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Some Commutativity Theorems for Non-Associative Rings

Research Article

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Abstract: In this section we prove some elementary results on commutativity of non-associative rings. In an elementary commutativity theorem for rings, Johnsen, Outcalt and Yaqub [1] showed that a non-associative ring R with unity satisfying $(xy)^2 = x^2y^2$ for all x, y in R is necessarily commutative. Boers [2] extended this to show that such a ring is also associative provided it is 2 and 3-torsion free. Without using that the ring is 2 and 3-torsion free, we prove that any assosymmetric ring in which $(xy)^2 = x^2y^2$is commutative and associative. Further, if Z(R) denotes the center of the ring R, we prove the commutativity of a 2-torsion free non-associative ring R satisfying any one of the following identities:
(1). $(xy)^2 \in Z(R)$
(2). $(xy)^2 - xy \in Z(R)$
(3). $((xy)z)^2 - (xy)z \in Z(R)$
(4). $\left[(xy)^2 - yx, x\right] = 0$or $\left[(xy)^2 - yx, y\right] = 0$
(5). $\left[x^2y^2 - xy, x\right] = 0$ or $\left[x^2y^2 - xy, y\right] = 0$
(6). $\left[(xy)^2 - x^2y - xy^2 + xy, x\right] = 0$ or $\left[(xy)^2 - x^2y - xy^2 + xy, y\right] = 0$ for all x, y, z in R.
At the end of this section we also give some example which show that the existence of the unity and 2-torsion free are essential in some results. We know that an assosymmetric ring R is a non-associtative ring in which $(x, y, z) = (P(x), P(y), P(z))$, where P is any permutation of x, y, z in R.

Keywords: Center, periodic ring, direct sum, left and right ideals.

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1. Introduction

During the last seven or eight decades abstract algebra has been developing very rapidly. In algebra the theory of rings serves as the building blocks for all branches of mathematics. In ring theory, the study of both associative and non-associative rings has evoked great interest and assumed importees. The results on non-associative rings in which one does assume a type of partial associative have been scattered though out the literature. Many mathematicians of recent years studies a certain special properties. Their general non-associative rings. Among those mathematicians Max Zorn, A.A. Albert, N.Jocobson, R.D.Schafer, Erwin Kleinfeld, R.L.Sansoucie, A.H.Boers and Armin They are the ones whose contributions to this filed are out standing.

2. Preliminaries

Definition 2.1 (Associative Ring)**.** *An associative ring R, sometimes called a ring in short, is an algebraic system with two binary operations addition '+' and multiplication '.' Such that*

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(1). The elements of R form an abelian group under addition and a semi group under multiplication,

(2). Multiplication is distributive on the right as well as on the left over addition i,e., $(x+y)z = xz+yz$, $z(x+y) = zx+zy$ for all x, y, z in R.

Definition 2.2 (Non-associative Ring)**.** *A non-associative ring R is an additive abelian group in which multiplication is defined, which is distributive over addition on the left as well as on the right, i.e., $(x+y)z = xz+yz$, $z(x+y) = zx+zy$, for all x, y, z in R. A non-associative ring differs from an associative ring in that the full associative law of multiplication is no longer assumed to be associative, i.e., it is not necessarily associative. Strictly speaking the associative law of multiplication has not been done away with, it has merely weakened.*

The well known examples of non-associative rings are alternative rings, Lie rings and Jordan rings. In 1930 Artin and Max Zorn defined alternative rings.

Definition 2.3 (Alternative Ring)**.** *An alternative ring R is a ring in which $(xx)y = x(xy)$, $y(xx) = (yx)x$ for all x, y in R. These equations are known as the left and right alternative laws respectively.*

Definition 2.4 (Lie Ring)**.** *A Lie ring R is a ring in which the multiplication is anti-commutative, i.e., $x^2 = 0$ (implying $xy = -yx$) and the Jacobi identity $(xy)z + (yz)x + (yz)x + (zx)y = 0$ for all x,y,z in R is satisfied.*

Definition 2.5 (Jordan Ring)**.** *A Jordan Ring R is a ring in which products are commutative, i.e., $xy = yx$ and satisfy the Jordan identify $(xy)\,x^2 = x\left(yx^2\right)$ for all x, y in R.*

Definition 2.6 (Associator)**.** *The associator (x, y, z) is defined by $(x, y, z) = (xy)z - x(yz)$ for all x, y, z in a ring.*

This plays a key role in the study of non-associative rings. It can be viewed as a measure of the non-associativity of a ring. This definition is due to Maxzorn wherein he proved that a finite alternative division ring is associative. In terms of associators, a ring R is called left alternative if $(x, x, y) = 0$, right alternative if $(y, x, x) = 0$ for all x, y in R and alternative if both the conditions hold.

Definition 2.7 (Commutator)**.** *The commutator $[x, y]$ is defined by $[x, y] = xy - yx$ for all x, y in a ring.*

Definition 2.8 (Commutative Ring)**.** *If the multiplication in a ring R is such that $xy = yx$ for all x, y in R then we call R a commutative ring.*

A non-commutative ring differs from commutative ring in that the multiplication is not assumed to be commutative. i.e., we do not assume $xy = yx$ for all x, y in R as an axiom. However, it does not mean that there always exist elements x, y in R such that $xy \neq yx$. The ring of 2×2 matrices over rationales and the ring of real quaternion due to Hamilton are the examples of non-commutative rings.

