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Wiener Index of Total Graph of Some Graphs

Research Article

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Abstract: Let $G = (V, E)$ be a graph. The *total graph* $T(G)$ of G is that graph whose vertex set is $V \cup E$, and two vertices are adjacent if and only if they are adjacent or incident in G. For a graph $G = (V, E)$, the graph $G.S_m$ is obtained by identifying each vertex of G by a root vertex of S_m and the graph $S_m.G$ is obtained by identifying each vertex of S_m except root vertex by any vertex of G, where S_m is a star graph with m vertices. In this paper, we consider G as the cycle graph C_n with n vertices and investigate the Wiener index of the total graphs of $C_n.S_m$ and $S_n.C_m$.

MSC: 05C12, 05C76

Keywords: Topological index, Wiener index, total graph.

1. Introduction

For a graph $G = (V, E)$ if u, $v \in V(G)$, then the *distance* $d(u, v)$ between u and v is defined as the length of a shortest u-v path in G. A *topological index* is a numerical quantity mathematically derived from the graph structure. It is a graph invariant i.e., it does not depend on the labeling or pictorial representation of the graph. The topological indices of molecular graphs are widely used for establishing association between the structure of a molecular compound and its physico-chemical properties or biological activity (e.g., pharmacology). Some topological indices are Wiener index, Schultz index etc. The *Wiener index* of a graph $G = (V, E)$ is defined as

$$W(G) = \sum_{\{u,v\} \subset V} d(u, v).$$

It is the oldest topological index and its mathematical properties and chemical applications have been extensively studied. The Wiener index was introduced by the chemist Harold Wiener in 1947 for explaining the correlation between the boiling points of paraffins and the structure of their molecules. It is a graph invariant much studied in both mathematical and chemical literature (*see* [11], [12], [13] and [18]). The concept of graph operator has found various applications in chemical research (*see* [7], [8], [9], [10], [15] and [17]). The *total graph* $T(G)$ of G is that graph whose vertex set is $V(G) \cup E(G)$, and two vertices are adjacent if and only if they are adjacent or incident in G. The notion of total graph was introduced by Behzad & Chartrand [4]. Several properties of total graphs are investigated in the literature (*see* [1], [2], [5] and [6]). Behzad obtained a characterization of total graphs [3]. Gavril established a linear time algorithm for the recognition of the total graphs in [14]. The total graph $H = T(G)$ of G is shown in Figure 1.

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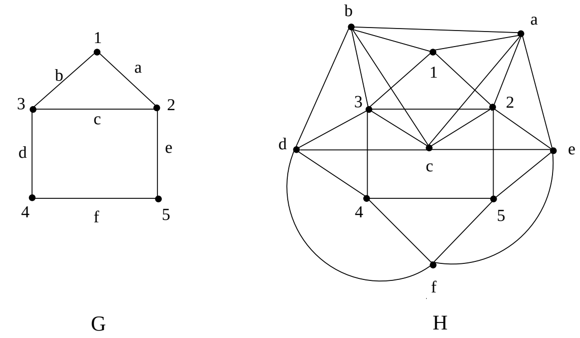

Figure 1. **A graph G and its total graph $H = T(G)$**

We define two new graph operators $G.S_m$ and $S_n.G$ defined as follows: For a graph $G = (V, E)$, the graph $G.S_m$ is obtained by identifying each vertex of G by a root vertex of S_m and the graph $S_m.G$ is obtained by identifying each vertex of S_m except root vertex by any vertex of G, where S_m is a star graph with m vertices. Now we consider G as the cycle graph C_n with n vertices. The graphs $C_4.S_5$ and $S_5.C_4$ are shown in Figure 2.

