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Oscillation Criteria of Second Order Difference Equation With Negative Nonlinear Neutral Term

Research Article

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Abstract: In this paper, the authors obtain sufficient conditions for the oscillatory behavior of neutral delay difference equation of the form

$$\Delta(a_n\Delta(x_n - p_n x^{\alpha}_{n-k})) + q_n x^{\beta}_{n-l} = 0$$

where $\{a_n\}$, $\{p_n\}$ and $\{q_n\}$ are positive real sequences, α and β are ratios of odd positive integers. Examples are provided to illustrate the main results.

MSC: 39A11

Keywords: Oscillation, negative nonlinear neutral term, second order difference equation.

1. Introduction

In this paper, we are concerned with the following second order nonlinear neutral difference equation of the form

$$\Delta(a_n\Delta(x_n - p_n x^{\alpha}_{n-k})) + q_n x^{\beta}_{n-l} = 0,\ n \ge n_0 \tag{1}$$

where n_0 is a nonnegative integer, subject to the following conditions:

(H_1) $\{a_n\}$, $\{p_n\}$ and $\{q_n\}$ are real positive sequences;

(H_2) $0 < \alpha \le 1$ and β are ratios of odd positive integers;

(H_3) l and k are positive integers;

(H_4) $0 \le p_n \le p < 1$ for all $n \ge n_0$.

Let $\theta = \max\{k, l\}$. By a solution of equation (1), we mean a real sequence $\{x_n\}$ which is defined for all $n \ge n_0 - \theta$, and satisfies equation (1) for all $n \ge n_0$. A solution of equation (1) is said to be oscillatory if it is neither eventually positive nor eventually negative, and nonoscillatory otherwise.
Recently, there has been a great interest in investigating the oscillatory behavior of difference equations, see [1, 2] and the references cited therein. There are number of results concerning oscillatory and asymptotic behavior of solutions of neutral

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difference equations of the form (1) with linear neutral term, and very few results are available for the neutral difference equations with nonlinear neutral term in the literature, see for example [4, 5, 7–12], and the references cited therein.

In [9], the authors investigated the oscillation of all solutions of equation (1) with $\sum_{n=n_0}^{\infty}\frac{1}{a_n}=\infty$. In order to solve the problem completely, we examine the other case where $\sum_{n=n_0}^{\infty}\frac{1}{a_n}<\infty$, which appears to be more difficult than the former. To accomplish this is the main purpose of this paper. After establishing necessary preliminary results in Section 2, we obtain sufficient conditions for the oscillation of all solutions of equation (1) in Section 3. Finally in Section 4, we provide some examples to illustrate the main results. Thus, the results presented in this paper are new and complement to the existing results reported in [5, 8–12].

2. Preliminary Lemmas

In this section, we present some lemmas which are useful to prove our main results. Define

$$\begin{aligned} z_n &= x_n - p_n x_{n-k}^{\alpha} \\ R_n &= \sum_{s=n_1}^{n-1}\frac{1}{a_s}, \text{ and } A_n=\sum_{s=n}^{\infty}\frac{1}{a_s}. \end{aligned}$$

Note that from the assumptions and the form of the equation (1), it is enough to state and prove the results for the case of eventually positive solutions only since the proof for the eventually negative is similar. We begin with the following lemma.

Lemma 2.1. *Let $\{x_n\}$ be an eventually positive solution of equation (1). Then one of the following three cases holds for all sufficiently large n:*

(I) $z_n>0,\ a_n\Delta z_n>0,\ \Delta(a_n\Delta z_n)\le 0$;

(II) $z_n>0,\ a_n\Delta z_n<0,\ \Delta(a_n\Delta z_n)\le 0$;

(III) $z_n<0,\ a_n\Delta z_n>0,\ \Delta(a_n\Delta z_n)\le 0$.