Definition 2.9 (Assosymmetric Ring)**.** *An Assosymmetric ring R is one in which $(x, y, z) = (P(x), P(y)P(z))$, where P is any permutation of x, y, z in R.*

Definition 2.10 (Standard Ring)**.** *A ring R is defined to be standard if it satisfies the following two identities:*

(1). $(wx, y, z) + (xz, y, w) + (wz, y, x) = 0$

(2). $(x, y, z) + (z, x, y) - (x, z, y) = 0$*, for all w, x, y and z in R.*

Definition 2.11 (Accessible Ring)**.** *A ring R is called accessible in case it satisfies the identities:*

(1). $(x, y, z) + (z, x, y) - (x, z, y) = 0$

(2). $((w, x), y, z) = 0$, *for all w, x, y and z in R.*

Definition 2.12 (Periodic Ring)**.** *A ring R is called a periodic ring if for every x in R, there exists distinct positive integers $m = m(x)$, $n = n(x)$ such that $x^m = x^n$. Due to Chacron R is periodic if and only if for each $x \in R$, there exists a positive integers $k = k(x)$ and a polynomial $f(\lambda) = f_x(\lambda)$ with integer co-efficient such that $x^k = x^{k+1} f(x)$.*

Definition 2.13 (s-Unital Ring)**.** *A ring R is called a left (respectively right) s-unital ring if $x \in Rx$ (respectively $x \in xR$) for each $x \in R$. Further R is called s-unital if it is both left as well as right s-unital, i.e., if $x \in xR \cap Rx$, for each $x \in R$.*

Definition 2.14 (Weakly Periodic Ring)**.** *A ring R is called a weakly periodic ring if every element of R is expressible as a sum of a nilpotent element and a potent element of R, $R = N + P$, where N is the set of nilpotent elements of R and P is the set of potent elements of R. It is well-known that if R is periodic, then it is weakly periodic.*

Definition 2.15 (Prime Ring)**.** *A ring R is called a primo ring if whenever A and B are ideals of R such that $AB = 0$, then either $A = 0$ or $B = 0$.*

Definition 2.16 (Semi Prime Ring)**.** *A ring R is semi prime if for any ideal A of R, $A^2 = 0$ implies $A = 0$. These rings are also referred to as rings free from trivial ideals.*

Definition 2.17 (Flexible Ring)**.** *If in a ring R, the identity $(x, y, x) = 0$ i.e., $(xy)x = x(yx)$ for all x, y in R holds then R is called flexible.*

Alternative, commutative, anti-commutative and there by Jordan and Lie rings are flexible.

Definition 2.18 (Nilpotent Ring)**.** *A ring is called nilpotent if there is a fixed positive integer t such that every product involving t elements is zero.*

Definition 2.19 (Torsion-free Ring)**.** *A ring R is said to be m-torsion free if $mx = 0$ implies $x = 0$ for all x in R.*

Definition 2.20 (Reduced Ring)**.** *A ring R is called reduced if $N = \{0\}$, where N is the set of nilpotent elements of R.*

Definition 2.21 (Center)**.** *In a ring R, the center denoted by Z(R) is the set of all elements $x \in R$ such that $xy = yx$ for all $y \in R$.*

Definition 2.22 (Derivation)**.** *A derivation of a ring R is an additive group homomorphism $d : R \to R$ satisfying $d(r_1, r_2) = (dr_1) r_2 + r_1 dr_2$.*

It is important to note that this definition does not depend on the associative of multiplication and in fact, we shall have occasion to deal with derivations of non-associative algebras.

3. Main Results

Theorem 3.1. *Let R be a non-associative assosymmetric ring with unity such that $(xy)^2 = x^2y^2$ for all x, y in R is commutative and associative.*

Proof. Suppose x, y in R. Then $(xy)^2 = x^2y^2$. Moreover

$$[x(1+y)]^2 = x^2(1+y)^2 = x^2 + 2x^2y + x^2y^2.$$

But also $[x(1+y)]^2 = (x+xy)^2 = x^2 + x(xy) + (xy)x + (xy)^2$. Hence,

$$x(xy) + (xy)x = 2x^2y \tag{1}$$

Now replacing x by $1+x$ in (1) we obtain.

$$(1+x)[(1+x)y] + [(1+x)y](1+x) = 2(1+x)^2 y$$
$$(1+x)(y+xy) + (y+xy)(1+x) = 2(1+2x+x^2y)$$
$$y + xy + xy + x(xy) + y + yx + xy + (xy)x = 2y + 4xy + 2x^2y.$$

This reduces to, $x(xy) + yx + (xy)x = xy + 2x^2y$. Using (1) in this, we get, $yx = xy$. Thus R is commutative. In an asssymmetric ring the following identity holds: $(xy, z) = x(y, z,) + (x, z)y + (x, y, z)$. Now this identity together with commutativity imply that, $(x, y, z) = 0$. Hence R is associative. Quadric and Ashraf [5] have proved that a semi prime ring R satisfying the identity $[(xy)^n, z] = 0$ is commutative. We prove the partial generalization of this result by choosing $n = 2$ in a non-associative ring. □

Theorem 3.2. *Let R be 2-torsion free non-associative ring in which $(xy)^2 \in Z(R)$ for all x, y in R. Then R is commutative.*

Proof. By hypothesis

$$(xy)^2 \in Z(R) \tag{2}$$

Replacing x by $(x+1)$ in (2) and using it, we obtain

$$(xy)y + y(xy) + y^2 \in Z(R) \tag{3}$$

Replacing x by $x+1$ in (3) and using it and applying 2-torsion free condition, we get

$$y^2 \in Z(R) \tag{4}$$

Replacing y by $y+1$ in (4) and applying 2-torsion free condition, we get $y \in Z(R)$. Hence R is commutative. □

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