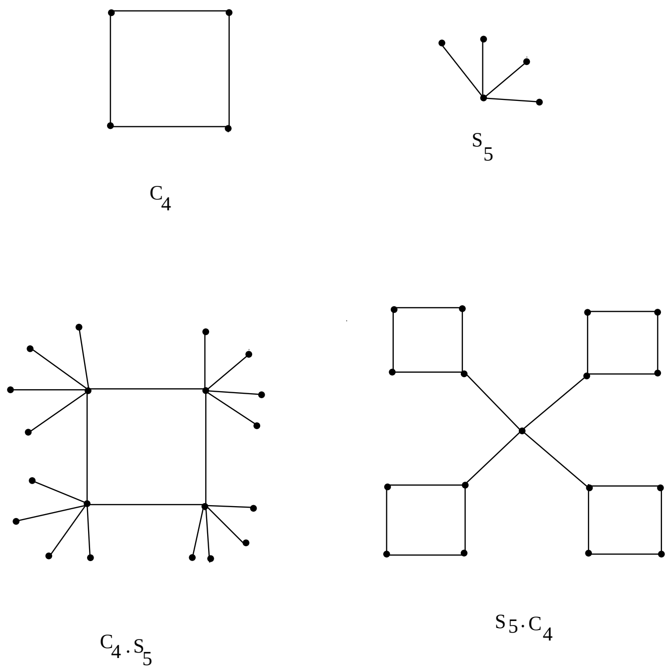

Figure 2. **The graphs C_4, S_5, $C_4.S_5$ and $S_5.C_4$**

2. Main Section

2.1. Wiener index of total graph of $C_n.S_m$

Lemma 2.1 ([16]). *The Wiener index of the graph $G = C_n$,*

$$W(G) = \begin{cases} \frac{n^3}{8}; & \text{if } n \text{ is even} \\ \frac{n^3-n}{8}; & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 2.2. *The Wiener index of the graph $G = T(C_n)$,*

$$W(G) = \frac{n^2(n+1)}{2}.$$

Proof. Since,

$$\begin{aligned} W(T(C_n)) &= \sum_{u,v \in V(T(C_n))} d(u,v) \\ &= \sum_{u,v \in V(C_n)} d(u,v) + \sum_{e,f \in E(C_n)} d(e,f) + \sum_{\substack{u \in V(C_n),\\ e \in E(C_n)}} d(u,e) \\ &= W(C_n) + W(C_n) + \sum_{\substack{u \in V(C_n),\\ e \in E(C_n)}} d(u,e). \end{aligned}$$

To calculate $\sum_{\substack{u \in V(C_n),\\ e \in E(C_n)}} d(u,e)$, we consider two cases:

Case 1: Suppose n is even. Then,

$$\sum_{\substack{u \in V(C_n),\\ e \in E(C_n)}} d(u,e) = 2n\left(1 + 2 + ... + \frac{n}{2}\right).$$

Thus,

$$W(T(C_n)) = W(C_n) + W(C_n) + 2n\left(1 + 2 + ... + \frac{n}{2}\right).$$

Now using Lemma 2.1, we have

$$\begin{aligned} W(T(C_n)) &= 2\left(\frac{n^3}{8}\right) + 2n\left(1 + 2 + ... + \frac{n}{2}\right) \\ &= \frac{n^2(n+1)}{2}. \end{aligned}$$

Case 2: Suppose n is odd. Then,

$$\sum_{\substack{u \in V(C_n),\\ e \in E(C_n)}} d(u,e) = 2n\left(1 + 2 + ... + \frac{(n-1)}{2}\right) + \frac{n(n+1)}{2}.$$

Therefore,

$$W(T(C_n)) = W(C_n) + W(C_n) + 2n\left(1 + 2 + ... + \frac{(n-1)}{2}\right) + \frac{n(n+1)}{2}.$$

Now using Lemma 2.1, we have

$$\begin{aligned} W(T(C_n)) &= 2\left(\frac{n^3 - n}{8}\right) + 2n\left(1 + 2 + ... + \frac{(n-1)}{2}\right) + \frac{n(n+1)}{2} \\ &= \frac{n^2(n+1)}{2}. \end{aligned}$$

Hence,

$$W(T(C_n)) = \frac{n^2(n+1)}{2}.$$

□

Note: Let $V(S_n^*)$ denotes the set of vertices of star graph S_n except root vertex.

