International Journal of Mathematics And its Applications
Volume 5, Issue 2–C (2017), 421–434.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Properties of Stone Almost Distributive Fuzzy Lattice

Research Article

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Abstract: The concept of properties of stone Almost distributive fuzzy lattice (Stone ADFL) as an extension of properties of Stone Almost distributive lattice (Stone ADL) is introduced. Necessary and sufficient conditions for properties of stone Almost distributive lattice (Stone ADL) to become properties of stone Almost distributive fuzzy lattice (Stone ADFL) are given.Different characterization of Stone Almost distributive fuzzy lattice (Stone ADFL) are investigated.

Keywords: Almost Distributive Lattice,Almost Distributive Fuzzy Lattice,Fuzzy poset, Fuzzy Lattice, Ston Almost Distributive Lattice,Stone Almost Distributive Fuzzy Lattice.

1. Introduction

The concept of an Almost distributive lattice was introduced by U.M.Swamy and G.C.Rao in [10] as a common abstraction of most of the existing ring theoretic and lattice theoretic generalization of a Boolean algebra. The notion of pseudo-complementation in an Almost distributive lattice was introduced by U.M.Swamy,G.C.Rao and G.N.Rao in [12] and they observe that an Almostdistributive lattice have more than one pseudo-complementation while it is unique in case of distributive lattice. U.M.Swamy,G.C.Rao and G.N.Rao introduce the concept of Stone Almost distributive lattice in [13] with respect to a pseudo-complementation on it, then it is a stone Almost distributive lattice (*Stone* ADL) with any other pseudo-complementation.The notion of properties of Stone Almost distributive lattice was introduced by G.C.Rao and Mihret Alemneh in [6]. The concept of a fuzzy set was introduced by Zadeh in [14] and this concept was adapted by Goguen in [3] and Sanchez in [11] use to define and study fuzzy relations. In this paper we use fuzzy partial order relation defined in [4] and the idea of fuzzy lattice in [4] to extend Properties of Stone Almost distributive lattice (*Stone* ADL) in [6] to Properties of Stone Almost distributive fuzzy lattice (*Stone* ADFL) and we characterized Properties of Stone Almost distributive fuzzy lattice (*Stone* ADFL).

2. Preliminaries

Definition 2.1. *An algebra* $(R, \vee, \wedge, 0)$ *of type* $(2, 2, 0)$ *is called an Almost distributive lattice* (ADL)*,if the following condition holds:*

(1). $a \vee 0 = a$,$0 \wedge a = 0$

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(2). $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

(3). $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$

(4). $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(5). $(a \vee b) \wedge b = b$

Lemma 2.2 ([7]). *Let $(R, \vee, \wedge)$ be an ADL with 0. For any $a, b, c \in R$ we have the following:*

(1). $a \wedge b = b \Leftrightarrow a \vee b = a$

(2). $a \wedge b = a \Leftrightarrow a \vee b = b$

(3). $\bigwedge$ *is associative.*

(4). $a \wedge b \wedge c = b \wedge a \wedge c.$

(5). $(a \vee b) \wedge c = (b \vee a) \wedge c.$

(6). $a \wedge b = 0 \Leftrightarrow b \wedge a = 0.$

(7). $a \wedge (a \vee b) = a$,$(a \wedge b) \vee b = b$ *and* $a \vee (b \wedge a) = a.$

(8). $a \wedge a = a$ *and* $a \vee a = a.$

(9). $a = a \vee 0$ *and* $0 \vee a = a.$

(10). $0 \wedge a = 0$ *and* $a \wedge 0 = 0.$

Definition 2.3 ([7]). *Let R be an ADL with 0. For any $a, b \in R$,define $a \leq b$ if and only if $a \wedge b = a$ or equivalently $a \vee b = b$.Then "$\leq$" is a partial order relation on R.*

Lemma 2.4 ([7]). *Let R be an ADL with 0, and $m \in R$. Then the following are equivalent:*

(1). *m is maximal with respect to partial order "$\leq$".*

(2). $m \vee a = m$,*for all* $a \in R.$

(3). $m \wedge a = a$, *for all* $a \in R.$

Definition 2.5 ([7]). *Let $(R, \vee, \wedge, 0)$ be an ADL with 0. Anon-empty subset I of R is an ideal of R,if $a \vee b \in R$ and $a \wedge x \in R$. Whenever $a, b \in I$ and $x \in R$.*

Proposition 2.6 ([7]). *For any $a, b \in R$.*

(1). $(a] \vee (b] = (a \vee b] = (b \vee a].$

(2). $(a] \wedge (b] = (a \wedge b] = (b \wedge a].$

Definition 2.7 ([6]). *Let $(R, \vee, \wedge)$ be an ADL with 0.Then a unary operation $\star$ on R is called a pseudo- complementation on R if,*

(1). $a \wedge a^\star = 0.$

(2). $a \wedge b = 0 \Rightarrow a^\star \wedge b = b.$

(3). $(a \vee b)^\star = a^\star \wedge b^\star$, *for all* $a, b \in R$.

The unary operation $\star$ is called a pseudo-complementation of a in R.

Lemma 2.8 ([6]). *Let R be an ADL with 0, and $\star$ be a pseudo-complementation on R.Then for any $a, b \in R$,the following condition holds:*

(1). $0^\star$ *is maximal.*

(2). If a is maximal,then $a^\star = 0$.

(3). $0^{\star\star} = 0$.

(4). $a^{\star\star} \wedge a = a$.

(5). $a^\star = 0 \Leftrightarrow a^{\star\star}$ *is maximal.*

(6). $a^\star \leq 0^\star$.

(7). $a^\star \wedge b^\star = b^\star \wedge a^\star$.

(8). $a^\star \leq (a \wedge b)^\star$ *and* $b^\star \leq (a \wedge b)^\star$.

(9). $a \leq b \Rightarrow b^\star \leq a^\star$.

(10). $a^\star \leq b^\star \Leftrightarrow b^{\star\star} \leq a^{\star\star}$.

(11). $a = 0 \Leftrightarrow a^{\star\star} = 0$.

Definition 2.9 ([6]). *Ahomomorphism between ADL_s, R_1 and R_2 is a mapping $f : R_1 \rightarrow R_2$ satisfying the following condition:*

(1). $f(a \wedge b) = f(a) \wedge f(b)$.

(2). $f(a \vee b) = f(a) \vee f(b)$.

(3). $f(0) = 0$, *for all* $a, b, 0 \in R$.

Definition 2.10 ([6]). *Let R be an ADL with 0, and $\star$ be a pseudo- complementation on R.Then R is called a stone ADL if,for any $a \in R$,$a^\star \vee a^{\star\star} = 0^\star$.*

Lemma 2.11 ([6]). *Let R be a stone ADL, and $a, b \in R$.Then the following condition holds:*

(1). $0^\star \wedge a = a$ *and* $0^\star \vee a = 0^\star$.

(2). $a^{\star\star\star} = a^\star$.

(3). $(a \wedge b)^\star = a^\star \vee b^\star$.

(4). $(a \wedge b)^{\star\star} = a^{\star\star} \wedge b^{\star\star}$.

(5). An element $a \in [0, 0^\star]$ is complemented if and only if $a = b^\star$ for some $b \in R$.

