0$*,for all* $x, y \in H$*. If A is a fuzzy partial order relation on a set X,then (X,A) is called a fuzzy partially ordered set or a fuzzy poset.If A is a fuzzy total order relation in a set X,then (X,A) is called a fuzzy totally ordered set or a fuzzy chain.*

Definition 2.4. *Let (X,A) be a fuzzy poset and B* $\subseteq$ *X.An element* $u \in X$ *is said to be an upper bound for a subset B iff* $A(b,u) > 0, \forall b \in B$*. An upper bound* u_0 *for a subset B is least upper bound of B iff* $A(u_0, u) > 0$ *for every upper bound u for B.An element* $v \in X$ *is said to be an lower bound for a subset B iff* $A(v,b) > 0, \forall b \in B$*. A lower bound* v_0 *for a subset B is the greatest lower bound of B iff* $A(v, v_0) > 0$ *for every lower bound v for B. We denote the lub of the set* $\{x,y\} = x \vee y$ *and glb of the set* $\{x,y\} = x \wedge y$.

Definition 2.5. *Let (X,A) be a fuzzy poset.(X,A) is a fuzzy lattice iff* $x \vee y$ *and* $x \wedge y$ *exists for all x,y* $\in X$.

Definition 2.6. *A bounded distributive fuzzy lattice(H,A) is said to be a Fuzzy Heyting Algebra if there exists a binary operation* $\to$ *such that ,for any x,y,z*$\in H$, $A(x \wedge z, y) > 0 \Leftrightarrow A(z, x \to y) > 0$.

3. Fuzzy Congruence Relation

In this section, we introduce the notion of fuzzy congruence relations on HA and give some properties about fuzzy congruence relations.

Definition 3.1. *A fuzzy set A of* $H \times H$ *is called a fuzzy relation on H. A fuzzy relation A on H is called a fuzzy equivalence relation if it satises the following conditions:*

(1). $A(x,x) = 1$ *for all x of H (fuzzy relexive),*

(2). $A(x,y) = A(y,x)$ *for all* x,y *of* H *(fuzzy symmetric),*

(3). $A(x,y) \geq Sup_{z\in H}(min[A(x,z), A(z,y)])$ *for all* x,y *of* H *(fuzzy transitive).*

Definition 3.2. *A relation* A *on the set* H *is called left compatible if* $(a,b) \in A$ *implies* $(x \vee a, x \vee b) \in A$, $(x \wedge a, x \wedge b) \in A$ *and*$(x \to a, x \to b) \in A$ *, for all* a,b,x *of H, and is called right compatible if* $(a,b) \in A$ *implies* $(a \vee x, b \vee x) \in A$ *,* $(a \wedge x, b \wedge x) \in A$, *and* $(a \to x, b \to x) \in A$ *for all* a,b,x *of* H.

Remark 3.3. *A compatible equivalence relation on H is called a congruence relation on H.*

Definition 3.4. *Let H be a HA and A be a fuzzy equivalence relation on H.Then for x, y* $\in H, A_x = A_y$ *if and only if* $A(x,y) = 1$.

Definition 3.5. *A fuzzy equivalence relation A on H is called a fuzzy congruence relation if the following conditions are satisfied,for all x,y,z,t of H.*

(1). $A(x \vee y, z \vee t) \geq min(A(x,z), A(y,t))$

(2). $A(x \wedge y, z \wedge t) \geq min(A(x,z), A(y,t))$

(3). $A(x \to y, z \to t) \geq min(A(x,z), A(y,t))$

We denote the set of all fuzzy congruence relations on H by FC(H).

