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An Application of Weakly Generalized Closed Graphs

Research Article

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Abstract: We defined some new spaces called as WG-Compact and WG-Connected spaces in order to characterize these spaces by using the notion of weakly generalized closed graphs. And, we investigate the interrelationship among various graph conditions.

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Keywords: wg-closed graphs, wg-compact space and wg-connected space.

1. Introduction

In 1991, Balachandran et al. introduced the notion of GO-compactness and GO-connectedness by involving generalized closed [1] (briefly g-closed) sets. In 2008, Caldas et al. [3] investigated GO-compact spaces in the context of multi-functions. In 1969, Long studied the properties of closed graphs [5]. Since the advent of these notions, several research papers with interesting results in different respects came to existence ([2], [11-13]). Recently, Nagaveni. et al. introduced and investigated the graph functions such as g-closed graph [8], wg-closed graph [8], g*closed graph [9], strongly g-closed graph [9] and *wg-closed graph [10]. In this paper, we defined some new spaces called as WG-Compact and WG- Connected spaces in order to characterize these spaces with the notion of weakly generalized closed graphs are used. The interrelationships among various graphs conditions are also discussed. Throughout the paper (X, τ) and (Y, σ) (or simply X and Y) are denoted by topological spaces. The interior and the closure of a subset A of (X, τ) are denoted by Int(A) and Cl(A) respectively.

2. Preliminaries

In this section, we list some definitions which are used in this sequel.

Definition 2.1. *A subset A of a space (X, τ) is called a*

(1). generalized closed (i.e. g-closed) set [4] if $Cl(A) \subset U$ whenever $A \subset U$ and U is open set.

(2). strongly generalized closed (i.e. g^* closed) set [14] if $Cl(A) \subset U$, whenever $A \subset U$ and U is g-open set in X.

(3). weakly generalized closed (i.e. wg-closed) set [6] if $Cl\ (Int(A)) \subset U$ whenever $A \subset U$ and U is open set in X.

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The complement of g-closed set (resp. g^* closed set and wg-closed) is said to be g-open set (resp. g^* open set and wg-open set). The family of all g-open sets (resp. g^* open set and wg-open set) is denoted by GO(X) (resp. $G^*O(X)$ and WGO(X)). We set $GO(X,x) = \{V \in GO(X)/x \in V\}$ for $x \in X$. We define similarly $G^*O(X,x) = \{V \in G^*O(X)/x \in V\}$ for $x \in X$.

Definition 2.2.

(1). The g-closure of a subset A of X is, denoted by Cl(A) [1], defined to be the intersection of all g-closed sets containing A. (Recently, it was denoted by g-Cl(A) [3] or $Cl_g(A)$ [13])*

(2). The wg-closure of a subset A of X is, denoted by wg-Cl(A) [7], defined to be the intersection of all wg-closed sets containing A.

Definition 2.3 ([5]). *Let $f : (X,\tau) \to (Y,\sigma)$ be any function. Then the subset $\{(x, f(x))/x \in X\}$ of the product space $(X \times Y, \tau \times \sigma)$ is called the graph of f and is denoted by G(f).*

Definition 2.4. *A function $f : (X,\tau) \to (Y,\sigma)$ is said to have a closed [5] (resp. g-closed [9], g*closed [9] and wg-closed [8]) graphs if for each $(x,y) \in X \times Y - G(f)$, there exist a open (resp. g-open, g*open and wg-open) sets U and V containing x and y respectively, such that $(U \times V) \cap G(f) = \emptyset$.*

Definition 2.5. *Let $f : (X,\tau) \to (Y,\sigma)$ be a function then the graph $G(f)$ is closed [6] (resp. g-closed [9], g*closed [9] and wg-closed [8]) in $X \times Y$ if and only if for each $(x,y) \in X \times Y - G(f)$, there exist a open (resp. g-open, g^* open and wg-open) sets U and V containing x and y respectively, such that $f(U) \cap V = \emptyset$.*

Definition 2.6. *A function $f : (X,\tau) \to (Y,\sigma)$ is said to have a strongly closed [12] (resp. strongly g-closed [13] and *wg-closed) graphs if for each $(x,y) \in X \times Y - G(f)$, there exist a open (resp. g-open and wg-open) sets U and V containing x and y respectively, such that $U \times Cl(V)) \cap G(f) = \Phi$ (resp. $(U \times g - Cl(V)) \cap G(f) = \Phi$ and $(U \times wg - Cl(V)) \cap G(f) = \Phi$).*