Definition 2.12 ([6]). *If R is an ADL with a maximal element,then the element $a \in R$ is called a complemented element if there exist an element $b \in R$ such that $a \wedge b = 0$ and $a \vee b$ is a maximal element of R. Here b is called a complement of a.*

Definition 2.13 ([6]). *Let R be an ADL with 0 and maximal element. The center of R is defined as the set of all complemented elements of R and it is denoted by $B(R)$ or simply B.*

Theorem 2.14. *An ideal I of an a ADL R is complemented if and only if $I = (a]$ for some $a \in B(R)$.*

Definition 2.15 ([4]). *Let X be a set ,A function $A : X \times X \to [0, 1]$ is said to be fuzzy partial order relation if it satisfies the following condition;*

(1). $A(x, x) = 1,$ for all $x\ inX$ that is A is reflexive.

(2). $A(x, y) > 0,$ and $A(y, x) > 0$ implies that $x = y$ .That is A is antisymmetric.

(3). $A(x, z) \geq sup_{y\ inX} min[A(x, y), A(y, z)] > 0$. That is A is transitive.

If A is a fuzzy partial order relation in a set X,then (X, A) is called a fuzzy partial order relation or fuzzy poset.

Definition 2.16 ([4]). *Let (X, A) be a fuzzy poset.Then (X, A) is a fuzzy lattice if and only if $x \vee y$, and $x \wedge y$ exists for all $x, y \in X$.*

Definition 2.17 ([5]). *Let(X, A) be a fuzzy lattice and $Y \subseteq X$. Then Y is an ideal of (X, A).*

(1). If $x \in X$,$y \in Y$ and $A(x, y) > 0$,then $x \in Y$.

(2). If $x, y \in Y$,then $x \vee y \in Y$.

Definition 2.18 ([1]). *Let $(R, \vee, \wedge, 0)$ be an algebra of type $(2, 2, 0)$ and we call (R, A) is an Almost Distributive Fuzzy Lattice(ADFL) if the following condition satisfied:*

(1). $A(a, a \vee 0) = A(a \vee 0, a) = 1$.

(2). $A(0, 0 \wedge a) = A(0 \wedge a, 0) = 1$.

(3). $A((a \vee b) \wedge c, (a \wedge c) \vee (b \wedge c)) = A((a \wedge c) \vee (b \wedge c), (a \vee b) \wedge c) = 1$.

(4). $A(a \wedge (b \vee c), (a \wedge b) \vee (a \wedge c)) = A((a \wedge b) \vee (a \wedge c), a \wedge (b \vee c)) = 1$.

(5). $A(a \vee (b \wedge c), (a \vee b) \wedge (a \vee c)) = A((a \vee b) \wedge (a \vee c), a \vee (b \wedge c)) = 1$.

(6). $A((a \vee b) \wedge b, b) = A(b, (a \vee b) \wedge b) = 1$, for all $a, b, c \in R$.

Definition 2.19 ([1]). *Let (R, A) be an ADFL. Then for any $a, b \in R$ $a \leq b$ if and only if $A(a, b) > 0$.*

Definition 2.20 ([2]). *Let (R_1, A_1) and (R_2, A_2) be two $ADFL_s$. Then $(A_1 \times A_2)((a, b), (c, d)) = min\{A_1(a, c), A_2(b, d)\}$.*

Definition 2.21 ([2]). *Let (R_1, A_1) and (R_2, A_2) be two $ADFL_s$. Then a mapping $f : (R_1, A_1) \to (R_2, A_2)$ is said to be a fuzzy latice homomorphism.If it satisfy the following condition for any $x, y, 0 \in R_1$:*

(1). $A_2(f(x \wedge y), f(x) \wedge f(y)) = A_2(f(x) \wedge f(y), f(x \wedge y)) = 1$.

(2). $A_2(f(x \vee y), f(x) \vee f(y)) = A_2(f(x) \vee f(y), f(x \vee y)) = 1$.

(3). $A_2(f(0), 0) > 0$.

3. Properties of Stone Almost Distributive Fuzzy Lattice

Definition 3.1. *Let (R, A) be an ADFL with 0 . Then a unary operation $\star$ on (R, A) is called a pseudo-complemented if,any $a, b \in R$ satisfy the following condition:*

(1). $A(a \wedge a^\star, 0) > 0$.

(2). $A(a \wedge b, 0) > 0 \Rightarrow A(b, a^\star \wedge b) > 0$.

(3). $A(a^\star \wedge b^\star, (a \vee b)^\star) > 0$.

Here $a^\star$ is called a pseudo-complement of a in (R, A) and an ADFL with a pseudo-complementation is called a pseudo-complemented ADFL.

Lemma 3.2. *Let (R, A) be a pseudo-complemented Almost distributive fuzzy lattice ,then $A(a \wedge a^\star, 0) > 0$ if and only if $a \wedge a^\star = 0$.*

Definition 3.3. *Let (R, A) be an ADFL. Then an element $a \in R$ is called a dense element if $A(a^\star, 0) > 0$.*

Definition 3.4. *Let (R, A) be an ADFL with 0 and $\star$ be a pseudo- complementation on (R, A).Then (R, A) is called a stone ADFL if, any $a \in R$,$A(0^\star, a^\star \vee a^{\star\star}) > 0$.*

Example 3.5. *Let $(R, +, ., 0)$ be a commutative regular ring with unity.Let (R, A) be an ADFL and for any $a \in R$,let a^0 be the unique idempotent element in R such that $A(a, a^0 a a^0) = A(a^0 a a^0, a) = 1$.For any $a, b \in R$, define by:*

(1). $A(a \wedge b, a^0 b) = A(a^0 b, a \wedge b) = 1$

(2). $A(a \vee b, a + (1 - a^0)b) = A(a + (1 - a^0)b, a \vee b) = 1$

(3). $A(a^\star, 1 - a^0) = A(1 - a^0, a^\star) = 1$. *Then clearly (R, A) is an ADFL under the given condition.*

(1). Let $a \in R$. Then $A(a \wedge a^\star, 0) = A(a^0 a^\star, 0) = A(a^0(1 - a^0), 0) = A(a^0 - a^0 a^0, 0) = A(a^0 - a^0, 0)$ since $a^0 a^0 = a^0 =$ $A(0, 0) = 1$. Similarly $A(0, a \wedge a^\star) = 1$. Hence $A(a \wedge a^\star, 0) = A(0, a \wedge a^\star) = 1$. Therefore we have $A(a \wedge a^\star, 0) > 0$.

(2). Let $a, b \in R$ and $A(a \wedge b, 0) > 0$. That is $A(a^0 b, 0) > 0$. Now, $A(a^\star \wedge b, b) = A((a^\star)^0 b, b) = A((1 - a^0)^0 b, b) = A((1 - a^0)b, b) = A(b - a^0 b, b) = A(b - a \wedge b, b) = A(b, b) = 1$ since $a \wedge b = 0$. Similarly $A(b, a^\star \wedge b) = 1$. Hence $A(a^\star \wedge b, b) = A(b, a^\star \wedge b) = 1$. Let $a, b \in R$, so that $A(ab, 0) > 0$,then $a^0 + b^0$ is also idempotent and $A((a+b)^0, a^0 + b^0) = A(a^0 + b^0, (a + b)^0) = 1$.