Theorem 2.3. *The Wiener index of the graph* $G = C_n.S_m$,

$$W(G) = \begin{cases} \frac{n^3}{8}(m+1)^2 + mn(n(1+m)-1); & \textit{if } n \textit{ is even} \\ \frac{n^3}{8}(m+1)^2 - \frac{n}{8}(m^2+10m+1) + mn^2(m+1); & \textit{if } n \textit{ is odd.} \end{cases}$$

Proof. Since,

$$\begin{aligned} W(C_n.S_m) &= \sum_{u,v \in V(C_n.S_m)} d(u,v) \\ &= \sum_{u,v \in V(C_n)} d(u,v) + \sum_{u,v \in V(S_m^*)} d(u,v) + \sum_{\substack{u \in V(C_n),\\ v \in V(S_m^*)}} d(u,v) \\ &= W(C_n) + \sum_{u,v \in V(S_m^*)} d(u,v) + \sum_{\substack{u \in V(C_n),\\ v \in V(S_m^*)}} d(u,v) \end{aligned}$$

To calculate $\sum_{u,v \in V(S_m^*)} d(u,v)$ and $\sum_{\substack{u \in V(C_n),\\ v \in V(S_m^*)}} d(u,v)$, we consider two cases:

Case 1: Suppose n is even. Then,

$$\begin{aligned} \sum_{u,v \in V(S_m^*)} d(u,v) &= n({}^mC_2).2 + nm^2\,(3+4+...+(n+1)) + \frac{nm^2}{2} + \left(\frac{n}{2}+2\right) \\ \sum_{\substack{u \in V(C_n),\\ v \in V(S_m^*)}} d(u,v) &= nm\left(1+2(2)+2(3)+...2\left(\frac{n}{2}\right)+\left(\frac{n}{2}+1\right)\right). \end{aligned}$$

Thus,

$$W(C_n.S_m) = W(C_n) + n({}^mC_2).2 + nm^2\,(3+4+...+(n+1)) + \frac{nm^2}{2} + \left(\frac{n}{2}+2\right) + nm\left(1+2(2)+2(3)+...2\left(\frac{n}{2}\right)+\left(\frac{n}{2}+1\right)\right).$$

Now using Lemma 2.1, we have

$$\begin{aligned} W(C_n.S_m) &= \frac{n^3}{8} + n({}^mC_2).2 + nm^2\,(3+4+...+(n+1)) + \frac{nm^2}{2} + \left(\frac{n}{2}+2\right) + nm\left(1+2(2)+2(3)+...2\left(\frac{n}{2}\right)+\left(\frac{n}{2}+1\right)\right) \\ &= \frac{n^3}{8}(m+1)^2 + mn(n(1+m)-1). \end{aligned}$$

Case 2: Suppose n is odd. Then,

$$\begin{aligned} \sum_{u,v \in V(S_m^*)} d(u,v) &= n({}^mC_2).2 + nm^2\left(3+4+...+\left(\frac{n}{n}+2\right)\right) \\ \sum_{\substack{u \in V(C_n),\\ v \in V(S_m^*)}} d(u,v) &= nm + 2nm\left(2+3+...\left(\frac{n}{2}+1\right)\right). \end{aligned}$$

Thus,

$$W(C_n.S_m) = W(C_n) + n({}^mC_2).2 + nm^2\left(3+4+...+\left(\frac{n}{n}+2\right)\right) + nm + 2nm\left(2+3+...\left(\frac{n}{2}+1\right)\right).$$

Now using Lemma 2.1, we have

$$\begin{aligned} W(C_n.S_m) &= \frac{n^3-n}{8} + n({}^mC_2).2 + nm^2\left(3+4+...+\left(\frac{n}{n}+2\right)\right) + nm + 2nm\left(2+3+...\left(\frac{n}{2}+1\right)\right) \\ &= \frac{n^3}{8}(m+1)^2 - \frac{n}{8}(m^2+10m+1) + mn^2(m+1). \end{aligned}$$

Hence,

$$W(C_n.S_m) = \begin{cases} \frac{n^3}{8}(m+1)^2 + mn(n(1+m)-1); & \text{if } n \text{ is even} \\ \frac{n^3}{8}(m+1)^2 - \frac{n}{8}(m^2+10m+1) + mn^2(m+1); & \text{if } n \text{ is odd.} \end{cases}$$

□

Theorem 2.4. *The Wiener index of the graph $G = T(C_n.S_m)$,*

$$W(G) = \begin{cases} \frac{1}{2}\left[mn^2(2+m)(4+n) + mn(3mn-5) + n^2(n+1)\right]; & \textit{if } n \textit{ is even} \\ \frac{1}{2}\left[mn^2(2+m)(4+n) + mn(3mn-m-5) + n^2(n+1)\right]; & \textit{if } n \textit{ is odd.} \end{cases}$$