Lemma 2.7. *Let $f : (X,\tau) \to (Y,\sigma)$ be a function then the graph $G(f)$ is strongly closed [12] (resp. strongly g-closed [13] and *wg-closed) in $X \times Y$ if and only if for each $(x,y) \in X \times Y - G(f)$, there exist a open (resp. g-open and wg-open) sets U and V containing x and y respectively, such that $f(U) \cap Cl(V) = \emptyset$ (resp. $f(U) \cap g - Cl(V) = \Phi$ and $f(U) \cap wg - Cl(V) = \Phi$).*

3. Compact and Connected Spaces with wg-closed Graph

In this section, we introduced new type of compact and connected spaces and we characterize these spaces with wg-closed graph.

Definition 3.1. *A collection $\{A_i : i \in I\}$ of wg-open sets in a topological space X is called wg-open cover of a subset B if $B \subset \cup\{A_i : i \in I\}$.*

Definition 3.2. *A space X is called WG-Compact if every wg-open cover of X admits a finite subcover. A subset A of X is said to be WG-Compact if every wg-open covering of A contains a finte sub collection that also covers A.*

Remark 3.3.

(1). Every WG-Compact space of topological space (X,τ) is Compact.

(2). Any finite subset of topological space (X,τ) is WG- Compact.

Example 3.4. *Let (X, τ) be infinite cofinite topological space. Then $WGO(X) = \{X, \emptyset, A/A^c$ is finite$\}$. Let $\{G_i : i \in I\}$ be an arbitary wg-open cover for X. let G_{i_0} be a wg-open set in the wg-open cover $\{G_i : i \in I\}$. Then $X - G_{i_0}$ is finite, say $\{x_1, x_2, \ldots, x_n\}$. Choose G_{i_j} such that $x_j \in G_{i_j}$ where $j = 1, 2, \ldots, n$. Then $X = G_{i_0} \cup G_{i_1} \cup \cdots \cup G_{i_n}$. The space is WG-Compact. Hence it is Compact.*

Lemma 3.5. *Every wg-closed subset of WG-Compact space is WG-Compact space.*

Proof. Let $\mathcal{G} = \{G_i\}$ be a wg-open cover of F, i.e. $F \subset \cup_i G_i$. Then $X = (\cup_i G_i) \cup F^c$, that is, $\mathcal{G}^* = \{G_i\} \cup F^c$ is a cover of X. But F^c is wg-open since F is wg-closed, so $\mathcal{G}^*$ is an wg-open cover of X. by hypothesis, X is WG-compact, hence $\mathcal{G}^*$ is reducible to a finite cover of X, say $X = (G_{i_1} \cup G_{i_2} \cup \cdots \cup G_{i_m}) \cup F^c$, $G_{i_m} \in \mathcal{G}$. But F and F^c are disjoint. Hence, $F \subset (G_{i_1} \cup G_{i_2} \cup \cdots \cup G_{i_m})$, $G_{i_m} \in \mathcal{G}$. We have just shown that any wg-open cover $\mathcal{G} = \{G_i\}$ of F contains a finite subcover, i.e. F is WG-Compact. □

Remark 3.6. *The converse of this theorem is need be not true as seen from the following example.*

Example 3.7. *A set which is wg-compact but not wg-closed. Let $X = \{a, b, c, d\}$ be endowed with the topology $\tau = \{X, \emptyset, \{c\}, \{a, b\}, \{a, b, c\}\}$. Then (X, τ) is wg-compact. The subset $A = \{c\}$ is WG-Compact but not wg-closed set.*

Theorem 3.8. *Let X be a space such that WGO(X) is a topology. If for function $f : (X, \tau) \to (Y, \sigma)$ where Y is WG-Compact, $G(f) \in WGC(X \times Y)$, then f is wg-continuous.*