(3).
$$\begin{aligned}
A(a^\star \wedge b^\star, (a \vee b)^\star) &= A(a^\star \wedge b^\star, 1 - (a \vee b)^0) \\
&= A(a^\star \wedge b^\star, 1 - (a + (1 - a^0)b)^0 \\
&= A(a^\star \wedge b^\star, 1 - (a^0 + (1 - a^0)^0 b^0 \\
&= A(a^\star \wedge b^\star, 1 - (a^0 + b^0 - a^0 b^0) \text{ since } (1 - a^0)^0 = 1 - a^0 \\
&= A(a^\star \wedge b^\star, 1 - a^0 - b^0 + a^0 b^0) \\
&= A(a^\star \wedge b^\star, (1 - a^\star)(1 - b^\star) \\
&= A(a^\star \wedge b^\star, a^\star \wedge b^\star) = 1.
\end{aligned}$$

Hence $A(a^\star \wedge b^\star, (a \vee b)^\star) = 1$. So that we get $A(a^\star \wedge b^\star, (a \vee b)^\star) > 0$. Thus $\star$ is a pseudo- complementation on (R, A). Now, for any $a \in R$,

$$\begin{aligned}
A(1, a^\star \vee a^{\star\star}) &= A(1, (1-a^0) \vee (1-a^0)^\star) \\
&= A(1, (1-a^0) \vee (1-(1-a^0)^0) \\
&= A(1, (1-a^0) \vee (1-1+a^0) \ \text{ since } (1-a^0)^0 = 1-a^0 \\
&= A(1, (1-a^0) \vee a^0) \\
&= A(1, (1-a^0) + (1-(1-a^0)^0)a^0) \\
&= A(1, (1-a^0) + (1-1+a^0)a^0) \quad \text{since } (1-a^0)^0 = 1-a^0 \\
&= A(1, (1-a^0) + a^0a^0) \\
&= A(1, 1-a^0+a^0) \quad \text{since } a^0a^0 = a^0 \\
&= A(1,1) = 1 > 0.
\end{aligned}$$

Hence $A(1, a^\star \vee a^{\star\star}) > 0$. Therefore (R, A) is a stone ADFL with respect to $\star$ a pseudo- complementation. Now,we prove some properties of a pseudo-complementation in a stone ADFL (R, A).

Lemma 3.6. *Let (R, A) be a stone ADFL and $a, b \in R$. Then the following condition holds:*

(1). $A(a, 0^\star) > 0$

(2). $A(a^\star, a^{\star\star\star}) = A(a^{\star\star}, a^\star) = 1$ and $A(0^{\star\star}, 0) > 0$

(3). $A((a \wedge b)^\star, a^\star \vee b^\star) > 0$.

(4). $A(a^{\star\star} \wedge b^{\star\star}, (a \wedge b)^{\star\star}) > 0$ and $A((a \vee b)^{\star\star}, a^{\star\star} \vee b^{\star\star}) > 0$

(5). An element $a \in [0, 0^\star]$ is complemented if and only if $A(a, b^\star) = A(b^\star, a) = 1$, for some $b \in R$.

Proof.

(1). Let (R, A) be an ADFL. Then for $a \in R$ and from definition of ADL $0 \wedge a = 0 \Rightarrow A(0 \wedge a, 0) > 0 \Rightarrow A(a, 0^\star) > 0$, since $a \leq a \vee 0^\star = (a \wedge 0^\star) \vee 0^\star = 0^\star$.

Hence $A(a, 0^\star) > 0$.

(2). Let $a \in R$. Since (R, A) is pseudo-complemented.

$$\begin{aligned}
A(a^\star, a^{\star\star\star}) &= A(a^\star, (a^{\star\star})^\star) \\
&= A(a^\star, (a^{\star\star} \vee a)^\star), \quad \text{since } a^{\star\star} = a^{\star\star} \vee a \\
&= A(a^\star, a^{\star\star\star} \wedge a^\star) \\
&= A(a^\star, a^\star) = 1 > 0 \ \text{ since } a^{\star\star} \wedge a^\star = 0 \Rightarrow a^{\star\star\star} \wedge a^\star = a^\star.
\end{aligned}$$

Hence $A(a^\star, a^{\star\star\star}) = 1$. Similarly $A(a^{\star\star\star}, a^\star) = 1$. Therefore $A(a^\star, a^{\star\star\star}) = A(a^{\star\star\star}, a^\star) = 1$. Again from stone ADL we have $a^\star \vee a^{\star\star} = 0^\star$ and $a \wedge a^\star = 0$. Now,$A(0^{\star\star}, 0) = ((0^\star)^\star, 0) = A((a^\star \vee a^{\star\star})^\star, 0) = A(a^{\star\star} \wedge a^{\star\star\star}, 0) = A(0, 0) = 1$ since $a^{\star\star} \wedge a^{\star\star\star} = 0$. Similarly $A(0, 0^{\star\star}) = 1$. Hence $A(0^{\star\star}, 0) = A(0, 0^{\star\star}) = 1$. Hence $A(0^{\star\star}, 0) > 0$.

(3). $A((a\wedge b)^{\star}, a^{\star}\vee b^{\star}) = A(a^{\star}\vee b^{\star}, a^{\star}\vee b^{\star}) = 1$ since $(a\wedge b)^{\star} = a^{\star}\vee b^{\star}$ from pseudo-complemented ADL. Hence $A((a\wedge b)^{\star}, a^{\star}\vee b^{\star}) > 0$

(4). $(4)A(a^{\star\star}\wedge b^{\star\star}, (a\wedge b)^{\star\star}) = A(a^{\star\star}\wedge b^{\star\star}, ((a\wedge b)^{\star})^{\star})$

$$= A(a^{\star\star}\wedge b^{\star\star}, (a^{\star}\vee b^{\star})^{\star}), \quad \text{since } (a\wedge b)^{\star} = a^{\star}\vee b^{\star}$$

$$= A(a^{\star\star}\wedge b^{\star\star}, a^{\star\star}\wedge b^{\star\star}) = 1 > 0, \quad \text{since } (a^{\star}\vee b^{\star})^{\star} = a^{\star\star}\wedge b^{\star\star}.$$

Hence $A(a^{\star\star}\wedge b^{\star\star}, (a\wedge b)^{\star\star}) > 0$ and

$$\begin{aligned} A((a\vee b)^{\star\star}, a^{\star\star}\vee b^{\star\star}) &= A(((a\vee b)^{\star})^{\star}, a^{\star\star}\vee b^{\star\star}) \\ &= A((a^{\star}\wedge b^{\star})^{\star}, a^{\star\star}\vee b^{\star\star}), \quad \text{since } (a\vee b)^{\star} = a^{\star}\wedge b^{\star}. \\ &= A(a^{\star\star}\wedge b^{\star\star}, a^{\star\star}\wedge b^{\star\star}) = 1 > 0, \quad \text{since } (a^{\star}\wedge b^{\star})^{\star} = a^{\star\star}\wedge b^{\star\star}. \end{aligned}$$

Hence $A((a\vee b)^{\star\star}, a^{\star\star}\vee b^{\star\star}) > 0$.

(5). Suppose $a \in [0, 0^{\star}]$ is a complement element of (R, A). Then there exist $b \in R$ such that $A(a\wedge b, 0) > 0$ and $A(0^{\star}, a\vee b) > 0$.

$$\begin{aligned} A(b\wedge a, 0) = A(b\wedge(a\wedge(a\vee b), 0) &= A((b\wedge a)\wedge(a\vee b), 0) \\ &= A(((b\wedge a)\wedge a)\vee((b\wedge a)\wedge b), 0), \quad \text{since } b\wedge a\wedge c = a\wedge b\wedge c \\ &= A((b\wedge a)\vee(a\wedge b\wedge b), 0), \quad \text{since } b\wedge b = b \\ &= A(a\wedge((b\wedge a)\vee b), 0) = A((a\wedge(b\wedge a)\vee(a\wedge b), 0) \\ &= A(((a\wedge b)\wedge a)\vee(a\wedge b), 0) \\ &= A((0\wedge a)\vee 0, 0) = A(0, 0) = 1 > 0. \end{aligned}$$