Proof. Assume that $x_n>0$, $x_{n-k}>0$, and $x_{n-l}>0$ for all $n\ge n_1$ for some $n_1\ge n_0$. From equation (1), we have

$$\Delta(a_n\Delta z_n) = -q_n x_{n-l}^{\beta}\le 0$$

for all $n\ge n_1$. Hence $\{z_n\}$ and $\{a_n\Delta z_n\}$ are eventually of one sign for all $n\ge n_1$. Then $\{z_n\}$ satisfying one of the following four cases for all $n\ge n_1$:

(I) $z_n>0,\ a_n\Delta z_n>0,\ \Delta(a_n\Delta z_n)\le 0$;

(II) $z_n>0,\ a_n\Delta z_n<0,\ \Delta(a_n\Delta z_n)\le 0$;

(III) $z_n<0,\ a_n\Delta z_n>0,\ \Delta(a_n\Delta z_n)\le 0$;

(IV) $z_n<0,\ a_n\Delta z_n<0,\ \Delta(a_n\Delta z_n)\le 0$.

Now, we shall prove that case (IV) cannot happen. If so, then we have $\lim_{n\to\infty} z_n=-\infty$. From the definition of z_n, we obtain $x_n>(-\frac{z_{n+k}}{p})^{1/\alpha}$, and therefore $\lim_{n\to\infty}\sup x_n=\infty$. Thus, there exists a subsequence $\{n_j\}$ of positive integers such that $\lim_{j\to\infty} n_j=\infty$ and $x_{n_j}=\max_{n_0\le n\le n_j} x_n\to\infty$ as $j\to\infty$. Then

$$z_{n_j}=x_{n_j}-p_{n_j}x_{n_j-k}^{\alpha}\ge x_{n_j}-p x_{n_j}^{\alpha}=(1-p x_{n_j}^{\alpha-1})x_{n_j}\to\infty$$

as $j\to\infty$ since $0<\alpha\le 1$. This contradiction completes the proof. □

Lemma 2.2. *Let $\{x_n\}$ be an eventually positive solution of equation (1) such that Case (I) of Lemma 2.1 holds. Then*

$$x_n \geq z_n \geq R_n a_n \Delta z_n, \ n \geq n_1 \geq n_0, \tag{2}$$

and $\{\frac{z_n}{R_n}\}$ is eventually decreasing.

Proof. The proof is similar to that of Lemma 2 in [10], and hence the details are omitted. □

Lemma 2.3. *Let $\{x_n\}$ be an eventually positive solution of equation (1) such that Case (II) of Lemma 2.1 holds. Then*

$$x_n \geq z_n \geq -A_n a_n \Delta z_n, \ n \geq n_1 \geq n_0, \tag{3}$$

and $\{\frac{z_n}{A_n}\}$ is eventually increasing.

Proof. The proof is similar to that of Lemma 2.3 in [5], and hence the details are omitted. □

3. Oscillation Results

In this section, we present some sufficient conditions for the oscillation of all solutions of equation (1).

Theorem 3.1. *Assume that $\beta \leq \alpha < 1$. If $l > k$,*

$$\sum_{n=n_1}^{\infty} q_n \left(M^{1-\alpha} + p_{n-l}\right)^{\beta} R_{n-k-l}^{\alpha\beta} = \infty \tag{4}$$

$$\lim_{n\to\infty} \sup \sum_{s=n-l+k}^{n-1} \frac{1}{a_s} \sum_{t=s}^{n-1} q_t > 0 \tag{5}$$

and

$$\lim_{n\to\infty} \sup \sum_{s=n_1}^{n-1} \left[M_1 A_{s+1} q_s - \frac{1}{4 a_s A_{s+1}} \right] = \infty \tag{6}$$

for any constants M and $M_1 > 0$, then every solution of equation (1) is oscillatory.