Lemma 3.6. *Let (H,A) be FHA,then a fuzzy equivalence relation A is called a fuzzy congruence relation on H if*

(1). $A(a \vee x, b \vee x) \geq A(a,b)$

(2). $A(a \wedge x, b \wedge x) \geq A(a,b)$

(3). $A(a \to x, b \to x) \geq A(a,b)$

(4). $A(x \to a, x \to b) \geq A(a,b)$

Proposition 3.7. *Let A and B be any fuzzy compatible relations on H. Then* $A \circ B$ *is also a fuzzy compatible relation on H.*

Proof. Let $a,b,x \in H$. Since A and B are fuzzy compatible equivalence relations on H,

$$\begin{aligned}
A \circ B(x \vee a, x \vee b) &= \sup_{z\in H}(\min(A(x \vee a, z), B(z, x \vee b)) \\
&\geq \min\left[A(x \vee a, x \vee z), B(x \vee z, x \vee b)\right] \\
&\geq \min(A(a,z), B(z,b)) \\
&\geq \sup_{z\in H}(\min(A(a,z), B(z,b)) \\
&= A \circ B(a,b) \\
(A \circ B)(x \wedge a, x \wedge b) &= \sup_{z\in H}(\min(A(x \wedge a, z), B(z, x \wedge b)) \\
&\geq \min\left[A(x \wedge a, x \wedge z), B(x \wedge z, x \wedge b)\right] \\
&\geq \min(A(a,z), B(z,b)) \\
&\geq \sup_{z\in H}(\min(A(a,z), B(z,b)) \\
&= A \circ B(a,b)
\end{aligned}$$

Finally,

$$\begin{aligned}(A\circ B)(x\to a,x\to b)&=\sup_{x\to z\in H}(\min(A(x\to a,x\to z),B(x\to z,x\to b))\\&\geq \min(A(x\to a,x\to z),B(x\to z,x\to b)\\&\geq \min(A(a,z),B(z,b))\\&\geq A\circ B(a,b).\end{aligned}$$

Therefore, $A\circ B$ is a fuzzy left compatible relation.Similarly $A\circ B$ is a fuzzy right compatible relation.Thus,we obtain $A\circ B$ is a fuzzy compatible relation. □

Example 3.8. *Let $H=[a,b],a,b\in H$ be an interval.Then H is a HA with its operations.Then the fuzzy relation A on H defined by*

$$A(x,y)=\begin{cases}1 & \text{if } x=y;\\ 0.5 & \text{if } x\neq y;\\ 0 & \text{otherwise.}\end{cases}$$

is a fuzzy congruence relation on H.

Proposition 3.9. *Let A and B be fuzzy congruence relations on H. Then $A\circ B$ is a fuzzy congruence relation on H if and only if $A\circ B\ =B\circ A$.*

Let A be a fuzzy relation on H. For each $\alpha\in[0,1]$, we put $H_A(\alpha)=\{(a,b):(a,b)\in H\times H,A(a,b)\geq\alpha\}$.This set is called the α level set of A.

Theorem 3.10. *A fuzzy relation A is a fuzzy congruence relation on H if and only if for each $\alpha\in[0,1]$, the α level set $H_A(\alpha)$ is a congruence relation on H.*

4. Quotient Heyting Algebras Induced by Fuzzy Congruence Relations

In this section, we introduce the notion of quotient HA's by fuzzy congruence relations and give the Fuzzy First, Second and Third Isomorphism Theorems of HA's by means of fuzzy congruence relations.

Definition 4.1. *Let A be a fuzzy congruence relation on H. For every element $x\in H$, we define a subset $A_x=\{y\in H: A(x,y)=1\}$ of H and the quotient HA of H is $H/A=\{A_x:x\in H\}$.*

Theorem 4.2. *If A is a fuzzy congruence relation of H,then H/A is a HA under the binary operations defined by $A_x\vee A_y= A_{x\vee y} and A_x\wedge A_y=A_{x\wedge y}$ and $A_x\to A_y=A_{x\to y}\ x,y\in H$.*

Proof. First we show that the above binary operations are well defined.In fact,if $A_x=A_{x'}$ and $A_y=A_{y'}$,then $A(x,x')=1$ and $A(y,y')=1$. Since $A(x,x')\leq A(x\vee y,x'\vee y)$ and $A(y,y')\leq A(x'\vee y,x'\vee y')$. Also,

$$\begin{aligned}A(x\vee y,x'\vee y')&\geq \sup_{x'\vee y\in H}(\min(A(x\vee y,x'\vee y),A(x'\vee y,x'\vee y'))\\&\geq \sup_{x'\vee y\in H}(\min(A(x,x'),A(y,y'))=1.\end{aligned}$$