Hence $A(b\wedge a, 0) > 0$. Since 0 is the least element $0 \le b\wedge a \Rightarrow A(0, b\wedge a) > 0$. So that we get $b\wedge a = 0$. Which implies $A(b^{\star}\wedge a, a) = A(a, b^{\star}\wedge a) = 1$. Now,

$$\begin{aligned} A(b^{\star}, a) &= A((b\vee 0)^{\star}, a), \quad \text{since } b = b\vee 0 \\ &= A(b^{\star}\wedge 0^{\star}, a) \\ &= A(b^{\star}\wedge(a\vee b), a), \quad \text{since } a\vee b = 0^{\star} \\ &= A((b^{\star}\wedge a)\vee(b^{\star}\wedge b), a) \\ &= A((b^{\star}\wedge a)\vee 0, a) \\ &= A(a\vee 0, a) \\ &= A(a, a) = 1, \quad \text{since } a\vee 0 = a. \end{aligned}$$

We have $A(b^{\star}, a) = 1$. Similarly $A(a, b^{\star}) = 1$. Hence $A(b^{\star}, a) = A(a, b^{\star}) = 1$. Conversely,assume that $A(b^{\star}, a) = A(a, b^{\star}) = 1$ for some $b \in R$. Then

$$\begin{aligned} A(a\wedge b^{\star\star}, 0) &= A(a\wedge(b^{\star})^{\star}, 0) \\ &= A(a\wedge a^{\star}, 0) \\ &= A(0, 0) = 1, \quad \text{since } a\wedge a^{\star} = 0 \end{aligned}$$

Similarly $A(0, a \wedge b^{\star\star}) = 1$. Hence $A(a \wedge b^{\star\star}, 0) = A(0, a \wedge b^{\star\star}) = 1$.

$$\begin{aligned} A(a \vee b^{\star\star}, o^{\star}) &= A(b^{\star} \vee b^{\star\star}, 0^{\star}) \quad \text{since } a = b^{\star} \\ &= A(0^{\star}, 0^{\star}) = 1 \quad \text{since } b^{\star} \vee b^{\star\star} = 0^{\star} \ . \end{aligned}$$

Similarly ,$A(0^{\star}, a \vee b^{\star\star}) = 1$. Hence $A(a \vee b^{\star\star}, 0^{\star}) = A(0^{\star}, a \vee b^{\star\star}) = 1$. Therefore $b^{\star\star}$ is the complement of a in $[0, 0^{\star}]$.

□

In general,if (R, A) is an ADFL with a maximal element,then an element $a \in R$ is called a complemented element if there exists an element $b \in R$ such that $A(a \wedge b, 0) > 0$ and $A((a \vee b) \vee x, a \vee b) > 0$ for all $x \in R$. Here b is called a complement of a.Unlike in the distributive fuzzy lattice ,a complemented element in an ADFL (R, A) need not have a unique element. If a is a complemented element in R,then we denote the set of all complements of a by $B_A(a)$.

Definition 3.7. *Let (R, A) be an ADFL with maximal element. Then the center of (R, A) is defined by $B_A(R) = \{a \in R | A(a \wedge b, 0) > 0$ and $A(x, (a \vee b) \wedge x) > 0$ for some $b \in R$, for all $x \in R\}$ and it is denoted by $B_A(R)$. So that $B_A(R)$ is called Birkhoff center of an ADFL (R, A).*

Lemma 3.8. *If (R, A) is an ADFL with a maximal element and a is a complemented element in (R, A),then $B_A(a)$ is a sub-ADFL of (R, A).*

Lemma 3.9. *Let (R, A) be a pseudo-complemented ADFL with center$B_A(R)$ and $a \in R$.Then $a \in B_A(R)$ if and only if $A((a \vee a^{\star}) \vee x, a \vee a^{\star}) > 0$ for all $x \in R$.*

Proof. Let (R, A) be a pseudo-complemented ADFL and $a \in B_A(R)$.Then there exists an element $b \in R$ such that $A(a \wedge b, 0) > 0$ and $A((a \vee b) \vee x, a \vee b) > 0$. Since $0 \leq a \wedge b \Rightarrow A(0, a \wedge b) > 0$. Hence $a \wedge b = 0$ by anti symmetry property of A. So $a \wedge b = 0 \Rightarrow A(b, a^{\star} \wedge b) > 0$ by Definition 3.1. Then

$$\begin{aligned} A((a \vee a^{\star}) \wedge (a \vee b), a \vee b) &= A([(a \vee a^{\star}) \wedge a] \vee [(a \vee a^{\star}) \wedge b], a \vee b) \\ &= A([(a \wedge a) \vee (a^{\star} \wedge a)] \vee [(a \wedge b) \vee (a^{\star} \wedge b)], a \vee b) \\ &= A([a \vee 0] \vee [0 \vee b], a \vee b) \ \ \text{since } a^{\star} \wedge a = 0,\ a \wedge b = 0 \text{ and } a^{\star} \wedge b = b \\ &= A(a \vee b, a \vee b) = 1 > 0 \quad \text{since } a \vee 0 = a \text{ and } 0 \vee b = b. \end{aligned}$$

Hence $A((a \vee a^{\star}) \wedge (a \vee b), a \vee b) = 1$. Similarly $A(a \vee b, (a \vee a^{\star}) \wedge (a \vee b)) = 1 > 0$. So that $(a \vee a^{\star}) \wedge (a \vee b) = a \vee b$ which implies that $a \vee a^{\star}$ is maximal element. Therefor $A((a \vee a^{\star}) \vee x, a \vee a^{\star}) > 0$ for all $x \in R$.
Conversely, suppose $A((a \vee a^{\star}) \wedge (a \vee b), a \vee b) > 0$ for all $x = a \vee b \in R$. Clearly $A(a \wedge a^{\star}, 0) > 0$ and $0 \leq a \wedge a^{\star}$,we have $A(0, a \wedge a^{\star}) > 0$. Hence $a \wedge a^{\star} = 0$ by antisymmetry property of A.

$$\begin{aligned} A(a \vee b, (a \vee a^{\star}) \wedge (a \vee b)) &= A(a \vee b, [(a \vee a^{\star}) \wedge a] \vee [(a \vee a^{\star}) \wedge b]) \\ &= A(a \vee b, [(a \wedge a) \vee (a^{\star} \wedge a)] \vee [(a \wedge b) \vee (a^{\star} \wedge b)]) \\ &= A(a \vee b, [(a \vee 0) \vee (0 \vee b)]) \text{ since } a^{\star} \wedge a = 0 \text{ and } a \wedge b = 0 \\ &= A(a \vee b, a \vee b) = 1 > 0 \text{ since } a \vee 0 = a \text{ and } 0 \vee b = b. \end{aligned}$$

Hence $A(a \vee b, (a \vee a^{\star}) \wedge (a \vee b)) > 0$. Which implies that $(a \vee a^{\star}) \wedge (a \vee b) = a \vee b$ by antisymmetry property of A. So that we have $a \vee a^{\star}$ is maximal element of (R, A). Therefore $a \in B_A(R)$. □

Theorem 3.10. *Let (R, A) be a stone-ADFL. Then the center $B_A(R)$ of (R,A) coincides with the center of the distributive fuzzy lattice $([0, 0^\star], A)$ and hence $(B_A(R), \vee, \wedge)$ is a Fuzzy Boolean Algebra.*

Proof. Let C be the center of $[0, 0^\star]$ and $a \in B_A(R)$.Then there exists $b \in R$ such that $A(a \wedge b, 0) > 0$ and $A(x, (a \vee b) \wedge x) > 0$ for all $x \in R$. Now, $A(a \wedge b, 0) > 0 \Rightarrow A(b, a^\star \wedge b) > 0$

$$\begin{aligned}\Rightarrow A(a^{\star\star} \wedge b, 0) &= A(a^{\star\star} \wedge a^\star \wedge b, 0) \text{ since } a^\star \wedge b = b\\ &= A((a^{\star\star} \wedge a^\star) \wedge b, 0)\\ &= A(0 \wedge b, 0)\\ &= A(0, 0) = 1\end{aligned}$$

