

A New Representation of the Extended Fermi-Dirac and Bose-Einstein Functions

Research Article

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Abstract: In [22] the authors discuss the extended Fermi-Dirac and Bose-Einstein functions. In this paper, we discuss a new representation of these functions as a series of complex delta functions. This leads to a similar representation for the Hurwitz and Riemann zeta functions.

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1. Introduction and Preliminaries

The Fermi-Dirac (FD) function $\mathfrak{F}_{s-1}(x)$ defined by [6]:

$$\mathfrak{F}_{s-1}(x) := \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^{t-x} + 1} dt \quad (\Re(s) > 0; x \geq 0) \quad (1)$$

and Bose-Einstein (BE) function $\mathfrak{B}_{s-1}(x)$ defined by [6]:

$$\mathfrak{B}_{s-1}(x) := \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^{t-x} - 1} dt \quad (\Re(s) > 1; x \geq 0) \quad (2)$$

find applications in several areas of Physics notably in Statistical [16] and Quantum Mechanics [20]. Some extensions of the FD and BE functions are given in [22]. The extended Fermi-Dirac (eFD) defined by Srivastva et al [22]

$$\Theta_\nu(s; x) = \frac{e^{-(\nu+1)x}}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-\nu t}}{e^t + e^{-x}} dt \quad (\Re(\nu) > -1; \Re(s) > 0; x \geq 0) \quad (3)$$

and the extended Bose-Einstein (eBE) function defined by Srivastva et al [22]

$$\Psi_\nu(s; x) = \frac{e^{-(\nu+1)x}}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-\nu t}}{e^t - e^{-x}} dt \quad (\Re(\nu) > -1; \Re(s) > 1 \text{ when } x = 0; \Re(s) > 0 \text{ when } x > 0) \quad (4)$$

reduce to the original FD and BE functions, when $\nu = 0$. Throughout this paper s is a complex number and we use \mathbb{C} , usual notation of set of complex numbers. Several mathematicians developed tables of integrals of products of special functions.

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Among these, especially the tables [8, 10, 11, 13, 18, 19, 21] are remarkable. More recently, in [25], a Fourier transform representation of the eFD and eBE functions has been used to evaluate some integrals of products of these functions and family of zeta functions. In our present investigation, we obtain a series representation of the eFD and eBE functions in terms of delta functional of complex argument acting on the space \mathcal{Z} of test functions. Similar representation for the Riemann and Hurwitz zeta functions has also obtained as a special case. This leads to some new integral formulae for these functions and the family of zeta function. Before going on to obtain these representations for the extended functions, we review their definition and relationship with the zeta and other related functions by using their integral representations for the purposes of our investigation. The Riemann zeta function, $\zeta(s)$ defined by the integral representations [5]

$$\zeta(s) := \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} dt \quad (s = \sigma + i\tau, \sigma > 1) \tag{5}$$

and ([5])

$$\zeta(s) := \frac{1}{C(s)} \int_0^\infty \frac{t^{s-1}}{e^t + 1} dt \quad (s = \sigma + i\tau, \sigma > 0), \quad \text{where } C(s) = \Gamma(s)(1 - 2^{1-s}), \tag{6}$$

has a meromorphic continuation to the whole complex s -plane (except for a simple pole at $s = 1$). There have been several generalizations of the Riemann zeta function, like the Hurwitz zeta function

$$\zeta(s, \nu) := \frac{1}{\Gamma(s)} \int_0^\infty \frac{e^{-(\nu-1)t} t^{s-1}}{e^t - 1} dt \quad (\Re(\nu) > 0; \sigma > 1). \tag{7}$$

Another generalization of the Riemann zeta function is the polylogarithm function, or Jonquire's function $\phi(z, s)$, defined by [29]

$$\phi(z, s) := \sum_{n=1}^\infty \frac{z^n}{n^s} \quad (s \in \mathbb{C} \text{ when } |z| < 1; \Re(s) > 1 \text{ when } |z| = 1). \tag{8}$$

It has the integral representation

$$\phi(z, s) = \frac{z}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - z} dt \quad (|z| \leq 1 - \delta, \delta \in (0, 1) \text{ and } \Re(s) > 0; z = 1 \text{ and } \Re(s) > 1). \tag{9}$$

Note that if z lies anywhere except on the segment of the real axis from 1 to ∞ , where a cut is imposed, the above integral defines an analytic function of z for $\Re(s) > 1$. If $z = 1$ then this coincides with the zeta function in the half plane for $\Re(s) > 1$. The Hurwitz-zeta and the polylogarithm functions are further generalized to the Hurwitz-Lerch zeta function by [9]

$$\Phi(z, s, \nu) := \sum_{n=0}^\infty \frac{z^n}{(n + \nu)^s} \quad (\nu \neq 0, -1, -2, -3, \dots; s \in \mathbb{C} \text{ when } |z| < 1; \Re(s) > 1 \text{ when } |z| = 1). \tag{10}$$