Proof. Assume that there is a nonoscillatory solution $\{x_n\}$ of equation (1), say, $x_n > 0$, $x_{n-k} > 0$, and $x_{n-l} > 0$ for all $n \geq n_1 \geq n_0$, where n_1 is chosen so that all three cases of Lemma 2.1 are hold for all $n \geq n_1$.
Case(I): From the definition of z_n, we have

$$\begin{aligned} x_n &\geq z_n + p_n z_{n-k}^{\alpha} \geq (z_{n-k}^{1-\alpha} + p_n) z_{n-k}^{\alpha} \\ &\geq (M^{1-\alpha} + p_n) z_{n-k}^{\alpha} \end{aligned} \tag{7}$$

where we have used $\{z_n\}$ is increasing and $z_n \geq M > 0$ for all $n \geq n_1$. Using (7) in equation (1), we obtain

$$\Delta(a_n \Delta z_n) + q_n (M^{1-\alpha} + p_{n-l})^{\beta} z_{n-k-l}^{\alpha\beta} \leq 0, \ n \geq n_1. \tag{8}$$

Combining (2) with (8), we have

$$\Delta(a_n \Delta z_n) + q_n (M^{1-\alpha} + p_{n-l})^{\beta} R_{n-k-l}^{\alpha\beta} (a_{n-k-l} \Delta z_{n-k-l})^{\alpha\beta} \leq 0.$$

Let $w_n = a_n\Delta z_n$. Then $w_n > 0$ and $\{w_n\}$ is an eventually positive solution of the inequality

$$\Delta w_n + q_n(M^{1-\alpha} + p_{n-l})^{\beta} R^{\alpha\beta}_{n-k-l} w^{\alpha\beta}_{n-k-l} \le 0. \tag{9}$$

But by Theorem 1 of [6] and (4), the inequality (9) has no eventually positive solution, a contradiction.

Case(II): Define

$$w_n = \frac{a_n\Delta z_n}{z_n},\ n \ge n_1. \tag{10}$$

Then $w_n < 0$ for all $n \ge N$. From (3) and (10), we have

$$-1 \le A_n w_n \le 0,\ n \ge n_1. \tag{11}$$

From the equation (1) and $x_n \ge z_n$, we have

$$\Delta(a_n\Delta z_n) + q_n z^{\beta}_{n-l} \le 0,\ n \ge n_1. \tag{12}$$

From (10) and (12), we obtain

$$\begin{aligned}\Delta w_n &\le -q_n\frac{z^{\beta}_{n-l}}{z_{n+1}} - \frac{a_n(\Delta z_n)^2}{z_n z_{n+1}}\\ &\le -M_1^{\beta-1}q_n - \frac{w_n^2}{a_n},\ n \ge n_1,\end{aligned} \tag{13}$$

where we have used $\{z_n\}$ is positive decreasing, $\beta < 1$ and $M_1 = z^{\beta-1}_{n_1-l}$. Multiplying (13) by A_{n+1} and then summing it from n_1 to $n-1$, we have

$$\sum_{s=n_1}^{n-1} A_{s+1}\Delta w_s + \sum_{s=n_1}^{n-1} M_1^{\beta-1}A_{s+1}q_s + \sum_{s=n_1}^{n-1} A_{s+1}\frac{w_s^2}{a_s} \le 0. \tag{14}$$

Using summation by parts formula in the first term of (14) and then rearranging, we obtain

$$A_n w_n - A_{n_1}w_{n_1} + \sum_{s=n_1}^{n-1} M_1^{\beta-1}A_{s+1}q_s + \sum_{s=n_1}^{n-1}\left(\frac{w_s}{a_s} + \frac{w_s^2}{a_s}\right) \le 0,$$

which on using completing the square yields

$$\sum_{s=n_1}^{n-1}\left[M_1^{\beta-1}A_{s+1}q_s - \frac{1}{4a_sA_{s+1}}\right] \le 1 + A_{n_1}w_{n_1}$$

when using (11). This contradicts with (6) as $n \to \infty$.