Similarly $A(0, a^{\star\star} \wedge b) = 1$. Hence $A(a^{\star\star} \wedge b, 0) = A(0, a^{\star\star} \wedge b) = 1$.

$$\begin{aligned}A(a^{\star\star}, a) &= A((a \vee b) \wedge a^{\star\star}, a)\\ &= A((a \wedge a^{\star\star}) \vee (b \wedge a^{\star\star}), a)\\ &= A((a \wedge a^{\star\star}) \vee (b \wedge a^{\star\star}), a)\\ &= A(a \vee 0, a)\\ &= A(a, a) = 1 \quad \text{since } b \wedge a^{\star\star} = 0.\\ A(a, a^{\star\star}) &= A(a, (a \vee b) \wedge a^{\star\star})\\ &= A(a, (a \wedge a^{\star\star}) \vee (b \wedge a^{\star\star}))\\ &= A(a, a \vee 0) = A(a, a) = 1 \quad \text{since } a \wedge a^\star = 0 \text{ and } b \wedge a^{\star\star} = 0.\end{aligned}$$

Hence $A(a^{\star\star}, a) = A(a, a^{\star\star}) = 1$. Since (R, A) is a stone ADFL, we have $A(0^\star, a^\star \vee a^{\star\star}) > 0$. $A(a^\star \vee a, 0^\star) = A(a^\star \vee a^{\star\star}, 0^\star)$ since $a = a^{\star\star}$ and $a^\star \vee a^{\star\star} = 0^\star$ in stone ADL.$= A(0^\star, 0^\star) = 1$. Hence $A(a^\star \vee a, 0^\star) = 1$. Similarly $A(0^\star, a^\star \vee a) = 1$. So that we have $A(a^\star \vee a, 0^\star) = A(0^\star, a^\star \vee a) = 1$. Hence $a^\star \vee a = 0^\star$ by antisymmetry property A.Therefore $a^\star \vee a$ is maximal. Thus $a \in C$and hence $B_A(R) \subseteq C$. Let $a \in C$. Then there exist $b \in [0, 0^\star]$ such that$A(a \wedge b, 0) > 0$ and $A(a \vee b, 0^\star) = A(0^\star, a \vee b) = 1$ which implythat $a \vee b = 0^\star$ is maximal. Thus $a \in B_A(R)$.We have $B_A(R) = C$. Since C is a Boolean algebra. We get $(B_A(R), \vee, \wedge)$ is a Fuzzy Boolean algebra. ☐

Corollary 3.11. *Let (R, A) be a stone ADFL with center $B_A(R)$.Then $B_A(R) = \{a \in R : A(a^{\star\star}, a) = A(a, a^{\star\star}) = 1\}$.*

Corollary 3.12. *If (R, A) is a stone ADFL and $a \in R$ is complemented,then $B_A(a) = \{a^\star\}$.*

Proof. Let a be a complemented element of (R, A).Then $a \in B_A(a)$ and hence $A(a, b^\star) = A(b^\star, a) = 1$ for some $b \in R$ by Corollary 3.11 $A(a \wedge a^\star, 0) > 0$ and $A(a \vee a^\star, 0^\star) = A(b^\star \vee b^{\star\star}, 0^\star) = A(0^\star, 0^\star) = 1$ since $a = b^\star$. Similarly $A(0^\star, a \vee a^\star) = 1$. Hence $A(a \vee a^\star, 0^\star) = A(0^\star, a \vee a^\star) = 1 \Rightarrow A(0^\star, a \vee a^\star) > 0$ and $A(a \vee a^\star, 0^\star) > 0$. Hence $a \vee a^\star = 0^\star$ by antisymmetry property of A. So that we have $a \vee a^\star$ is maximal. Therefore $a^\star \in B_A(a)$. Let $c \in B_A(a)$.Then $a^\star, c \in B_A(a)$,so that $A(a^\star \wedge c, c \wedge a^\star) = A(c \wedge a^\star, a^\star \wedge c) = 1$. Again since $c \in B_A(a)$, we have $A(a \wedge c, 0) > 0$ and $A(x, (a \vee c) \wedge x) > 0$ for all $x \in R$

$$\begin{aligned}A(c, a^\star) &= A(0^\star \wedge c, a^\star) \quad \text{since } 0^\star \text{ is maximal.}\\ &= A((a \vee a^\star) \wedge c, a^\star) \text{since } a \vee a^\star = 0^\star.\\ &= A((a \wedge c) \vee (a^\star \wedge c), a^\star)\\ &= A(0 \vee (a^\star \wedge c), a^\star) \quad \text{since } a \wedge c = 0.\end{aligned}$$

$$= A(a^\star \wedge c, a^\star) = A(c \wedge a^\star, a^\star)$$

$$= A((a \wedge a^\star) \vee (c \wedge a^\star), a^\star) = A((a \vee c) \wedge a^\star, a^\star)$$

$$= A(a^\star, a^\star) = 1 \text{ since } a \vee c \text{ is maximal.}$$

As a result we have $A(c, a^\star) = 1$. Similarly $A(a^\star, c) = 1$. Implies that $A(a^\star, c) = A(c, a^\star) = 1$. Therefore $B_A(a) = \{a^\star\}$. □

Theorem 3.13. *R is a stone ADL with 0 if and only if (R, A) is a stone ADFL with 0.*

Proof. Assume R be a stone ADL with 0.Then $\star$ is a pseudo-complementation on R. Let (R, A) be an ADFL. For any $a \in R$, we have

(1). $A(a \wedge a^\star, 0) > 0$ since $a \wedge a^\star = 0$.

(2). Let $a, b \in R$ and $A(a \wedge b, 0) > 0$ implies that $A(b, a^\star \wedge b) > 0$ since $a \wedge b = 0 \Rightarrow a^\star \wedge b = b$.

(3). $A((a \vee b)^\star, a^\star \wedge b^\star) = A(a^\star \wedge b^\star, a^\star \wedge b^\star) = 1$, since $(a \vee b)^\star = a^\star \wedge b^\star$.

Hence $A((a \vee b)^\star, a^\star \wedge b^\star) = 1$. Similarly $A(a^\star \wedge b^\star, (a \vee b)^\star) = 1$. So that we have $A((a \vee b)^\star, a^\star \wedge b^\star) = A(a^\star \wedge b^\star, (a \vee b)^\star) = 1$. Implies $A(a^\star \wedge b^\star, (a \vee b)^\star) > 0$. Thus $\star$ is a pseudo-complement on (R, A). Let $a \in R$.Then $a^\star \vee a^{\star\star} = 0^\star$ by definition of stone ADL. Now,

$$\begin{aligned} A(0^\star, a^\star \vee a^{\star\star}) &= A(0^\star, a^\star \vee (0^\star \wedge a^{\star\star})) \\ &= A(0^\star, (a^\star \vee 0^\star) \wedge (a^\star \vee a^{\star\star})) \\ &= A(0^\star, (a^\star \vee 0^\star) \wedge 0^\star) \quad \text{since } a^\star \vee a^{\star\star} = 0^\star \\ &= A(0^\star, 0^\star) = 1 > 0. \end{aligned}$$

Hence $A(0^\star, a^\star \vee a^{\star\star}) > 0$. Therefore (R, A) is a stone ADFL.
Conversely,Suppose (R, A) is a stone ADFL and $a \in R$. Then $\star$ is a pseudo- complementation on (R, A). Now, for any $a \in R$,

(1). $A(a \wedge a^\star, 0) > 0$. Since $0 \leq a \wedge a^\star$ implies that $A(0, a \wedge a^\star) > 0$. Hence $a \wedge a^\star = 0$ by antisymmetry property of A.