This function has the integral representation [9]:

$$\Phi(z, s, \nu) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-(\nu-1)t}}{e^t - z} dt \quad (\Re(\nu) > 0; \text{ and either } |z| \leq 1; z \neq 1; \Re(s) > 0 \text{ or } z = 1; \Re(s) > 1). \tag{11}$$

If a cut is made from 1 to ∞ along the positive real z -axis, Φ is an analytic function of z in the cut z -plane provided that $\Re(s) > 0$ and $\Re(\nu) > 0$. In particular, the Hurwitz-Lerch zeta function is related to the eFD and eBE functions by [22]

$$\Theta_\nu(s; x) := e^{-(\nu+1)x} \Phi(-e^{-x}, s, \nu + 1) \tag{12}$$

and

$$\Psi_\nu(s; x) := e^{-(\nu+1)x} \Phi(e^{-x}, s, \nu + 1). \quad (13)$$

Further to these relations we have following [22]

$$\Psi_0(s; x) = \text{Li}_s(z); \quad (14)$$

$$\Psi_0(s; -x) = \mathfrak{B}_{s-1}(x); \quad (15)$$

$$\Psi_\nu(s; 0) = \zeta(s, \nu + 1); \quad (16)$$

$$\Psi_0(s; 0) = \zeta(s, 1) = \zeta(s); \quad (17)$$

$$\Theta_0(s; x) = F_{s-1}(-x); \quad (18)$$

$$\Theta_0(s; 0) = (1 - 2^{1-s})\zeta(s). \quad (19)$$

For more detailed study of zeta and related functions, I refer the interested reader to [1–4, 7, 12, 17, 23, 24, 26–28]. The plan of the paper as follows. A brief introduction to “distributions and test functions” is given in Section 2. We obtain a new representation of the eFD and eBE functions in Section 3. This representation is given as a series of complex delta functionals. We discuss the convergence of this series in Section 4. Some integral formulae involving these functions are obtained in Section 5. Further insights and concluding remarks are given in Section 6.

2. Distributions and Test Functions

Generalized functions (or distributions) have played an important role to extend the theory of Fourier and other integral transforms. These are *continuous linear functionals acting on some space of test functions*. In other words the space of distributions is the dual of some space of test functions. In this section, we will discuss some spaces of test functions and distributions, which are necessary in understanding the concepts used in the sequel. For any function $\phi : \mathbb{R} \rightarrow \mathbb{C}$, the set

$$\text{supp } \phi = \overline{\{t \in \mathbb{R} : \phi(t) \neq 0\}}$$

is called the *support* of ϕ . The space of test functions is the space of all complex valued functions $\phi(t)$, which are C^∞ (infinitely differentiable) and has compact support, for example

$$\phi(t) = \begin{cases} \exp \frac{1}{t^2-1} & (|t| < 1) \\ 0 & (|t| \geq 1) \end{cases}.$$

The space of such test functions is denoted by \mathcal{D} , see [30]. Every complex valued function, $f(t)$, that is continuous for all t and has compact support, can be approximated uniformly by some test function. That is, given an $\epsilon > 0$ there will exist a $\phi(t) \in \mathcal{D}$ such that

$$|f(t) - \phi(t)| \leq \epsilon \quad \forall t. \quad (20)$$

The space of all continuous linear functionals acting on the space \mathcal{D} is called its dual space denoted by \mathcal{D}' . Distributions can be generated by using the following method. Let $f(t)$ be a locally integrable function then corresponding to it, one can define a distribution f through the convergent integral

$$\langle f, \phi \rangle = \langle f(t), \phi(t) \rangle := \int_{-\infty}^{+\infty} f(t)\phi(t)dt \quad \forall \phi \in \mathcal{D}. \quad (21)$$

It is easy to show that it is a continuous linear functional by using the definition of ϕ and the details are omitted here, [30]. Distributions that can be generated in this way (from locally integrable functions) are called the *regular distributions*. The study of distributions is important because they not only contain representations of locally integrable functions but also include many other entities that are not regular distributions. Therefore many operations like limits, integration and differentiation, which were originally defined for functions, can be extended to these new entities. All distributions that are not regular are called *singular distributions*, for example, the delta functional is a singular distribution. It is defined by

$$\langle \delta(t - c), \phi(t) \rangle = \phi(c) \quad (\forall \phi \in \mathcal{D}, c \in \mathbb{R}). \tag{22}$$