Proof. Since,

$$\begin{aligned} W(T(C_n.S_m)) &= \sum_{u,v\in V(T(C_n.S_m))} d(u,v) \\ &= \sum_{u,v\in V(C_n.S_m)} d(u,v) + \sum_{e,f\in E(C_n.S_m)} d(e,f) + \sum_{\substack{u\in V(C_n.S_m),\\ e\in E(C_n.S_m)}} d(u,e) \\ &= W(C_n.S_m) + \sum_{e,f\in E(C_n.S_m)} d(e,f) + \sum_{\substack{u\in V(C_n.S_m),\\ e\in E(C_n.S_m)}} d(u,e). \end{aligned}$$

To calculate $\sum_{e,f\in E(C_n.S_m)} d(e,f)$ and $\sum_{\substack{u\in V(C_n.S_m),\\ e\in E(C_n.S_m)}} d(u,e)$ we consider two cases:

Case 1: Suppose n is even. Then,

$$\begin{aligned} \sum_{e,f\in E(C_n.S_m)} d(e,f) &= W(C_n) + 2nm\left(1+2+...\frac{n}{2}\right) + {}^mC_2 + nm^2\left(2+3+...+\frac{n}{2}\right) + \frac{nm^2}{2}\left(\frac{n}{2}+1\right) \\ \sum_{\substack{u\in V(C_n.S_m),\\ e\in E(C_n.S_m)}} d(u,e) &= 2nm\left(2+3+\cdots+\left(\frac{n}{2}+1\right)\right) + nm + nm(m-1).2 + 2nm^2\,(3+4 \\ &+ \cdots + \left(\frac{n}{2}+1\right)\Big) + nm^2\left(\frac{n}{2}+2\right) + 2n\left(1+2+...+\frac{n}{2}\right) + nm \\ &+ 2nm\left(2+3+...+\frac{n}{2}\right) + nm\left(\frac{n}{2}+1\right). \end{aligned}$$

Thus,

$$\begin{aligned} W(T(C_n.S_m)) &= W(C_n.S_m) + W(C_n) + 2nm(1+2+...n) + {}^mC_2 + nm^2\left(2+3+...+\frac{n}{2}\right) \\ &+ \frac{nm^2}{2}\left(\frac{n}{2}+1\right) + 2nm\left(2+3+...\left(\frac{n}{2}+1\right)\right) + nm + nm(m-1).2 \\ &+ 2nm^2\left(3+4+...+\left(\frac{n}{2}+1\right)\right) + nm^2\left(\frac{n}{2}+2\right) \\ &+ 2n\left(1+2+...+\frac{n}{2}\right) + nm + 2nm\left(2+3+...+\frac{n}{2}\right) + nm\left(\frac{n}{2}+1\right). \end{aligned}$$

Now using Theorem 2.3 and Lemma 2.1, we have

$$\begin{aligned} W(T(C_n.S_m)) &= \frac{n^3}{8}(m+1)^2 + mn(n(1+m)-1) + \frac{n^3}{8} \\ &+ 2nm(1+2+...n) + {}^mC_2 + nm^2\left(2+3+...+\frac{n}{2}\right) + \frac{nm^2}{2}\left(\frac{n}{2}+1\right) \\ &+ 2nm\left(2+3+...\left(\frac{n}{2}+1\right)\right) + nm + nm(m-1).2 + 2nm^2\,(3+4 \\ &+ \cdots + \left(\frac{n}{2}+1\right)\Big) + nm^2\left(\frac{n}{2}+2\right) + 2n\left(1+2+...+\frac{n}{2}\right) + nm \\ &+ 2nm\left(2+3+...+\frac{n}{2}\right) + nm\left(\frac{n}{2}+1\right) \\ &= \frac{1}{2}\left[mn^2(2+m)(4+n) + mn(3mn-5) + n^2(n+1)\right]. \end{aligned}$$