(2). $A(a \wedge b, 0) > 0$ implies that $A(b, a^\star \wedge b) > 0$. Since $0 \leq a \wedge b$. We have $A(0, a \wedge b) > 0$. So that $a \wedge b = 0$ by antisymmetry property of A and $a^\star \wedge b = b$.

(3). $A((a \vee b)^\star, a^\star \wedge b^\star) = A(a^\star \wedge b^\star, (a \vee b)^\star) = 1 > 0$ imply that $A((a \vee b)^\star, a^\star \wedge b^\star) > 0$ and $A(a^\star \wedge b^\star, (a \vee b)^\star) > 0$. So that we have $(a \vee b)^\star = a^\star \wedge b^\star$ by antisymmetry property of A. Hence $\star$ is a pseudo complementation on R.

(4). $A(0^\star, a^\star \vee a^{\star\star}) > 0$ by definition of stone Almost distributive fuzzy lattice(stone ADFL). Since $0^\star$ is a maximal element, we have $a^\star \vee a^{\star\star} \leq 0^\star$. Implies $A(a^\star \vee a^{\star\star}, 0^\star) > 0$. Hence $a^\star \vee a^{\star\star} = 0^\star$ by antisymmetry property of A. Therefore R is a stone ADL.

□

In this paper $(a]_A$ represents a principal ideal of an ADFL (R, A) generated by a.

Definition 3.14. *Let $(PI(R), A)$ be the principal ideal of an ADFL (R, A). Then $(a]_A = \{x \in R | A(x, a \wedge x) > 0\}$ for all $(a] \in PI(R)$.*

Lemma 3.15. *Let $(PI(R), A)$ be a principal ideal of an ADFL (R, A). Then $A(y, x_\alpha) > 0$ if and only if $(y]_A \subseteq (x_\alpha]_A$ for all $\alpha \in T$. For the family of sets $\{x_\alpha : \alpha \in T\}$ elements of (R, A).*

Definition 3.16. *Let (R, A) be an ADFL.Then (R, A) is said to be complete if $(PI(R), A)$ is a complete fuzzy sub lattice of $(I(R), A)$. That is for any family $\{x_\alpha : \alpha \in T\}$ of elements of (R, A), there exist $y, z \in R$ such that $\bigvee_\alpha (x_\alpha]_A = (y]_A$and $\bigwedge_\alpha (x_\alpha]_A = (z]_A$. Where $\bigvee_\alpha (x_\alpha]_A$ is the least upper bound and $\bigwedge_\alpha (x_\alpha]_A$ is the greatest lower bound of the family $\{x_\alpha : \alpha \in T\}$ in $PI((R, A))$.*

Lemma 3.17. *Let (R, A) be a complete ADFL and $y, z, x_\alpha \in R$ for all $\alpha \in T$. Then*

(1). $\bigvee_\alpha (x_\alpha]_A = (y]_A$ if and only if there exist elements $\alpha_1, \alpha_2, ..., \alpha_n \in T$ such that $A(y, (x_{\alpha_1} \vee x_{\alpha_2} \vee ... \vee x_{\alpha_n}) \wedge y) > 0$ and $A(x_\alpha, y) > 0$ for all $\alpha \in T$.

(2). $\bigwedge_\alpha (x_\alpha]_A = (z]_A$ if and only if

(i). $A(z, x_\alpha) > 0$ for all $\alpha \in T$. and

(ii). $x \in R$ and $A(x, x_\alpha) > 0$ for all $\alpha \in T$ implies that $A(x, z) > 0$.

Proof.

(1). Assume $\bigvee_\alpha (x_\alpha]_A = (y]_A$. Then $y \in (y]_A$ implies that $y \in (x_\alpha]_A \Rightarrow A(y, x_\alpha \wedge y) > 0$. Since $x_\alpha \wedge y \leq y$ implies that $A(x_\alpha \wedge y, y) > 0$. Hence we get $x_\alpha \wedge y = y$. So that we get $y \leq x_\alpha$, for all $\alpha \in T$ since $y \in (x_\alpha] \Rightarrow y \leq x_\alpha$. Hence $y \leq x_{\alpha_i}$, for $1 \leq i \leq n$.

$$A(y, (x_{\alpha_1} \vee x_{\alpha_2} \vee ... \vee x_{\alpha_n}) \wedge y) = A(y, (x_{\alpha_1} \wedge y) \vee (x_{\alpha_2} \wedge y) \vee ... \vee (x_{\alpha_n} \wedge y)) \text{ by } RD\wedge$$
$$= A(y, y \vee y \vee y... \vee y) = A(y, y) = 1 > 0, \quad \text{since } y \leq x_{\alpha_i}, 1 \leq i \leq n.$$

Hence $A(y, (x_{\alpha_1} \vee x_{\alpha_2} \vee ... \vee x_{\alpha_n}) \wedge y) > 0$. Now,let $\alpha \in T$. Then $x_\alpha \in \bigvee_\alpha (x_\alpha]_A$ imply that $x_\alpha \in (y]_A \Rightarrow A(x_\alpha, y \wedge x_\alpha) > 0$. As $y \wedge x_\alpha \leq x_\alpha \Rightarrow A(y \wedge x_\alpha, x_\alpha) > 0$ and hence we get $y \wedge x_\alpha = x_\alpha$ by antisymmetry property of A. Hence $x_\alpha \leq y$ for all $\alpha \in T$. Therefore $A(x_\alpha, y) > 0$ for all $\alpha \in T$.

Conversely, assume that there exist $\alpha_1, \alpha_2, ..., \alpha_n \in T$ such that $A(y, (x_{\alpha_1} \vee x_{\alpha_2} \vee .... \vee x_{\alpha_n}) \wedge y) > 0$ and $A(x_\alpha, y) > 0$, for all $\alpha \in T$. We need to show that $\bigvee_\alpha (x_\alpha]_A = (y]_A$. Now,since $A(x_\alpha, y) > 0$, for every $\alpha \in T$ and hence we get $x_\alpha \in (y]_A$, for every $\alpha \in T$. Hence $(x_\alpha]_A \subseteq (y]_A$ for every $\alpha \in T$. Therefore $(y]_A$ is an upper bound of $\{(x_\alpha]_A : \alpha \in T\}$ in the fuzzy lattice $(PI(R), A)$. Let $J \in I((R, A))$ be any upper bound of $\{(x_\alpha]_A : \alpha \in T\}$. So that $x_\alpha \in J$, for any $\alpha \in T$. By our assumption,there exists $\alpha_1, \alpha_2, ..., \alpha_n \in T$ such that $A(y, (x_{\alpha_1} \vee x_{\alpha_2} \vee .... \vee x_{\alpha_n}) \wedge y) > 0$. Now,for $1 \leq i \leq n$,$x_{\alpha_1}, x_{\alpha_2}, ..., x_{\alpha_n} \in J$ and $(x_{\alpha_1} \vee x_{\alpha_2} \vee ... \vee x_{\alpha_n}) \wedge y \leq x_{\alpha_i}$, $1 \leq i \leq n$. Therefore $(x_{\alpha_1} \vee x_{\alpha_2} \vee x_{\alpha_3} \vee ... \vee x_{\alpha_n}) \wedge y \in J$ and hence we get $(y]_A \subseteq J$. Thus $(y]_A$ is the least upper bound of $\{(x_\alpha]_A : \alpha \in T\}$ in $(I(R), A)$. That is $\bigvee_\alpha (x_\alpha]_A = (y]_A$.