Proof. Let $x \in X$, $V \in O(Y, f(x))$ and $y \in Y - V$. Then $(x, y) \in X \times Y - G(f)$. So there exist $U_y \in WGO(X, x)$, $V_y \in WGO(Y, y)$ such that

$$f(U_y) \cap V_y = \Phi. \tag{1}$$

This relation holds for every $y \in Y - V$. clearly $\mathcal{V} = \{V_y : y \in Y - V\}$ is a cover of $Y - V$ by wg-open sets. Now Y is WG-Compact and $Y - V$ is wg-closed. Hence by the Lemma 3.5, $Y - V$ is WG-Compact. So $\mathcal{V}$ has a finite subfamily $\{V_i : i = 1, 2, \ldots, n\}$ such that $Y - V \subset \bigcup_{i=1}^{n} V_{y_i}$. Let $\{U_{y_i} : i = 1, 2, \ldots, n\}$ be the corresponding sets of $WGO(X, x)$ satisfying the relation of type (1). Set $U = \bigcap_{i=1}^{n} U_{y_i}$. Then, $U \in WGO(X, x)$. If $\alpha \in U$, then $f(\alpha) \notin V_{y_i}$ for all $i = 1, 2, \ldots, n$. This implies that $f(\alpha) \notin Y - V$. So that $f(\alpha) \in V$. Since α is the arbitrary it follows that $f(U) \subset V$ which guarantees the wg-continuous of f. □

Definition 3.9. *Two subsets A and B of the space X are called WG-separated if $A \cap wg - Cl(B) = B \cap wg - Cl(A) = \Phi$.*

Lemma 3.10. *A space X is called WG-connected if X cannot be expressed as the union of two WG-separated sets.*

Definition 3.11. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be WG-Connected set if every image of WG-connected set U in X is WG-Connected in Y.*

Definition 3.12. *A subset A of X is called wg-clopen if A is both wg-open and wg-closed.*

Definition 3.13. *A mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be WG-Connected if and only if for every inverse image of wg-clopen in Y is wg-clopen in X.*

Definition 3.14. *A space X is called extremely WG-Disconnected if the wg-closures of every wg-open set is wg-open.*

Theorem 3.15. *Let $f : (X, \tau) \to (Y, \sigma)$ be a set WG-Connected surjection and Y be an extremely WG-Disconnected wg-T_2 space. Then G(f) is wg-closed.*

Proof. Let $(x, y) \in X \times Y - G(f)$. Now Y being wg-$T_2$ space, there is a $H \in WGO(Y, y)$ such that $f(x) \notin wg-Cl(H) = V$. Since Y is extremely WG-Disconnected, V is wg-clopen in Y not containging $f(x)$. Again since f is set WG-Connected surjection $f^{-1}(V)$ is wg-clopen in X and $x \notin f^{-1}(V)$. Then $U \in WGO(X, x)$ and $f(U) \cap V = \Phi$. Hence $G(f)$ is wg-closed. □

4. The Interrelationship Among Various Graph Conditions

In this section, we investigate the interrelationship among various graphs conditions. we list some results which are used in this sequel. Veerakumar proved the results that every closed sets are g*closed sets and every g*closed sets are g-closed sets. Nagaveni proved the results that every closed sets are wg-closed sets and every g-closed sets are wg-closed sets.

Theorem 4.1. *Every function with a closed graph has a wg-closed graph.*

Proof. It follows from the result that every closed set is wg-closed set. □

The converse need not be true as seen from the following example.

Example 4.2. *Let* $X = \{a, b\}$ *and* $Y = \{a, b, c\}$ *be endowed with the topologies* $\tau = \{X, \emptyset, \{a\}, \{b\}\}$ *and* $\sigma = \{Y, \emptyset, \{a\}, \{b, c\}\}$ *respectively and suppose that wg-open sets of* (X, τ) *are* $\{X, \emptyset, \{a\}, \{b\}\}$ *and wg-open sets of* (Y, σ) *are* $\{Y, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$*. Let* $f : (X, \tau) \to (Y, \sigma)$ *be the mapping defined by* $f(a) = a$*,* $f(b) = b$*. Then f has wg-closed graph.*

Theorem 4.3. *Every function with a closed graph has a g* closed graph.*

Proof. It follows from the result that every closed set is g*closed set. □

The converse need not be true as seen from the following example.