Some other properties of delta functional include

$$\delta(-t) = \delta(t), \tag{23}$$

$$\delta(\alpha t) = \frac{1}{|\alpha|} \delta(t) \quad (\alpha \neq 0). \tag{24}$$

The Fourier transform of an arbitrary distribution in \mathcal{D}' is not, in general, a distribution but is instead another kind of continuous linear functional which is defined over a new space of test functions. Such a functional is called an *ultradistribution*, for example delta functional of complex argument is an ultradistribution [30]. Therefore, we first discuss the space of test functions on which such distributions are defined. The space of test functions denoted by \mathcal{Z} consists of all those entire functions whose Fourier transforms are the elements of \mathcal{D} , [30]. Since ϕ is an entire function, it cannot be zero on any interval $a < t < b$ except when it is zero everywhere. Thus the spaces \mathcal{D} and \mathcal{Z} do not intersect except in the identically zero testing function. The corresponding dual to the testing function space \mathcal{Z} is \mathcal{Z}' whose elements are the Fourier transforms of the elements of \mathcal{D}' . Neither \mathcal{D} nor \mathcal{D}' are subspaces of \mathcal{Z} or \mathcal{Z}' and vice versa. However, the following inclusion holds

$$\mathcal{Z} \subset \mathcal{S} \subset \mathcal{S}' \subset \mathcal{Z}' \tag{25}$$

and

$$\mathcal{D} \subset \mathcal{S} \subset \mathcal{S}' \subset \mathcal{D}'. \tag{26}$$

Here \mathcal{S} is the space of testing functions of rapid descent satisfying

$$\lim_{t \rightarrow \infty} |t|^{-N} \phi(t) = 0 \quad \text{for some integer } N \tag{27}$$

and \mathcal{S}' is the space of distributions of slow growth satisfying [30]

$$|t|^m \phi^k(t) \leq C_{mk} \quad \text{for each pair of nonnegative integers } m, k; -\infty < t < \infty. \tag{28}$$

The space \mathcal{S}' is also called the space of *tempered distributions*. From these spaces of test functions and distributions only space \mathcal{S} and its dual are closed under Fourier transform. \mathcal{S}' is a proper subspace of \mathcal{D}' ; that is there are distributions in \mathcal{D}' that are not in \mathcal{S}' . For example, the series

$$g(t) = \sum_{\mu=1}^{\infty} e^{\mu^2} \delta(t - \mu) \tag{29}$$

defines a distribution in \mathcal{D}' . Indeed, given any ϕ in \mathcal{D} ,

$$\langle g, \phi \rangle = \sum_{\mu=1}^{\infty} e^{\mu^2} \phi(\mu). \tag{30}$$

The last series possesses only a finite number of non zero terms and therefore converges. On the other hand, there are testing functions e^{-t^2} in \mathcal{S} , such as $\phi(t) = e^{-t^2}$, for which the series (30) does not converge. Hence (29) is not a distribution of slow growth. An important example of ultradistributions is [30]

$$\mathcal{F}[e^{\alpha t}; \omega] = 2\pi \sum_{n=0}^{\infty} \frac{(i\alpha)^n}{n!} \delta^n(\omega) = 2\pi \delta(\omega + i\alpha), \tag{31}$$

where α is a complex constant. Complex delta functional is a distribution acting on the testing function space \mathcal{Z} . A sequence $\{f_\mu\}_{\mu=1}^{\infty}$ is said to be convergent in \mathcal{Z}' if every f_μ is in \mathcal{Z}' and if, for each ϕ in \mathcal{Z} , the sequence $\{\langle f_\mu, \phi \rangle\}_{\mu=1}^{\infty}$ converges in the ordinary sense of convergence of numbers. As ϕ traverses \mathcal{Z} , the limits of $\{\langle f_\mu, \phi \rangle\}_{\mu=1}^{\infty}$ define a functional f on \mathcal{Z} , and f is said to be the limit of $\{f_\mu\}_{\mu=1}^{\infty}$. The dual space \mathcal{Z}' is closed under convergence. As usual, an infinite series is said to converge in \mathcal{Z}' if the sequence of its partial sums converges in \mathcal{Z}' . For further discussion of the spaces, \mathcal{Z} and \mathcal{Z}' , see [14] and [15].

3. A New Representation of the eFD and eBE Functions

We will proceed to obtain a representation of the eFD and eBE functions in terms of delta functionals. In view of the relationships of these extended functions with the zeta family and other related functions (discussed in Section 1), similar representations for the FD, BE, polylogarithm and zeta functions are also obtained here.