(2). Assume that $\bigwedge_\alpha (x_\alpha]_A = (z]_A$. Then we need to show $A(x, z) > 0$.

(i). Now,$(z]_A \subseteq \bigwedge_\alpha (x_\alpha]_A$, for all $\alpha \in T$ and $z \in (z]_A \subseteq \bigwedge_\alpha (x_\alpha]_A$ which implies that $z \in (x_\alpha]_A \Rightarrow A(z, x_\alpha \wedge z) > 0$. Since $x_\alpha \wedge z \leq z \Rightarrow A(x_\alpha \wedge z, z) > 0$ and hence we get $x_\alpha \wedge z = z$ by antisymmetry property of A. Hence we get $z \leq x_\alpha$ since$z \in (x_\alpha] \Rightarrow z \leq x_\alpha$. Therefore $A(z, x_\alpha) > 0$, for all $\alpha \in T$.

(ii). Since $\bigwedge_\alpha (x_\alpha]_A \subseteq (z]_A$,$x \in \bigwedge_\alpha (x_\alpha]_A x \in (z]_A \Rightarrow A(x, z \wedge x) > 0$. Since $z \wedge x \leq x \Rightarrow A(z \wedge x, x) > 0$. Hence $z \wedge x = x$ and we get $x \leq z$.Therefore $A(x, z) > 0$. On the other hand, from (i), $A(z, x_\alpha) > 0$ implies $z \in (x_\alpha]_A$. Hence we get $(z]_A \subseteq \bigwedge_\alpha (x_\alpha]_A$. Let $x \in \bigwedge_\alpha (x_\alpha]_A \Rightarrow x \in (x_\alpha]_A \Rightarrow A(x, x_\alpha \wedge x) > 0$ and $A(x, z) > 0$ implies $x \in (z]_A$. Hence $\bigwedge_\alpha (x_\alpha]_A \subseteq (z]_A$'. Therefore $\bigwedge_\alpha (x_\alpha]_A = (z]_A$.

□

Definition 3.18. *An ADFL (R, A) is said to be relatively complemented if $([a, b], A)$ is a complemented fuzzy lattice for any $a, b \in R$ with $A(a, b) > 0$.*

Definition 3.19. *Let (R, A) be an ADFL.Then a pseudo-complemented distributive fuzzy lattice with 0 is called a stone fuzzy lattice if,for any $a \in R$,$A(1, a^\star \vee a^{\star\star}) > 0$.*

Theorem 3.20. *Let (R, A) be an ADFL and (R, A) is a complete relatively complemented ADFL with maximal element m. Then the following condition holds:*

(1). The set $(I(R), A)$ of all ideals of (R, A) is a stone fuzzy lattice.

(2). The center of $(I(R), A) = (PI(R), A)$.

Lemma 3.21. *Let (R, A) be an ADFL and $a \in R$. Then (R_a, A) is a sub-ADFL of (R, A).Where $R_a = \{a \wedge x : x \in R\}$.*

Theorem 3.22. *Let (R, A) be a stone-ADFL and $a \in R$. Then the map $f : (R, A) \to (R_a \times R_a, A_1 \times A_1)$ defined by $(A_1 \times A_1)(f(x), (a \wedge x, a^\star \wedge x)) = (A_1 \times A_1)((a \wedge x, a^\star \wedge x), f(x)) = 1,$ for all $x \in R$ is an isomorphism if and only if $a \in B_A(R)$. Where (R_a, A_1) is a sub-ADFL of (R, A).*

Proof. Let (R, A) be a stone-ADFL and $a \in R$. Suppose the map $f : (R, A) \to (R_a \times R_a, A_1 \times A_1)$ defined by $(A_1 \times A_1)(f(x), (a \wedge x, a^\star \wedge x)) = (A_1 \times A_1)((a \wedge x, a^\star \wedge x), f(x)) = 1,$ for all $x \in R$ is an isomorphism. Now,

$$\begin{aligned}
(A_1 \times A_1)(f(a), (a \wedge a, a^\star \wedge a)) &= (A_1 \times A_1)((a \wedge a, a^\star \wedge a), (a, 0)) \\
&= (A_1 \times A_1)((a, 0), (a, 0)) \\
&= \min\{A_1(a, a), A_1(0, 0)\} \\
&= \min\{1, 1\} = 1, \text{ since } a \wedge a = a \text{ and } a^\star \wedge a = 0. \\
(A_1 \times A_1)(f(a^{\star\star}), (a \wedge a^{\star\star}, a^\star \wedge a^{\star\star})) &= (A_1 \times A_1)((a \wedge a^{\star\star}, a^\star \wedge a^{\star\star}), (a, 0)) \\
&= (A_1 \times A_1)((a, 0), (a, 0) \\
&= \min\{A_1(a, a), A_1(0, 0)\} \\
&= \min\{1, 1\} = 1, \text{since } a \wedge a^{\star\star} = a \text{ and } a^\star \wedge a^{\star\star} = 0.
\end{aligned}$$

Hence $(A_1 \times A_1)(f(a), (a, 0)) = A_1 \times A_1)(f(a^{\star\star}), (a, 0)) = 1$. Again,

$$\begin{aligned}
(A_1 \times A_1)(f(a), f(a^{\star\star})) &= (A_1 \times A_1)((a \wedge a, a^\star \wedge a), (a \wedge a^{\star\star}, a^\star \wedge a^{\star\star})) \\
&= (A_1 \times A_1)((a, 0), (a, 0)) \\
&= \min\{A_1(a, a), A_1(0, 0)\} \\
&= \min\{1, 1\} = 1.
\end{aligned}$$

Hence $(A_1 \times A_1)(f(a), f(a^{\star\star})) > 0$. Similarly $(A_1 \times A_1)((f(a^{\star\star}), f(a)) > 0$. Implies $f(a) = f(a^{\star\star})$ by antisymetry property of $A_1 \times A_1$. Hence $a = a^{\star\star}$ since f is one-to-one. Thus $a \in B_A(R)$ by Corollary 3.11.

Conversely, assume $a \in B_A(R)$. We need to show f is an isomorphism. Let $x, y \in R$. Then (i). $(A_1\times A_1)(f(x\wedge y),f(x)\wedge f(y))=(A_1\times A_1)((a\wedge(x\wedge y),a^\star\wedge(x\wedge y)),(a\wedge x,a^\star\wedge x)\wedge(a\wedge y,a^\star\wedge y))$

$$\begin{aligned}&=(A_1\times A_1)((a\wedge a)\wedge(x\wedge y),(a^\star\wedge a^\star)\wedge(x\wedge y)),((a\wedge x)\wedge(a\wedge y),(a^\star\wedge x)\wedge(a^\star\wedge y))\\&=(A_1\times A_1)((a\wedge(x\wedge y),(a^\star\wedge(x\wedge y)),((a\wedge x)\wedge(a\wedge y),(a^\star\wedge x)\wedge(a^\star\wedge y))\\&=\min\{A_1((a\wedge(x\wedge y),(a\wedge x)\wedge(a\wedge y)),A_1(a^\star\wedge(x\wedge y),(a^\star\wedge x)\wedge(a^\star\wedge y))\}\\&=\min\{A_1(a\wedge(x\wedge y),a\wedge(x\wedge y)),A_1(a^\star\wedge(x\wedge y),a^\star\wedge(x\wedge y))\}\\&=\min\{1,1\}=1.\end{aligned}$$

Hence $(A_1\times A_1)(f(x\wedge y),f(x)\wedge f(y))=1$. Similarly $(A_1\times A_1)(f(x)\wedge f(y),f(x\wedge y))=1$. We get $(A_1\times A_1)(f(x\wedge y),f(x)\wedge f(y))=(A_1\times A_1)(f(x)\wedge f(y),f(x\wedge y))=1$.