Case 2: Suppose n is odd. Then,

$$\begin{aligned}
\sum_{e,f\in E(C_n.S_m)} d(e,f) &= W(C_n) + 2nm\left(1+2+\ldots\frac{n-1}{2}\right) + nm\left(\frac{n+1}{2}\right) + n({}^mC_2) + nm^2\left(2+3+\ldots+\frac{n+1}{2}\right)\\
\sum_{\substack{u\in V(C_n.S_m),\\ e\in E(C_n.S_m)}} d(u,e) &= 2n\left(1+2+\ldots+\frac{n-1}{2}\right) + n\left(\frac{n+1}{2}\right) + nm\,(1+2(2)+2(3)+\ldots\\
&+ 2\left(\frac{n+1}{2}\right)\bigg) + nm\left(2(2)+2(3)+\ldots+2\left(\frac{n+1}{2}\right)+\left(\frac{n+3}{2}\right)\right)\\
&+ nm + nm(m-1).2 + 2nm^2\left(3+4+\ldots+\frac{n+3}{2}\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
W(T(C_n.S_m)) &= W(C_n.S_m) + W(C_n) + 2nm\left(1+2+\ldots\frac{n-1}{2}\right) + nm\left(\frac{n+1}{2}\right)\\
&+ n({}^mC_2) + nm^2\left(2+3+\ldots+\frac{n+1}{2}\right) + 2n\left(1+2+\ldots+\frac{n-1}{2}\right)\\
&+ n\left(\frac{n+1}{2}\right) + nm\left(1+2(2)+2(3)+\ldots+2\left(\frac{n+1}{2}\right)\right)\\
&+ nm\left(2(2)+2(3)+\ldots+2\left(\frac{n+1}{2}\right)+\left(\frac{n+3}{2}\right)\right) + nm\\
&+ nm(m-1).2 + 2nm^2\left(3+4+\ldots+\frac{n+3}{2}\right).
\end{aligned}$$

Now using Theorem 2.3 and Lemma 2.1, we have

$$\begin{aligned}
W(T(C_n.S_m)) &= \frac{n^3}{8}(m+1)^2 - \frac{n}{8}(m^2+10m+1) + mn^2(m+1) + \frac{n^3-n}{8}\\
&+ 2nm\left(1+2+\ldots\frac{n-1}{2}\right) + nm\left(\frac{n+1}{2}\right) + n({}^mC_2)\\
&+ nm^2\left(2+3+\ldots+\frac{n+1}{2}\right) + 2n\left(1+2+\ldots+\frac{n-1}{2}\right)\\
&+ n\left(\frac{n+1}{2}\right) + nm\left(1+2(2)+2(3)+\ldots+2\left(\frac{n+1}{2}\right)\right)\\
&+ nm\left(2(2)+2(3)+\ldots+2\left(\frac{n+1}{2}\right)+\left(\frac{n+3}{2}\right)\right) + nm\\
&+ nm(m-1).2 + 2nm^2\left(3+4+\ldots+\frac{n+3}{2}\right)\\
&= \frac{1}{2}\left[mn^2(2+m)(4+n) + mn(3mn-m-5) + n^2(n+1)\right].
\end{aligned}$$

Hence,

$$W(T(C_n.S_m)) = \begin{cases} \frac{1}{2}\left[mn^2(2+m)(4+n) + mn(3mn-5) + n^2(n+1)\right]; & \text{if } n \text{ is even}\\ \frac{1}{2}\left[mn^2(2+m)(4+n) + mn(3mn-m-5) + n^2(n+1)\right]; & \text{if } n \text{ is odd.} \end{cases}$$

□

2.2. Wiener index of total graph of $S_n.C_m$

Note: Let $V(S_n^{**})$ denotes the set of root vertex of star graph S_n.

Theorem 2.5. *The Wiener index of the graph $G = S_n.C_m$,*

$$W(G) = \begin{cases} \frac{n-1}{8}\left[m^3(2n-3) + 2m^2(4n-7) + 8m\right]; & \textit{if } n \textit{ is even}\\ \frac{n-1}{8}\left[m^3(2n-3) + 2m^2(4n-7) - 2\right] - \frac{m}{8}(2n^2-15n+11); & \textit{if } n \textit{ is odd.} \end{cases}$$

Proof. Since,

$$
\begin{aligned}
W(S_n.C_m) &= \sum_{u,v\in V(S_n.C_m)} d(u,v)\\
&= \sum_{u,v\in V(S_n^{**})} d(u,v) + \sum_{u,v\in V(C_m)} d(u,v) + \sum_{\substack{u\in V(S_n^{**}),\\ v\in V(C_m)}} d(u,v)\\
&= 0 + \sum_{u,v\in V(C_m)} d(u,v) + \sum_{\substack{u\in V(S_n^{**}),\\ v\in V(C_m)}} d(u,v).
\end{aligned}
$$