Theorem 3.1. *The eBE function has a representation*

$$\Gamma(s)\Psi_\nu(s; x) = 2\pi \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m (n + \nu + 1)^m e^{-(n+\nu+1)x}}{m!} \delta(s + m) \tag{32}$$

($\Re(\nu) > -1; \Re(s) > 1$ when $x = 0; \Re(s) > 0$ when $x > 0$).

Proof. Consider the eBE function

$$\Gamma(s)\Psi_\nu(s; x) = e^{-(\nu+1)x} \int_{-\infty}^{+\infty} \frac{e^{-(\nu+1)t} t^{s-1}}{1 - e^{-(t+x)}} dt \quad (\Re(\nu) > -1; \Re(s) > 1 \text{ when } x = 0; \Re(s) > 0 \text{ when } x > 0). \tag{33}$$

Replacing t by e^{-y} in (33) gives

$$\Gamma(s)\Psi_\nu(s; x) = e^{-(\nu+1)x} \int_{-\infty}^{+\infty} \frac{e^{-i\tau y} e^{-\sigma y} \exp(-(\nu + 1)e^{-y})}{1 - \exp(-(e^{-y} + x))} dy \tag{34}$$

($\Re(\nu) > -1; \Re(s) = \sigma > 1$ when $x = 0; \Re(s) = \sigma > 0$ when $x > 0$), where y lies between $-\infty$ and ∞ . The expansion of the denominator in (34) yields

$$[1 - \exp(-(e^{-y} + x))]^{-1} = \sum_{n=0}^{\infty} \exp(-n(e^{-y} + x)) \quad (x \geq 0, -\infty < y < +\infty) \tag{35}$$

then, also

$$\exp(-n(e^{-y} + x)) = \sum_{m=0}^{\infty} \frac{(-1)^m (n + \nu + 1)^m}{m!} e^{-my} \quad (\Re(\nu) > -1, -\infty < y < +\infty). \tag{36}$$

Using (35)-(36) in (34) leads to the following double sum, denoted by $\psi_{(\nu, \sigma)}(x; y)$,

$$\psi_{(\nu, \sigma)}(x; y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m (n + \nu + 1)^m e^{-(n+\nu+1)x}}{m!} e^{-(\sigma+m)y} \quad (\Re(\nu) > -1; \sigma > 1 \text{ when } x = 0; \sigma > 0 \text{ when } x > 0). \tag{37}$$

Now term by term application of the Fourier transform and use of the relationship (31) in (37) leads to the following series of delta functionals.

$$\Gamma(s)\Psi_\nu(s; x) = 2\pi \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m (n + \nu + 1)^m e^{-(n+\nu+1)x}}{m!} \delta(\tau - i(\sigma + m)) \tag{38}$$

($\Re(\nu) > -1; \sigma > 1$ when $x = 0; \sigma > 0$ when $x > 0$). Now by making use of the basic property of delta function

$$\delta(\tau - i(\sigma + m)) = \delta\left[\frac{1}{i} (i\tau - i^2(\sigma + m))\right] = |i| \delta(\sigma + i\tau + m) = \delta(s + m), \tag{39}$$

we get (32). □

Remark 3.2. Note that this representation exists only if we consider the product $\Psi_\nu(s; x)\Gamma(s)$. On the other hand if we take Gamma function on the RHS then $\frac{\delta(s+m)}{\Gamma(s)} = \frac{1}{\Gamma(-m)}$, that means $\Psi_\nu(s; x)$ vanishes for all values of s and it leads to no result.

Corollary 3.3. The Hurwitz-Lerch zeta function $\Phi(z, s, \nu)$ has a representation

$$\Gamma(s)\Phi(z, s, \nu) = 2\pi \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m (n + \nu)^m z^n}{m!} \delta(s + m) \quad (\Re(\nu) > 0; \Re(s) > 0 \text{ when } 0 < z < 1; \Re(s) > 1 \text{ when } z = 1). \tag{40}$$

Proof. This follows by using the relationship $\Psi_\nu(s; x) = e^{-(\nu+1)x}\Phi(e^{-x}, s, \nu + 1)$ in (32) when one uses $z = e^{-x}$ and replaces $\nu \mapsto \nu - 1$. □

Corollary 3.4. The polylogarithm function has a representation

$$\Gamma(s)Li_s(z) = 2\pi z \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m (n + 1)^m z^n}{m!} \delta(s + m) \quad (\Re(s) > 0 \text{ when } 0 < z < 1, \sigma > 1 \text{ when } z = 1). \tag{41}$$