Case(III): From the definition of z_n, we have

$$x_{n-k} > \left(-\frac{z_n}{p}\right)^{\frac{1}{\alpha}}. \tag{15}$$

Using (15) in equation (1), we obtain

$$\Delta(a_n\Delta z_n) - \frac{1}{p^{\frac{\beta}{\alpha}}}q_n z^{\frac{\beta}{\alpha}}_{n+k-l} \le 0,\ n \ge n_1. \tag{16}$$

Summing (16) from s to $n-1$ for $n > s+1$, we have

$$a_n\Delta z_n - a_s\Delta z_s - \frac{1}{p^{\frac{\beta}{\alpha}}}\sum_{t=s}^{n-1} q_t z^{\frac{\beta}{\alpha}}_{t+k-l} \le 0. \tag{17}$$

Since z_n is negative and increasing, we obtain $\lim_{n\to\infty} z_n = c \le 0$. Let $c = 0$. Summing (17) from $n-l+k$ to $n-1$ for s, we have

$$z_{n-l+k} - z_n \le \frac{1}{p^{\frac{\beta}{\alpha}}} z_{n-l+k}^{\frac{\beta}{\alpha}} \sum_{s=n-l+k}^{n-1} \frac{1}{a_s} \sum_{t=s}^{n-1} q_t$$

or

$$\frac{z_{n-l+k}}{z_{n-l+k}^{\frac{\beta}{\alpha}}} \ge \frac{1}{p^{\frac{\beta}{\alpha}}} \sum_{s=n-l+k}^{n-1} \frac{1}{a_s} \sum_{t=s}^{n-1} q_t. \tag{18}$$

Since $\frac{z_{n-l+k}}{z_{n-l+k}^{\frac{\beta}{\alpha}}} = |z_{n-l+k}|^{1-\beta/\alpha}$ and $1 - \frac{\beta}{\alpha} > 0$, we have

$$\lim_{n\to\infty} \sup \sum_{s=n-l+k}^{n-1} \frac{1}{a_s} \sum_{t=s}^{n-1} q_t \le 0,$$

which contradicts (5). Next assume that $c > 0$. From (5), we claim that

$$\lim_{n\to\infty} \sup \sum_{s=n_1}^{n-1} \frac{1}{a_s} \sum_{t=s}^{n-1} q_t = \infty. \tag{19}$$

In fact, from (5), there is a subsequence $\{n_i\}$ and $n_{i+1} - n_i \ge l - k$ such that

$$\sum_{s=n_i-l+k}^{n_i-1} \frac{1}{a_s} \sum_{t=s}^{n_i-1} q_t \ge b > 0,$$

where b is a constant. Hence

$$\lim_{n\to\infty} \sum_{s=n_1}^{n-1} \frac{1}{a_s} \sum_{t=s}^{n-1} q_t \ge \lim_{j\to\infty} \sum_{i=1}^{j} \sum_{s=n_i-l+k}^{n_i-1} \frac{1}{a_s} \sum_{t=s}^{n_i-1} q_t = \infty,$$

where $n_j = \max\{n_i : n_i \le n\}$. From (17), we have

$$\Delta z_s + \frac{1}{p^{\frac{\beta}{\alpha}}} \frac{z_n^{\beta/\alpha}}{a_s} \sum_{t=s}^{n-1} q_t \ge 0.$$

Summing the last inequality from n_1 to $n-1$, we obtain

$$z_{n_1} - z_n \le \frac{z_n^{\beta/\alpha}}{p^{\frac{\beta}{\alpha}}} \sum_{s=n_1}^{n-1} \frac{1}{a_s} \sum_{t=s}^{n-1} q_t$$

or

$$\frac{p^{\frac{\beta}{\alpha}} z_{n_1}}{z_n^{\beta/\alpha}} \ge \sum_{s=n_1}^{n-1} \frac{1}{a_s} \sum_{t=s}^{n-1} q_t.$$