Which implies $A(x \vee y, x' \vee y') = 1$. Thus, $A_{x\vee y} = A_{x'\vee y'}$. Therefore, The operation $'\vee'$ is well defined.Similarly,the operation $'\wedge'$ is also well defined. Now,$A(x, x') \leq A(x \to y, x' \to y)$ and $A(y, y') \leq A(x' \to y, x' \to y')$, in similar approach, $A(x \to, x' \to y') \leq 1$. This implies $A(x \to y, x' \to y') \geq 1 \Rightarrow A_{x\to y} = A_{x'\to y'}$. Thus,$' \to'$ is well defined. We shall show that $H/A = \{A_x : x \in H\}$

(1). Let $A_x \in H/A$.Then $A_x \to A_x = A_{x\to x} = A_1$.

(2). Let $A_x, A_y \in H/A, x, y \in H$.Then $A_y \wedge (A_x \to A_y)$=$A_y \wedge (A_{x\to y})$=$A_{y\wedge(x\to y)}$=$A_y \Rightarrow A_y \leq A_x \to A_y$.

(3). Let A_x, A_y and $A_z \in H/A$. Then $A_x \to (A_y \wedge A_z) = A_x \to A_{y\wedge z} = A_{x\to(y\wedge z)} = A_{(x\to y)\wedge(x\to z)} = (A_x \to A_y) \wedge (A_x \to A_z)$.

(4). $A_x \wedge (A_x \to A_y) = A_x \wedge (A_x \to y)) = A_{x\wedge(x\to y)} = A_{x\wedge y} = A_x \wedge A_y$.

(5). $(A_x \vee A_y) \to A_z = A_{x\vee y} \to A_z = A_(x \vee y) \to z = A_{(x\to z)\wedge(y\to z)} = (A_x \to A_z) \wedge (A_y \to A_z)$. Hence, $(H/A, \wedge, \vee, \to, A_0, A_1)$ is a HA Wich is called a quotient HA.

□

Lemma 4.3. *Let H and H' be HA'S and f be a homomoriphism from H to H'.If A' is a fuzzy congruence relation on H',then the map defined by $f^{-1}(A')(x, y) = A'(f(x), f(y))$,for all $x, y \in H$ is a fuzzy congruence relation on H.*

Proof. For all $x, y, z \in H$, $f^{-1}(A')(x, x) = 1, f^{-1}(A')(x, y) = A'(f(x), f(y) = A'(f(y), f(x))$ which means $f^{-1}(A')$ is fuzzy reflexive and fuzzy symmetric relation on H. Since

$$\begin{aligned} f^{-1}(A')(x, y) = A'(f(x), f(y)) &\geq \sup_{f(z)\in H'} (\min(A'(f(x), f(z)), A'(f(z), f(y))) \\ &\geq \min(A'(f(x), f(z)), A'(f(z), f(y)) \\ &= \min(f^{-1}(A')(x, z), f^{-1}(A')(z, y)) \\ &\geq \sup_{z\in H} \min(f^{-1}(A')(x, z), f^{-1}(A')(z, y)). \end{aligned}$$

Therefore, $f^{-1}(A')$ is a transitive relation of H.So $f^{-1}(A')$ is a fuzzy equivalence relation. Again, $f^{-1}(A')(z \vee x, z \vee y) = A'(f(z \vee x), f(z \vee y)) = A'(f(z) \vee f(x), f(z) \vee f(y)) \geq A'(f(x), f(y)) = f^{-1}(A')(x, y)$. Similarly, $f^{-1}(A')(z \wedge x, z \wedge y) \geq f^{-1}(A')(x, y)$. Further, $f^{-1}(A')(z \to x, z \to y) = A'(f(z \to x), f(z \to y)) \geq A'(f(z) \to f(x), f(z) \to f(y)) \geq A'(f(x), f(y) = f^{-1}(A')(x, y)$. This means that $f^{-1}(A')$ is a fuzzy left compatible relation on a Heyting algebra H.By the same argument,we can see that $f^{-1}(A')$ is a fuzzy right compatible relation of H. Therefore, $f^{-1}(A')$ is a fuzzy congruence relation on H. □