Proof. This follows by putting $z = e^{-x}$, $\nu = 0$ in (32) and using $\Psi_0(s; x) = Li_s(z)$. □

Corollary 3.5. The BE function has a representation

$$\Gamma(s)\mathfrak{B}_{s-1}(x) = e^x \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m (n + 1)^m e^{nx}}{m!} \delta(s + m) \quad (x \geq 0; \Re(s) > 1). \tag{42}$$

Proof. This follows by putting $\nu = 0$ and replacing $x \mapsto -x$ in (3.1) and using $\Psi_0(s; -x) = \mathfrak{B}_{s-1}(x)$. □

Corollary 3.6. The Hurwitz zeta function has a representation

$$\Gamma(s)\zeta(s, \nu) = 2\pi \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m (n + \nu)^m}{m!} \delta(s + m) \quad (\Re(\nu) > 0; \Re(s) > 1). \tag{43}$$

Proof. It follows by setting $x = 0$, replacing $\nu \mapsto \nu - 1$ in (32) and making use of $\Psi_\nu(s; 0) = \zeta(s, \nu + 1)$. □

Corollary 3.7. The Riemann zeta function (5) has a representation

$$\Gamma(s)\zeta(s) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m (n + 1)^m}{m!} \delta(s + m) \quad (\Re(s) > 1). \tag{44}$$

Proof. Upon setting $\nu = x = 0$ in (32) and using $\Psi_0(s; 0) = \zeta(s, 1) = \zeta(s)$, one can arrive at (44). □

Theorem 3.8. *The eFD function $\Theta_\nu(s; x)$ has a representation*

$$\Gamma(s)\Theta_\nu(s; x) = 2\pi \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{m+n} (n+\nu+1)^m e^{-(n+\nu+1)x}}{m!} \delta(s+m) \quad (\Re(\nu) > -1; x \geq 0; \Re(s) > 0). \quad (45)$$

Proof. Consider the extended Fermi-Dirac function

$$\Theta_\nu(s; x) = \frac{e^{-(\nu+1)x}}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-\nu t}}{e^t + e^{-x}} dt \quad (\Re(\nu) > -1; \Re(s) > 0; x \geq 0) \quad (46)$$

Replacing t by e^{-y} and using $s = \sigma + i\tau$ on the RHS of (46) gives

$$\Gamma(s)\Theta_\nu(s; x) = e^{-(\nu+1)x} \int_{-\infty}^{+\infty} e^{-i\tau y} \frac{e^{-\sigma y} \exp(-(\nu+1)e^{-y})}{1 + \exp(-(e^{-y} + x))} dy \quad (\Re(\nu) > -1; \sigma > 0 \text{ when } x = 0; \sigma > 0 \text{ when } x > 0), \quad (47)$$

where y lies between $-\infty$ and ∞ . The expansion of the denominator in (47) yields

$$[1 + \exp(-(e^{-y} + x))]^{-1} = \sum_{n=0}^{\infty} (-1)^n \exp(-n(e^{-y} + x)) \quad (x \geq 0, -\infty < y < +\infty) \quad (48)$$

then, also

$$\exp(-(n+\nu+1)e^{-y}) = \sum_{m=0}^{\infty} \frac{(-1)^m (n+\nu+1)^m}{m!} e^{-my} \quad (\Re(\nu) > -1, -\infty < y < +\infty). \quad (49)$$

Using (48)-(49) in (47) leads to the following double sum, denoted by $\theta_{(\nu, \sigma)}(x; y)$,

$$\theta_{(\nu, \sigma)}(x; y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{m+n} (n+\nu+1)^m e^{-(n+\nu+1)x}}{m!} e^{-(\sigma+m)y} \quad (50)$$

$(\Re(\nu) > -1; \sigma > 0 \text{ when } x = 0; \sigma > 0 \text{ when } x > 0)$. Now term by term application of the Fourier transform and use of the relationship (31) in (50) leads to the following series of delta functionals.

$$\Gamma(s)\Theta_\nu(s; x) = 2\pi \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{m+n} (n+\nu+1)^m e^{-(n+\nu+1)x}}{m!} \delta(\tau - i(\sigma + m)) \quad (51)$$

$(\Re(\nu) > -1; \sigma > 0 \text{ when } x = 0; \sigma > 0 \text{ when } x > 0)$. Now by making use of the basic property of delta function

$$\delta(\tau - i(\sigma + m)) = \delta\left[\frac{1}{i}(\tau - i^2(\sigma + m))\right] = |i| \delta(\sigma + i\tau + m) = \delta(s + m), \quad (52)$$

we get (45). □

Corollary 3.9. *The FD function has a representation*

$$\Gamma(s)\mathfrak{F}_{s-1}(x) = 2\pi e^x \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{m+n} (n+1)^m e^{nx}}{m!} \delta(s+m) \quad (x \geq 0; s > 1). \quad (53)$$