In view of $c < 0$, $\frac{p^{\frac{\beta}{\alpha}} z_{n_1}}{z_n^{\beta/\alpha}}$ has an upper bound, so

$$\lim_{n\to\infty} \sum_{s=n_1}^{n-1} \frac{1}{a_s} \sum_{t=s}^{n-1} q_t < \infty$$

which contradicts (19). This completes the proof of the theorem. □

Theorem 3.2. *Assume that $\beta = 1$. If*

$$\lim_{n\to\infty} \inf \sum_{s=n-l}^{n-1} q_s \left(R_{s-l} + K^{\alpha-1} p_{s-l} R_{s-l-k}^{\alpha}\right) = \left(\frac{l}{l+1}\right)^{l+1}, \tag{20}$$

and

$$\lim_{n\to\infty} \sup \sum_{s=n_1}^{n-1} \left[q_s A_{s+1} - \frac{1}{4a_s A_{s+1}}\right] = \infty \tag{21}$$

for any constant $K > 0$, then every solution of equation (1) is either oscillatory or tends to zero as $n \to \infty$.

Proof. Assume that there exists a nonoscillatory solution $\{x_n\}$ of equation (1), say, $x_n > 0$, $x_{n-k} > 0$, and $x_{n-l} > 0$ for all $n \geq n_1 \geq n_0$, where n_1 is chosen so that all three cases of Lemma 2.1 are hold for all $n \geq n_1$.

Case(I): From the definition of z_n and $\frac{z_n}{R_n}$ is decreasing, we have

$$x_n \;\geq\; z_n + p_n z_{n-k}^{\alpha} \geq \left(1 + K^{\alpha-1} p_n \frac{R_{n-k}^{\alpha}}{R_n}\right) z_n \tag{22}$$

where we have used $\frac{z_n}{R_n} \leq K$ for some $K > 0$. Using (22) in equation (1), we obtain

$$\Delta(a_n \Delta z_n) + q_n \left(1 + K^{\alpha-1} p_{n-l} \frac{R_{n-k-l}^{\alpha}}{R_{n-l}}\right) z_{n-l} \leq 0, \; n \geq n_1. \tag{23}$$

From (2) in (23), we have

$$\Delta(a_n \Delta z_n) + q_n \left(1 + K^{\alpha-1} p_{n-l} \frac{R_{n-k-l}^{\alpha}}{R_{n-l}}\right) R_{n-l} a_{n-l} \Delta z_{n-l} \leq 0.$$

Let $w_n = a_n \Delta z_n$. Then $w_n > 0$ and $\{w_n\}$ is an eventually positive solution of the inequality

$$\Delta w_n + q_n (R_{n-l} + K^{\alpha-1} p_{n-l} R_{n-k-l}^{\alpha}) w_{n-l} \leq 0. \tag{24}$$

But by Theorem 7.6.1 of [3] and (20), the inequality (24) has no eventually positive solution, a contradiction.

Case(II): Define

$$w_n = \frac{a_n \Delta z_n}{z_n}, \; n \geq n_1.$$

Proceeding as in Case (II) of Theorem 3.1, we obtain (11) and

$$\Delta w_n \;\leq\; -q_n - \frac{w_n^2}{a_n}$$

where we have used $\{z_n\}$ is positive decreasing, and l is a positive integer. The remaining part of the proof is similar to that of Case (II) of Theorem 3.1 and hence the details are omitted.

Case(III): In this case $z_n < 0$ and $\Delta z_n > 0$ for all $n \geq n_1$ for some sufficiently large n_1. Hence $\lim_{n\to\infty} z_n$ exists, and $z_n \leq c \leq 0$ for all n sufficiently large. Then

$$x_n = p_n x_{n-k}^{\alpha} + z_n < p x_{n-k}^{\alpha} + c. \tag{25}$$

Next, we show that $\{x_n\}$ is bounded. If this is not the case, there is a sequence $\{n_k\}$ with $n_k \to \infty$ as $k \to \infty$ such that

$$x_{n_k} = \sup_{n_1 \leq j \leq n_k} x_j \text{ and } \lim_{k\to\infty} x_{n_k} = \infty.$$