(ii). $(A_1\times A_1)(f(x\vee y),f(x)\vee f(y))=(A_1\times A_1)((a\wedge(x\vee y),a^\star\wedge(x\vee y)),(a\wedge x,a^\star\wedge x)\vee(a\wedge y,a^\star\wedge y))$

$$\begin{aligned}&=(A_1\times A_1)(((a\wedge(x\vee y),(a^\star\wedge(x\vee y)),((a\wedge x)\vee(a\wedge y),(a^\star\wedge x)\vee(a^\star\wedge y))\\&=\min\{(A_1((a\wedge(x\vee y),(a\wedge x)\vee(a\wedge y)),A_1(a^\star\wedge(x\vee y),(a^\star\wedge x)\vee(a^\star\wedge y))\}\\&=\min\{A_1(a\wedge(x\vee y),a\wedge(x\vee y)),A_1(a^\star\wedge(x\vee y),a^\star\wedge(x\vee y))\}\\&=\min\{1,1\}=1.\end{aligned}$$

Hence $(A_1\times A_1)(f(x\vee y),f(x)\vee f(y))=1$. Similarly $(A_1\times A_1)(f(x)\vee f(y),f(x\vee y))=1$. Implies $(A_1\times A_1)(f(x\vee y),f(x)\vee f(y))=(A_1\times A_1)(f(x)\vee f(y),f(x\vee y))=1$.

(iii). If $0\in R$,

$$\begin{aligned}(A_1\times A_1)(f(0),(0,0))&=(A_1\times A_1)((a\wedge 0,a^\star\wedge 0),(0,0))\\&=\min\{A_1((0,0),A_1(0,0)\}\\&=\min\{1,1\}=1>0.\end{aligned}$$

Hence $(A_1\times A_1)(f(0),(0,0))>0$. Therefore f is a fuzzy lattice homomorphism.

Let $x,y\in R$ and $(A_1\times A_1)(f(x),f(y))=(A_1\times A_1)(f(y),f(x))=1$. Now,

$$\begin{aligned}(A_1\times A_1)(x,y)&=(A_1\times A_1)(0^\star\wedge x,y)\text{ since }0^\star\text{ is maximal.}\\&=(A_1\times A_1)((a\vee a^\star)\wedge x,y)=(A_1\times A_1)((a\wedge x)\vee(a^\star\wedge x),y)\\&=(A_1\times A_1)((a\wedge y)\vee(a^\star\wedge y),y),\text{replace x by y.}\\&=(A_1\times A_1)((a\vee a^\star)\wedge y,y)\\&=(A_1\times A_1)(y,y)=1>0.\end{aligned}$$

Hence $(A_1\times A_1)(x,y)>0$. Similarly $(A_1\times A_1)(y,x)>0$. Implies $x=y$ by antisymmetry property of $A_1\times A_1$. Hence f is one-to-one. To show f is onto. Suppose $(a\wedge x,a^\star\wedge y)\in R_a\times R_a$ for some $x,y\in R$. Write $w=(a\wedge x)\vee(a^\star\wedge y)$. Then

$$\begin{aligned}(A_1\times A_1)(f(w),(a\wedge x,a^\star\wedge y))&=(A_1\times A_1)((a\wedge w,a^\star\wedge w),(a\wedge x,a^\star\wedge y))\\&=(A_1\times A_1)(a\wedge[(a\wedge x)\vee(a^\star\wedge y)],a^\star\wedge[(a\wedge x)\vee(a^\star\wedge y)],(a\wedge x,a^\star\wedge y))\\&=(A_1\times A_1)([(a\wedge x)\vee a\wedge(a^\star\wedge y)],[(a^\star\wedge a)\wedge x\vee(a^\star\wedge y)],(a\wedge x,a^\star\wedge y))\\&=(A_1\times A_1)([(a\wedge x)\vee 0,0\vee(a^\star\wedge y)],(a\wedge x,a^\star\wedge y))\\&=\min\{A_1((a\wedge x,a\wedge x),A_1(a^\star\wedge y,a^\star\wedge y)\}\\&=\min\{1,1\}=1>0.\end{aligned}$$

Hence $(A_1 \times A_1)(f(w), (a \wedge x, a^\star \wedge y)) > 0$. Similarly $(A_1 \times A_1)((a \wedge x, a^\star \wedge y), f(w)) > 0$. We get $f(w) = (a \wedge x, a^\star \wedge y)$ by antisymmetry property of $A_1 \times A_1$. Hence f is onto. Therefore f is an isomorphism. □

Definition 3.23. *Let (R, A) be an ADFL with 0 and $a \in R$. Then an element b in (R, A) is said to be a semi-complement of the element a if, $A(a \wedge b, 0) > 0$. We denote the set of all semi-complement of the element a by $S(a)$.*

Definition 3.24. *Let (R, A) be an ADFL with a pseudo-complementation and $a \in R$. Then $S(a) = (a^\star]_A$ and $S(a)$ is also an ideal of (R, A).*

An ideal I in an ADFL (R, A) is called a direct factor if there exists an ideal J of (R, A) such that $I \wedge J = (0]_A$ and $R \subseteq I \vee J$.

References

[1] A.Berhanu, G.Yohannes and T.Bekalu, *Almost Distributive Fuzzy Lattice*, International Journal of Mathematics and its Application, 5(1-C)(2017), 307-316.

[2] A.Berhanu and T.Gerima, *Birkhoff center of Almost distributive Fuzzy Lattice*, Accepted for publication in International Journal of Computing science and applied Mathematics.

[3] J.A.Goguen, *L-fuzzy set*, J. Math. Anal. Appl., 18(1967), 145-174.

[4] Inheung Chon, *Fuzzy partial order relation and Fuzzy lattice*, Korean J. Math., 17(4)(2009), 361-374.

[5] Ivan Mezzomo, Benjamin C.Bedregas and Regivan H.N.Santiago, *Kinds of fuzzy lattices*, Rural Federal University of Semi-Arid-UFERSA,50,515,000,angicos,Rio Grande do.Norte. Brazil.

[6] G.C.Rao and Mihret Alamneh, *Properties of stone Almost distributive lattice*, Asian-European Journal of Mathematics, 8(1)(2015).

[7] G.C.Rao, *Almost distributive lattice*, thesis submitted to the Andra University for the a ward of the degree of Doctor of philosophy in Mathematics, (1980).

[8] G.C.Rao and G.N.Rao, *Dense element in Almost distributive lattice*, South East Asian Bull Math., 27(2004), 1081-1088.

[9] G.C.Rao and S.Ravi Kumer, *Normal Almost distributive lattice*, South east Asian Bull. Math. 32(5)(2008), 831-841.

[10] U.M.Swamy and G.C.Rao, *Almost distributive lattice*, J. Aust. Math. Soc. (ser A), 31(1981), 77-91.

[11] E.Sanchez, *Resolution of composite fuzzy relation equation*, Inform. and control, 30(1976), 38-48.

[12] U.M.Swamy, G.C.Rao and G.N.Rao, *Pseudo-complementation on Almost distributive lattice*, South east Asian Bull. Math., 24(2000), 95-104.

[13] U.M.Swamy, G.C.Rao and G.N.Rao, *Stone Almost distributive lattice*, South East Asian Bull. Math., 27(2003), 513-526.

[14] L.A.Zadeh, *Fuzzy Sets*, Inform and Control, 8(1965), 338-353.