Proof. This follows from (45), by putting $\nu = 0$, replacing $x \mapsto -x$ and making use of the relation $\Theta_0(s; x) = F_{s-1}(-x)$. □

Corollary 3.10. *The Riemann zeta function (6) has a representation*

$$C(s)\zeta(s) = 2\pi \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{n+m} (n+1)^m}{m!} \delta(s+m) \quad (\Re(s) > 0). \quad (54)$$

Proof. Upon setting $\nu = x = 0$ in (45) and making use of $\Theta_0(s; 0) = (1 - 2^{1-s})\zeta(s)$ we get (54). □

4. Convergence of the Series Representation

A representation of the eFD, eBE and related functions is obtained as a series of delta functionals, which is meaningful only if converges in the sense of distributions (Generalized Functions).

Remark 4.1. *Since complex delta functional is a continuous linear functional on the space of test functions \mathcal{Z} . Therefore, it is easy to prove that all the above series involving delta functional are continuous linear functionals acting on the space \mathcal{Z} .*

Linearity. For all $\phi_1, \phi_2 \in \mathcal{Z}$, (32) yields

$$\langle \Gamma(s)\Psi_\nu(s; x), \phi_1 + \phi_2 \rangle = 2\pi \sum_{n=0}^\infty \sum_{m=0}^\infty \frac{(-1)^m (n + \nu + 1)^m e^{-(n+\nu+1)x}}{m!} \langle \delta(s + m), \phi_1 + \phi_2 \rangle. \tag{55}$$

$$\begin{aligned} \langle \Gamma(s)\Psi_\nu(s; x), \phi_1 + \phi_2 \rangle &= 2\pi \sum_{n=0}^\infty \sum_{m=0}^\infty \frac{(-1)^m (n + \nu + 1)^m e^{-(n+\nu+1)x}}{m!} \left[\langle \delta(s + m), \phi_1 \rangle + \langle \delta(s + m), \phi_2 \rangle \right] \\ &= \langle \Gamma(s)\Psi_\nu(s; x), \phi_1 \rangle + \langle \Gamma(s)\Psi_\nu(s; x), \phi_2 \rangle, \end{aligned} \tag{56}$$

which shows that the eBE function is linear.

Continuity. Let $\{\phi_\mu\}_{\mu=1}^\infty$ be a sequence of test functions in \mathcal{Z} converges to zero. Then by the continuity of delta functional, the sequence $\{\langle \delta(s + m), \phi_\mu \rangle\}_{\mu=1}^\infty$ converges to zero, which implies that

$$\{\langle \Gamma(s)\Psi_\nu(s; x), \phi_\mu \rangle\}_{\mu=1}^\infty = 2\pi \sum_{n=0}^\infty \sum_{m=0}^\infty \frac{(-1)^m (n + \nu + 1)^m e^{-(n+\nu+1)x}}{m!} \{\langle \delta(s + m), \phi_\mu \rangle\}_{\mu=1}^\infty$$

converges to zero. Hence it is proved that the eBE function is a continuous linear functional acting on the space of test functions \mathcal{Z} . Similarly, this fact can be proved for (40)-(45) and (53)-(54). For all functions $\Lambda(s) \in \mathcal{Z}$, (32) directly yields

$$\langle \Gamma(s)\Psi_\nu(s; x), \Lambda(s) \rangle = 2\pi \sum_{n=0}^\infty \sum_{m=0}^\infty \frac{(-1)^m (n + \nu + 1)^m e^{-(n+\nu+1)x}}{m!} \langle \delta(s + m), \Lambda(s) \rangle \tag{57}$$

Similar representation for the eFD function is

$$\langle \Gamma(s)\Theta_\nu(s; x), \Lambda(s) \rangle = 2\pi \sum_{n=0}^\infty \sum_{m=0}^\infty \frac{(-1)^{m+n} (n + \nu + 1)^m e^{-(n+\nu+1)x}}{m!} \langle \delta(s + m), \Lambda(s) \rangle. \tag{58}$$