From (25) with k sufficiently large, we obtain

$$x_{n_k} \leq p x_k^{\alpha} + c$$

or

$$(1 - p x_k^{\alpha-1}) x_{n_k} \leq c$$

which, as $k \to \infty$, leads to a contradiction. Thus, $\{x_n\}$ is bounded. Let $\lim_{n\to\infty} \sup x_n = M_2 > 0$. Then there is a sequence $\{n_j\}$ such that $x_{n_j} \to M_2$ as $j \to \infty$. Now

$$z_{n_j} \geq x_{n_j} - p x_{n_j-k}^{\alpha}$$

and so

$$x_{n_j-k} \geq \frac{1}{p^{1/\alpha}}(x_{n_j} - z_{n_j})^{\frac{1}{\alpha}}.$$

Letting $j \to \infty$, we obtain

$$M_2 \geq \lim_{j\to\infty} x_{n_j-k} \geq \left(\frac{M_2}{p}\right)^{\frac{1}{\alpha}}.$$

Since $p \in (0,1)$, it follows that $M_2 = 0$, that is $\lim_{n\to\infty} x_n = 0$. This completes the proof. □

Theorem 3.3. *Assume that $\beta > 1$. If*

$$\sum_{n=n_1}^{\infty} q_n \left(1 + K^{\alpha-1} p_{n-l} \frac{R^{\alpha}_{n-k-l}}{R_{s-l}}\right)^{\beta} = \infty, \tag{26}$$

and

$$\sum_{n=n_1}^{\infty} \frac{1}{a_s} \sum_{s=n_1}^{n-1} q_s A^{\beta}_{s-l} = \infty \tag{27}$$

then every solution of equation (1) is either oscillatory or tends to zero as $n \to \infty$.

Proof. Proceeding as in the proof of Theorem 3.2, we see that Lemma 2.1 holds for all $n \geq n_1$.

Case(I): Proceeding as in the proof of Theorem 3.2(Case(I)), we have

$$\Delta(a_n \Delta z_n) + q_n \left(1 + K^{\alpha-1} p_{n-l} \frac{R^{\alpha}_{n-k-l}}{R_{n-l}}\right)^{\beta} z^{\beta}_{n-l} \leq 0, \; n \geq n_1.$$

Define

$$w_n = \frac{a_n \Delta z_n}{z^{\beta}_{n-l}}, \; n \geq n_1,$$

then $w_n > 0$, and

$$\begin{aligned} \Delta w_n &\leq -q_n \left(1 + K^{\alpha-1} p_{n-l} \frac{R^{\alpha}_{n-k-l}}{R_{n-l}}\right)^{\beta} - \frac{\beta a_{n+1} \Delta z_{n+1} \Delta z_{n-l}}{z^{\beta}_{n-l}} \\ &\leq -q_n \left(1 + K^{\alpha-1} \frac{R^{\alpha}_{n-k-l}}{R_{n-l}}\right)^{\beta}, \; n \geq n_1. \end{aligned}$$

Summing the last inequality from n_1 to $n-1$, we obtain

$$\sum_{s=n_1}^{n} q_s \left(1 + K^{\alpha-1} p_{s-l} \frac{R^{\alpha}_{s-k-l}}{R_{s-l}}\right)^{\beta} < w_{n_1} < \infty.$$

Letting as $n \to \infty$ in the last inequality, we obtain a contradiction to (26).

Case(II): From Lemma 2.3, we have

$$\begin{aligned} z_{n-l} &> -A_{n-l} a_n \Delta z_n \geq -A_{n-l} a_{n_1} \Delta z_{n_1}, \; n \geq n_1 \\ &\geq dA_{n-l} \end{aligned} \tag{28}$$

where $d = -a_{n_1} \Delta z_{n_1}$. From equation (1) and (28), we obtain

$$\Delta(-a_n \Delta z_n) \geq q_n d^{\beta} A^{\beta}_{n-l}, \; n \geq n_1.$$

Summing the last inequality from n_1 to $n-1$, we have

$$-a_n\Delta z_n \geq -a_{n_1}\Delta z_{n_1} + d^\beta \sum_{s=n_1}^{n-1} q_s A_{s-l}^\beta.$$

Dividing the last inequality by a_n and then summing it from n_1 to $n-1$, we obtain

$$z_{n_1} \geq z_{n_1} - z_n \geq d^\beta \sum_{s=n_1}^{n-1} \frac{1}{a_s} \sum_{t=n_1}^{s-1} q_t A_{t-l}^\beta.$$

Letting $n \to \infty$ in the last inequality, we obtain

$$\sum_{n=n_1}^{\infty} \frac{1}{a_n} \sum_{s=n_1}^{n-1} q_s A_{s-l}^\beta < \infty$$

a contradiction to (27).