Example 4.4. *Let* $X = \{a, b\}$ *and* $Y = \{a, b, c\}$ *be endowed with the topologies* $\tau = \{X, \emptyset, \{a\}, \{b\}\}$ *and* $\sigma = \{Y, \emptyset, \{c\}, \{a, c\}, \{b, c\}\}$ *respectively and suppose that g*-open sets of* (X, τ) *are* $\{X, \emptyset, \{a\}, \{b\}\}$ *and g* open sets of* (Y, σ) *are* $\{Y, \emptyset, \{c\}, \{b, c\}, \{a, c\}\}$*. Let* $f : (X, \tau) \to (Y, \sigma)$ *be the mapping defined by* $f\,(a) = a$*,* $f\,(b) = b$*. Then f has g*-closed graph. But it is not closed graph.*

Theorem 4.5. *Every function with a closed graph has a g-closed graph.*

Proof. It follows from the result that every closed set is g-closed set. □

The converse need not be true as seen from the following example.

Example 4.6. *Let* $X = \{a, b\}$*,* $Y = \{a, b, c, d\}$ *be two sets endowed with the discrete topology* τ *and* $\sigma = \{Y, \emptyset, \{c, d\}\}$ *respectively and suppose that g-open sets of* (X, τ) *are* $\{X, \emptyset, \{a\}, \{b\}\}$ *and g-open sets of* (Y, σ) *are* $\{Y, \emptyset, \{c\}, \{a\}, \{b\}, \{d\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, c\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$*. Let* $f : (X, \tau) \to (Y, \sigma)$ *be the mapping defined by* $f(a) = a$ *and* $f(b) = b$*. Then* $G(f)$ *is g-closed graph but it is not closed graph*

Theorem 4.7. *Every function with a g* closed graph has a g-closed graph.*

Proof. It follows from the result that every g*closed set is g-closed set. □

The converse need not be true as seen from the example.

Example 4.8. *Let $X = \{a, b\}$ and $Y = \{a, b, c\}$ be endowed with the topologies $\tau = \{X, \emptyset, \{a\}, \{b\}\}$ and $\sigma = \{Y, \emptyset, \{a\}, \{b, c\}\}$ respectively and suppose that g-open sets and g*open sets of (X, τ) are $\{X, \emptyset, \{a\}, \{b\}\}$ respectively. Similarly g-open sets and g* open sets of (Y, σ) are $\{Y, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$ and $\{Y, \emptyset, \{a\}, \{b, c\}\}$ respectively. Let $f : (X, \tau) \to (Y, \sigma)$ be the mapping defined by $f(a) = a$, $f(b) = b$. Then f has g-closed graph. But it is not g*closed graph.*

Theorem 4.9. *Every function with a g-closed graph has a wg-closed graph.*

Proof. It follows from the result that every g-closed set is wg-closed set. □

The converse need not be true as seen from the following example.

Example 4.10. *Let $X = \{a, b\}$ and $Y = \{a, b, c\}$ be endowed with the topologies $\tau = \{x, \Phi, \{a\}, \{b\}\}$ and $\sigma = \{X, \emptyset, \{a, b\}\}$ respectively. Suppose that g-open and wg-open sets of (X, τ) are $\{X, \emptyset, \{a\}, \{b\}\}$ respectively. Similarly g-open and wg-open sets of $\{x, \Phi, \{a\}, \{b\}, \{a, b\}\}$ and $\{x, \Phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$ respectively. Let $f : (X, \tau) \to (Y, \sigma)$ be the mapping defined by $f(a) = a$, $f(b) = b$. Then f has wg-closed graph. But it is not g-closed graph.*

Theorem 4.11 ([2])**.** *If $A \subset X$, $A \subset g - Cl(A) \subset Cl(A)$.*

Theorem 4.12. *If $A \subset X$, $A \subset wg - Cl(A) \subset g - Cl(A) \subset Cl(A)$.*

The Proof is Obvious from the result that every closed set is g-closed set and every g-closed set is wg-closed set.

Theorem 4.13. *Every Function with a strongly g-closed graph has a g-closed graph.*

The converse need not be true as seen from the following example.

Example 4.14. *Let $X = \{a, b\}$ and $Y = \{a, b, c\}$ be endowed with the topologies $\tau = \{X, \emptyset, \{a\}, \{b\}\}$ and $\sigma = \{Y, \emptyset, \{c\}, \{a, c\}, \{b, c\}\}$ respectively. Let $f : (X, \tau) \to (Y, \sigma)$ be the mapping defined by $f(a) = a$, $f(b) = b$. Suppose that, g-open sets of (X, τ) are all the subsets of X. Similarly, g-open sets of (Y, σ) are $(Y, \Phi, \{c\}, \{a, c\}, \{b, c\}\}$ respectively. Then f is g-closed graph, but it is not strongly g-closed graph.*

Theorem 4.15. *Every function with a *wg-closed graph has a wg-closed graph.*

The converse need not be true as seen from the following example.