Theorem 4.4 (Fuzzy First Isomorphism Theorem)**.** *Let H, H' be HA's, f be an epimorphism from H to H',and A' be a fuzzy congruence relation on H.Then $\frac{H}{f^{-1}(A')} \cong \frac{H'}{A'}$.*

Proof. It follows from Theorem 4.2 and Lemma 4.3, $H/f^{-1}(A')$ and H'/A' are both quotient HA's. We define a map h from $H/f^{-1}(A')$ to H'/A' by $h(f^{-1}(A')_x) = A'_x, x \in H$. By Definition 4.1,

(1). h is well defined. Suppose $f^{-1}(A')_x = f^{-1}(A')_y$,then $f^{-1}(A')(x, y) = 1 \Rightarrow A'(f(x), f(y)) = 1 \Rightarrow A'_{f(x)} = A'_{f(y)}$ [Definition 4.1]. Therefore h is well defined.

(2). h is homomorphism.

(a). $h(f^{-1}(A')_x \vee f^{-1}(A')_y) = h(f^{-1}(A')_{x\vee y}) = A'_{f(x\vee y)} = A'_{f(x)\vee f(y)} = A'_{f(x)} \vee A'_{f(y)} = h(f^{-1}(A')_x) \vee h(f^{-1}(A')_y)$.

(b). $h(f^{-1}(A')_x \wedge f^{-1}(A')_y) = h(f^{-1}(A')_{x\wedge y}) = A'_{f(x\wedge y)} = A'_{f(x)\wedge f(y)} = A'_{f(x)} \wedge A'_{f(y)} = h(f^{-1}(A')_x) \wedge h(f^{-1}(A')_y)$.

(c). $h(f^{-1}(A')_x \rightarrow f^{-1}(A')_y) = h(f^{-1}(A')_{x\rightarrow y}) = A'_{f(x\rightarrow y)} = A'_{f(x)\rightarrow f(y)} = A'_{f(x)} \rightarrow A'_{f(y)} = h(f^{-1}(A')_x) \rightarrow h(f^{-1}(A')_y)$.

Hence, From a, b and c we have h is a homomorphism.

(3). h is an epimorphism: For $A'_y \in H'/A', y \in H'$.Since f is epimorphic, there exists $x \in H$, such that $f(x) = y$ so $h(f^{-1}(A')_x) = A'_{f(x)} = A'_{y'}$.

(4). h is monomorphism. Suppose $h(f^{-1}(A')_x) = h(f^{-1}(A')_y)$, then $A'_{f(x)} = A'_{f(y)} \Rightarrow A'(f(x), f(y)) = 1 \Rightarrow f^{-1}(A')(x,y) = 1$. Hence, $f^{-1}(A')_x = f^{-1}(A')_y$ this means h is a monomorphism. In conclusion, $\frac{H}{f^{-1}(A')} \cong \frac{H'}{A'}$).

□

Corollary 4.5. *Let A be a fuzzy congruence relation on H.Then the mapping $f : H \rightarrow H/A$ defined by $f(x) = A_x, \forall x \in H$,is a homomorphism.*

Lemma 4.6. *Let A be a fuzzy congruence relation of H.Let $H_A = \{y \in H : A(0,y) = 1\}$.Then H_A is an ideal of H.*

Proof.

(1). Let $x \in H$ and $y \in H_A$ such that $A(x,y) \geq 0$. Then $A(0,y) = 1$. $A(0,x) = A(x \wedge 0, x \wedge y) \geq \min\{A(x,x), A(0,y)\} = \min(1,1) = 1$. Thus, $x \in H_A$.

(2). Let $x \in H_A$ and $y \in H_A$. Then $A(0,x) = 1, A(0,y) = 1$. Then $A_0 = A_x$ and $A_0 = A_y \Rightarrow A_0 \vee A_0 = A_x \vee A_y \Rightarrow A_{0\vee 0} = A_{x\vee y}$. $\Rightarrow A_0 = A_{x\vee y} \Rightarrow A(0, x\vee y) = 1$. Thus, $x \vee y \in H_A$. Hence H_A is an ideal of H.