These representations of the extended FD and BE functions are well defined for all those functions for which these infinite series converge. By using the shifting property of the delta functional, one can get the following equation

$$\langle \delta(s + m), \Lambda(s) \rangle = \Lambda(-m) \quad (m = 0, 1, 2, \dots), \tag{59}$$

which are slowly increasing (bounded by a polynomial) test functions. Further, note that the sums of co-efficients

$$\sum_{m=0}^\infty a_{m,n} = \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{(-1)^m (n + \nu + 1)^m e^{-(n+\nu+1)x}}{m!} = \frac{e^{-\nu(x+1)}}{e^{x+1} - 1} \tag{60}$$

$$\sum_{m=0}^\infty a'_{m,n} = \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{(-1)^{m+n} (n + \nu + 1)^m e^{-(n+\nu+1)x}}{m!} = \frac{e^{-\nu(x+1)}}{e^{x+1} + 1} \tag{61}$$

are convergent. As mentioned in Remark 4.1, (55) and (56) are continuous linear functionals acting on \mathcal{Z} . Therefore, the convergence of the series (55)-(56) in \mathcal{Z}' is proved in the ordinary sense of convergence of numbers. It proves that the series (32) and (45) are distributions in \mathcal{Z}' , which leads to a similar fact for (40)-(45) and (53)-(54). It is easy to prove now because

the sums of the co-efficients in these series are the special cases of the sums (60)-(61). A simple example is the Riemann zeta function for which we have

$$\sum_{m=0}^{\infty} a_{m,n} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^m (n+1)^m}{m!} = \frac{1}{e-1} \quad (62)$$

$$\sum_{m=0}^{\infty} a'_{m,n} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n} (n+1)^m}{m!} = \frac{1}{e+1} \quad (63)$$

5. Some Integral Formulae by Using New Representation

In this section by using the new representation of eFD and eBE functions we find some new integrals of products of these functions. Consider the product of these extended functions on the particular set of functions

$$\{a^{su}\} \quad (a > 0; s \in \mathbb{C}). \quad (64)$$

It gives

$$\begin{aligned} \int_{s \in \mathbb{C}} a^{su} \Gamma(s) \Psi_{\nu}(s; x) ds &= 2\pi \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m (n+\nu+1)^m e^{-(n+\nu+1)x} a^{-mu}}{m!} \\ &= \frac{2\pi a^u \exp(-\nu(x+a^{-u}))}{\exp(x+a^{-u})-1} \quad (\Re(\nu) > -1; x \geq 0; \Re(s) > 1). \end{aligned} \quad (65)$$

Similarly, for the eFD function

$$\begin{aligned} \int_{s \in \mathbb{C}} a^{su} \Gamma(s) \Theta_{\nu}(s; x) ds &= 2\pi \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{m+n} (n+\nu+1)^m e^{-(n+\nu+1)x} a^{-mu}}{m!} \\ &= \frac{2\pi a^u \exp(-\nu(x+a^{-u}))}{\exp(x+a^{-u})+1} \quad (\Re(\nu) > -1; x \geq 0; \Re(s) > 0) \end{aligned} \quad (66)$$

For the Hurwitz-Lerch zeta function

$$\int_{s \in \mathbb{C}} a^{su} \Gamma(s) \Phi(z, s, \nu) ds = \frac{2\pi a^u \exp(-(\nu-1)a^{-u})}{\exp(a^{-u})-z} \quad (\Re(\nu) > 0; \Re(s) > 0 \text{ when } 0 < z < 1; \Re(s) > 1 \text{ when } z = 1) \quad (67)$$

Similarly for the Polylogarithm function

$$\int_{s \in \mathbb{C}} a^{su} \text{Li}_s(z) ds = \frac{2\pi z a^u}{\exp(a^{-u})-z} \quad (\Re(s) > 0 \text{ when } 0 < z < 1; \Re(s) > 1 \text{ when } z = 1). \quad (68)$$

For the BE function

$$\int_{s \in \mathbb{C}} a^{su} \Gamma(s) \mathfrak{B}_{s-1}(x) ds = \frac{2\pi e^x a^u}{\exp(a^{-u})-e^x} \quad (x \geq 0; \Re(s) > 1). \quad (69)$$

Similarly for the FD function

$$\int_{s \in \mathbb{C}} a^{su} \Gamma(s) \mathfrak{F}_{s-1}(x) ds = \frac{2\pi e^x a^u}{\exp(a^{-u})+e^x} \quad (x \geq 0; \Re(s) > 0). \quad (70)$$

The following expression involving the *Hurwitz zeta function* holds

$$\int_{s \in \mathbb{C}} a^{su} \Gamma(s) \zeta(s, \nu) ds = \frac{2\pi a^u \exp(-(\nu - 1)a^{-u})}{\exp(a^{-u}) - 1} \quad (\Re(\nu) > 0; \Re(s) > 1). \tag{71}$$