Example 4.16. *Let $X = \{a, b\}$ and $Y = \{a, b, c\}$ be endowed with the topologies $\tau = \{X, \emptyset, \{a\}, \{b\}\}$ and $\sigma = \{Y, \emptyset, \{c\}, \{a, c\}, \{b, c\}\}$ respectively. Let $f : (X, \tau) \to (Y, \sigma)$ be the mapping defined by $f(a) = a$, $f(b) = b$. Suppose that, wg-open sets of (X, τ) are all the subsets of X. Similarly, wg-open sets of (Y, σ) are $(Y, \Phi, \{c\}, \{a, c\}, \{b, c\}\}$ respectively. Then f is wg-closed graph, but not *wg-closed graph.*

Theorem 4.17. *Every function with a g*closed graph has a wg-closed graph.*

Proof. It follows from the result that every g*closed set is wg-closed set. □

The converse need not be true as seen from the following example.

Example 4.18. *Let $X = \{a, b\}$ and $Y = \{a, b, c\}$ be endowed with the topologies $\tau = \{X, \emptyset, \{a\}, \{b\}\}$ and $\sigma = \{Y, \emptyset, \{a\}, \{b, c\}\}$ respectively. Let $f : (X, \tau) \to (Y, \sigma)$ be the mapping defined by $f(a) = a$, $f(b) = b$. Then f has wg-closed graph. But it is not g*closed graph.*

Theorem 4.19. *Every Function with a closed graph has a strongly g-closed graph.*

The converse need not be true as seen from the following example.

Example 4.20. *Let* $X = \{a, b\}$ *and* $Y = \{a, b, c, d\}$ *be endowed with the topologies* $\tau = \{X, \emptyset, \{a\}, \{b\}\}$ *and* $\sigma = \{Y, \emptyset, \{c, d\}\}$ *respectively. Let* $f : (X, \tau) \to (Y, \sigma)$ *be the mapping defined by* $f(a) = a$, $f(b) = b$. *Then* f *has strongly g-closed graph. But it is not closed graph.*

Theorem 4.21. *Every Function with a closed graph has a *wg-closed graph.*

The converse need not be true as seen from the following example.

Example 4.22. *Let* $X = \{a, b\}$ *and* $Y = \{a, b, c, d\}$ *be endowed with the topologies* $\tau = \{X, \emptyset, \{a\}, \{b\}\}$ *and* $\sigma = \{Y, \emptyset, \{a, b, c\}, \{d\}\}$ *respectively. Let* $f : (X, \tau) \to (Y, \sigma)$ *be the mapping defined by* $f(a) = a$, $f(b) = b$. *Suppose that, wg-open sets of* (X, τ) *are all the subsets of* X. *Similarly, wg-open sets of* (Y, σ) *are* $\{Y, \Phi, \{a\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$ *respectively. Then* f *has *wg-closed graph. But not closed graph.*

The above discussions can be summarized in the following diagram 1 where $A \to B$ means A imply B and $A \nrightarrow B$ means A does not imply B.

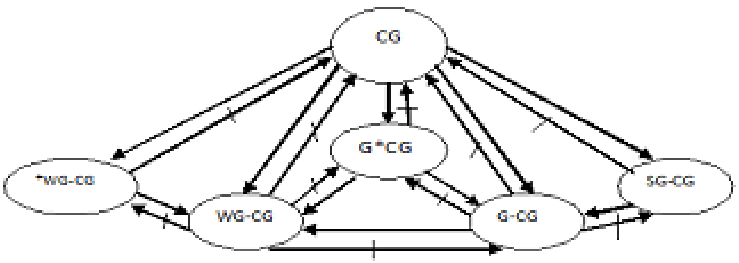

Diagram 1

The abbreviations in the above diagram have the following meanings. CG = Closed graph, G-CG = g-closed graph, G*CG = g*closed graph, WG-CG = wg-closed graph, SG-CG = strongly g-closed graph and *WG-CG = *wg-closed graph.

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