□

Lemma 4.7. *Let I be an ideal of H, A and B are fuzzy congruence relations of H.*

(1). If A is restricted to I, then A is a fuzzy congruence relation of I,

(2). $A \cap B$ is fuzzy congruence relation of H,

(3). I/A is an ideal of H/A.

Proof.

(1). is clear

(2). For any $x,y \in H$, since $(A\cap B)(x,y) = \min(A(x,y), B(x,y))$. Then $(A\cap B)(x,y)$ is fuzzy reflexive and fuzzy transitive relations. We only show that $A \cap B$ is a fuzzy transitive relations. Since

$$\begin{aligned}(A\cap B)(x,y) &= \min(A(x,y), B(x,y))\\ &\geq \min(A(x,z), A(z,y), B(x,z), B(Z,y))\\ &\geq \min(A(x,z), A(z,y), B(x,z), B(z,y))\\ &= \min(A(x,z), B(x,z), A(z,y), B(z,y))\\ &\geq \sup_{z\in H}(\min((A\cap B)(x,z), (A\cap B)(z,y)))\end{aligned}$$

Hence, $A \cap B$ is a fuzzy transitive relation on H. Further more; for every $a \in$ H,

$$\begin{aligned}(A \cap B)(a \vee x, a \vee y) &= \min(A(a \vee x, a \vee y), B(a \vee x, a \vee y)) \\ &\geq \min(A(x, y), B(x, y)) \\ &= (A \cap B)(x, y).\end{aligned}$$

Similarly, $(A \cap B)(a \wedge x, a \wedge y) \geq (A \cap B)(x, y)$.

$$\begin{aligned}(A \cap B)(a \to x, a \to y) &= \min(A(a \to x), B(a \to y)) \\ &= \min(A(a \to x, a \to y), B(a \to x, a \to y)) \\ &\geq \min(A(x, y), B(x, y)) \\ &= (A \cap B)(x, y).\end{aligned}$$

This means that $A \cap B$ is a fuzzy left compatible relation.Similarly, $A \cap B$ is a fuzzy right compatible relation. Hence ,$A \cap B$ is a fuzzy congruence relation.

(3). First, we show that $K = \{A_a : a \in I\}$ is an ideal of H/A. For any $A_a, A_b \in \{A_a; a \in I\}, a, b \in I$. Since I is an ideal $a \vee b \in I$, hence $A_a \vee A_b = A_{a \vee b} \in \{A_a : a \in I\}$. For any $A_a \in \{A_a : a \in H\}, A_x \in H/A, a \in I, x \in H$, then $a \wedge x, x \wedge a \in I$, hence $A_a \wedge A_x = A_{a \wedge x} \in K$ and $A_x \wedge\ A_a = A_{x \wedge a} \in K$. Thus K is an ideal of H/A. Next, we define $\phi : I/A \to H/A$ by $(I/A)_a \to A_a$, for all $(I/A)_a \in I/A$. It is easy to verify $I/A \cong K$ under the mapping. Hence, we may regard I/A is an ideal of H/A in isomorphic sense. This completes the proof.

□

Theorem 4.8 (Fuzzy Second Isomorphism Theorem)**.** *Let A and B be two fuzzy congreunce relations of a Heyting algebra H with* $A_0 \subseteq B_0$*.Then* $H_A \vee H_B / \cong H_A/A \cap B$.

Proof. By Lemma 4.7, B is a fuzzy congruence relation of $H_A \vee H_B$ and $A \cap B$ is fuzzy congruence relation of H_A. Thus $H_A \vee H_B$ and $H_A/A \cap B$ are both HA's. For any $x \in H_A \vee H_B$, then $x = a \vee b$ where $a \in H_A, b \in H_B$. It implies $A(0, a) = 1 and B(0, b) = 1$. Define $f : (H_A \vee H_B)/B \to H_A/A \cap B$ by $f(B_x) = (A \cap B)_a$.