Similarly one can get the following formulae for the *Riemann zeta function*

$$\int_{s \in \mathbb{C}} a^{su} \Gamma(s) \zeta(s) ds = \frac{2\pi a^u}{\exp(a^{-u}) - 1} \quad (\Re(s) > 1). \tag{72}$$

$$\int_{s \in \mathbb{C}} a^{su} C(s) \zeta(s) ds = \frac{2\pi a^u}{\exp(a^{-u}) + 1} \quad (\Re(s) > 0). \tag{73}$$

By putting $a = 1$ in above identities, one can get the following results

$$\begin{aligned} \int_{s \in \mathbb{C}} \Gamma(s) \Psi_\nu(s; x) ds &= 2\pi \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m (n + \nu + 1)^m e^{-(n+\nu+1)x}}{m!} \\ &= \frac{2\pi \exp(-\nu(x+1))}{\exp(x+1) - 1} \quad (\Re(\nu) > -1; x \geq 0; \Re(s) > 1). \end{aligned} \tag{74}$$

$$\begin{aligned} \int_{s \in \mathbb{C}} \Gamma(s) \Theta_\nu(s; x) ds &= 2\pi \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{m+n} (n + \nu + 1)^m e^{-(n+\nu+1)x}}{m!} \\ &= \frac{2\pi \exp(-\nu(x+1))}{\exp(x+1) + 1} \quad (\Re(\nu) > -1; x \geq 0; \Re(s) > 0) \end{aligned} \tag{75}$$

$$\int_{s \in \mathbb{C}} \Gamma(s) \Phi(z, s, \nu) ds = \frac{2\pi \exp(-(\nu - 1))}{e - z} \quad (\Re(\nu) > 0; \Re(s) > 0 \text{ when } 0 < z < 1; \Re(s) > 1 \text{ when } z = 1) \tag{76}$$

$$\int_{s \in \mathbb{C}} \Gamma(s) \text{Li}_s(z) ds = \frac{2\pi z}{e - z} \quad (\Re(s) > 0 \text{ when } 0 < z < 1; \Re(s) > 1 \text{ when } z = 1) \tag{77}$$

$$\int_{s \in \mathbb{C}} \Gamma(s) \mathfrak{B}_{s-1}(x) ds = \frac{2\pi}{e^{1-x} - 1} \quad (x \geq 0; \Re(s) > 1) \tag{78}$$

$$\int_{s \in \mathbb{C}} \Gamma(s) \mathfrak{F}_{s-1}(x) ds = \frac{2\pi e^x}{1 + e^x} \quad (x \geq 0; \Re(s) > 0) \tag{79}$$

$$\int_{s \in \mathbb{C}} \Gamma(s) \zeta(s, \nu) ds = \frac{2\pi \exp(-(\nu - 1))}{e - 1} \quad (\Re(\nu) > 0; \Re(s) > 1) \tag{80}$$

$$\int_{s \in \mathbb{C}} \Gamma(s) \zeta(s) ds = \frac{2\pi}{e - 1} \quad (\Re(s) > 1) \tag{81}$$

$$\int_{s \in \mathbb{C}} C(s) \zeta(s) ds = \frac{2\pi}{e + 1} \quad (\Re(s) > 0). \tag{82}$$

6. Further Insights and Concluding Remarks

In this paper we formalized the definition of eFD and eBE functions over the space of entire test functions denoted by \mathcal{Z} . These extended functions have elegant connections with the zeta family and other related functions, which have proved useful to obtain the representations of these functions in terms of delta functions by taking particular values of the involved parameters. It led to some new integrals of products of these functions and the family of zeta functions over complex plane. The methodology can hopefully be applied to other functions. For example the Riemann zeta function in the critical strip

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) dt \quad (0 < \Re(s) < 1) \tag{83}$$

has always been remained a source of interest. By using the methods in this paper we obtain

$$\zeta(s) \Gamma(s) = 2\pi \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m (n + 1)^m}{m!} \delta(s + m) - 2\pi \delta(s - 1) \quad (0 < \Re(s) < 1). \tag{84}$$

Further, we can get the following useful identities

$$\int_{s \in \mathbb{C}} a^{su} \Gamma(s) \zeta(s) ds = \frac{2\pi}{\exp(a^{-u}) - 1} - 2\pi a^u \quad (0 < \Re(s) < 1). \quad (85)$$

$$\int_{s \in \mathbb{C}} \Gamma(s) \zeta(s) ds = \frac{2\pi}{e - 1} - 2\pi \quad (0 < \Re(s) < 1). \quad (86)$$

We have used a space of test functions which is already available in the literature. It is hoped that this new representation can be used to develop the elements of a generalized theory for these functions by using delta functionals, which though not truly functions in the classical sense can, with some precautions, be treated as functions.

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