Case(III): In this case $z_n < 0$ and $\Delta z_n > 0$ for all $n \geq n_1$. Then proceeding as in the Case (III) of Theorem 3.2, we obtain $\lim\limits_{n\to\infty} x_n = 0$. This completes the proof. □

4. Examples

In this section, we provide some examples to illustrate the main results.

Example 4.1. *Consider the second order neutral difference equation*

$$\Delta\left(2^n\Delta\left(x_n - \frac{1}{2}x_{n-2}^{1/3}\right)\right) + 3(2^n)x_{n-3}^{1/5} = 0,\ n \geq 1. \tag{29}$$

Here $a_n = 2^n$, $p_n = \frac{1}{2}$, $q_n = 3(2^n)$, $l = 3$, $k = 2$, $\alpha = \frac{1}{3}$ and $\beta = \frac{1}{5}$. Since $R_n = 1 - \frac{1}{2^{n-1}}$ and $A_n = \frac{1}{2^{n-1}}$, one can easily verify that all conditions of Theorem 3.1 are satisfied and hence every solution of equation (29) is oscillatory. In fact $\{x_n\} = \{(-1)^{15n}\}$ is one such oscillatory solution of equation (29).

Example 4.2. *Consider the second order neutral difference equation*

$$\Delta\left(n(n+1)\Delta\left(x_n - \frac{1}{2}x_{n-1}^{1/3}\right)\right) + 6(n+1)^2 x_{n-1} = 0,\ n \geq 1. \tag{30}$$

Here $a_n = n(n+1)$, $p_n = \frac{1}{2}$, $q_n = 6(n+1)^2$, $l = k = 1$, $\alpha = \frac{1}{3}$ and $\beta = 1$. Since $R_n = 1 - \frac{1}{n}$ and $A_n = \frac{1}{n}$, one can easily verify that all conditions of Theorem 3.2 are satisfied and hence every solution of equation (30) is either oscillatory or tends to zero as $n \to \infty$. In fact $\{x_n\} = \{(-1)^{3n}\}$ is one such oscillatory solution of equation (30).

Example 4.3. *Consider the second order neutral difference equation*

$$\Delta\left(n(n+1)\Delta\left(x_n - \frac{1}{n^{2/3}}x_{n-1}^{1/3}\right)\right) + \frac{n^3(8(n+1)^2(n+2) - 2n - 3)}{(n+1)(n+2)}x_{n-1}^3 = 0,\ n \geq 1. \tag{31}$$

Here $a_n = n(n+1)$, $p_n = \frac{1}{n^{2/3}}$, $q_n = \frac{n^3(8(n+1)^2(n+2)-2n-3)}{(n+1)(n+2)}$, $\alpha = \frac{1}{3}$, $\beta = 3$, $k = 1$ and $l = 1$. Since $R_n = 1 - \frac{1}{n}$ and $A_n = \frac{1}{n}$, one can easily verify that all conditions of Theorem 3.3 are satisfied and hence every solution of equation (31) is either oscillatory or tends to zero as $n \to \infty$. In fact $\{x_n\} = \{\frac{(-1)^n}{n}\}$ is one such oscillatory solution of equation (31).

We conclude this paper with the following remark.

Remark 4.4. *In this paper, we have presented some new oscillation results for the equation* (1)*, and it would be interesting to improve the results of Theorem 3.2 and Theorem 3.3 to similar to that of Theorem 3.1.*

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