If $B_x = B_{x'}$, where $x' = a' \vee b' \in H_A$, $b' \in H_B, a' \in H_A$, then we have $A(0, a') = 1, B(0, b') = 1, B(x, x') = B(a \vee b, a' \vee b') = 1$. Since $A(a, a') \geq min(A(a, 0), A(0, a'))$. So A(a,a')=1. Similarly,B(b,b')=1, by Definition 4.1 and Lemma 4.7 and $A_0 \subseteq B_0$, we have $H_A \subseteq H_B$. Therefore, $a, a' \in H_B$, which is implies $B(0, a) = 1$, and $B(0, a') = 1$. Since $B(a, a') \geq \min(B(a, 0), B(0, a'))$. Thus, $B(a, a') = 1$. $(A \cap B)(a, a') \geq min(A(a, a'), B(a, a')) = 1$ $(A \cap B)(a, a') = 1 \Rightarrow (A \cap B)_a = (A \cap B)_{a'}$. Therefore, f is well defined. For any $B_x, B_y \in (H_A \vee H_B)/B$, where $x = a \vee b, y = a_1 \vee b_1, a, a_1 \in H_A$ and $b, b_1 \in H_B$

$x \vee y \in H_A = (a \vee b) \vee (a_1 \vee b_1)$=$(a \vee a_1) \vee (b \vee b_1)$. $x \wedge y = (a \vee b) \wedge (a_1 \vee b_1)$=$((a \vee b) \wedge a_1) \vee ((a \vee b) \wedge b_1)$=$(a \wedge a_1) \vee [(b \wedge a_1) \vee ((a \wedge b_1) \vee (b \wedge b_1))]$=$(a \wedge a_1) \vee b', b' = [(b \wedge a_1) \vee ((a \wedge b_1) \vee (b \wedge b_1))] \in H_B$.

$f(B_x \vee B_y) = f(B_{x \vee y}) = (A \cap B)_{a \vee a'} = (A \cap B)_a \vee (A \cap B)_{a_1} = f(B_x) \vee f(B_y)$. In similar fashion, we can have $f(B_x \wedge B_y) = f(B_{x \wedge y}) = (A \cap B)_{a \wedge a'} = (A \cap B)_a \wedge (A \cap B)_{a_1} = f(B_x) \wedge f(B_y)$. Also, $f(B_x \to B_y) = f(B_{x \to y}) = (A \cap B)_{a \to a'} = (A \cap B)_a \to (A \cap B)_{a_1} = f(B_x) \to f(B_y)$. Therefore, f is a homomorphism. □

Theorem 4.9 (Fuzzy Third Isomorphism Theorem)**.** *Let A and B be two congruence relations of a Heyting algebra H with* $A \leq B$*.Then* $(H/A)/(H_B/A) \cong H/B$.

Proof. By Lemma 4.6 and 4.7 H_B/A is an ideal of H/A. Define $f : H/A \to H/B$ by $f(A_x) = B_x, x \in H$. if $A_x = A_y$, then $A(x, y) = 1$. Since $A \leq B$,so $B(x, y) \geq A(x, y) = 1 \Rightarrow B(x, y) = 1 \Rightarrow B_x = B_y$. Hence f is well defined. $f(A_x \vee A_y) = f(A_{x\vee y}) = B_{x\vee y} = B_x \vee B_y = f(A_x) \vee f(A_y)$. Again $f(A_x \wedge A_y) = f(A_{x\wedge y}) = B_{x\wedge y} = B_x \wedge B_y = f(A_x) \wedge f(A_y)$. And $f(A_x \to A_y) = f(A_{x\to y}) = B_{x\to y} = B_x \to B_y = f(A_x) \to f(A_y)$. Which means f is a hommorphism for any $B_x \in H/B, \exists A_x \in H/A$ such that $f(A_x) = B_x$, so f is an epimorphism. Now, we show $kerf = H_B/A$. $kerf = \{A_x \in H/A : f(A_x) = B_0\} = \{A_x \in H/A : B_x = B_0\} = \{A_x \in H/A : B(0, x) = 1\} = \{A_x \in H/A : x \in H_B\} = H_B/A$. Hence, $(H/A)/(H_B/A) \cong H/B$. The proof is completed. □

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