To calculate $\sum_{u,v\in V(C_m)} d(u,v)$ and $\sum_{\substack{u\in V(S_n^{**}),\\ v\in V(C_m)}} d(u,v)$, we consider two cases:

Case 1: Suppose n is even. Then,

$$
\begin{aligned}
\sum_{u,v\in V(C_m)} d(u,v) &= (n-1).W(C_m) + {}^{(n-1)}C_2\left[\left(2+2(3)+2(4)+\cdots+2\left(\frac{m}{2}\right)+\left(\frac{m}{2}+1\right)\right)\right.\\
&+ \left(3+2(4)+2(5)+\cdots+2\left(\frac{m}{2}+1\right)+\left(\frac{m}{2}+2\right)\right)+\cdots+\left(\left(\frac{m}{2}+2\right)\right.\\
&+ \left.\left. 2\left(\frac{m}{2}+3\right)+2\left(\frac{m}{2}+4\right)+\cdots+2\left(\frac{m}{2}+\frac{m}{2}+1\right)+\left(\frac{m}{2}+\frac{m}{2}+2\right)\right)\right].
\end{aligned}
$$

and

$$
\sum_{\substack{u\in V(S_n^{**}),\\ v\in V(C_m)}} d(u,v) = (n-1)\left[1+2(2)+2(3)+\cdots+2\left(\frac{m}{2}\right)+\left(\frac{m}{2}+1\right)\right].
$$

Thus,

$$
\begin{aligned}
W(S_n.C_m) &= (n-1).W(C_m) + {}^{(n-1)}C_2\left[\left(2+2(2)+2(3)+\cdots+2\left(\frac{m}{2}\right)+\left(\frac{m}{2}+1\right)\right)\right.\\
&+ \left(3+2(4)+2(5)+\cdots+2\left(\frac{m}{2}+1\right)+\left(\frac{m}{2}+2\right)\right)+\cdots+\left(\left(\frac{m}{2}+2\right)\right.\\
&+ \left.\left. 2\left(\frac{m}{2}+3\right)+2\left(\frac{m}{2}+4\right)+\cdots+2\left(\frac{m}{2}+\frac{m}{2}+1\right)+\left(\frac{m}{2}+\frac{m}{2}+2\right)\right)\right]\\
&+ (n-1)\left[1+2(2)+2(3)+\cdots+2\left(\frac{m}{2}\right)+\left(\frac{m}{2}+1\right)\right].
\end{aligned}
$$

Now using Lemma 2.2 we have,

$$
\begin{aligned}
W(S_n.C_m) &= (n-1)\left(\frac{m^3}{8}\right) + {}^{(n-1)}C_2\left[\left(2+2(2)+2(3)+\cdots+2\left(\frac{m}{2}\right)+\left(\frac{m}{2}+1\right)\right)\right.\\
&+ \left(3+2(4)+2(5)+\cdots+2\left(\frac{m}{2}+1\right)+\left(\frac{m}{2}+2\right)\right)+\cdots+\left(\left(\frac{m}{2}+2\right)\right.\\
&+ \left.\left. 2\left(\frac{m}{2}+3\right)+2\left(\frac{m}{2}+4\right)+\cdots+2\left(\frac{m}{2}+\frac{m}{2}+1\right)+\left(\frac{m}{2}+\frac{m}{2}+2\right)\right)\right]\\
&+ (n-1)\left[1+2(2)+2(3)+\cdots+2\left(\frac{m}{2}\right)+\left(\frac{m}{2}+1\right)\right]\\
&= \frac{n-1}{8}\left[m^3(2n-3)+2m^2(4n-7)+8m\right].
\end{aligned}
$$

Case 2: Suppose n is odd. Then,

$$
\begin{aligned}
\sum_{u,v\in V(C_m)} d(u,v) &= (n-1).W(C_m) + {}^{(n-1)}C_2\left[\left(2+2(3)+2(4)+\cdots+2\left(\frac{m+3}{2}\right)\right)\right]\\
&+ 2\left({}^{(n-1)}C_2\right)\left[\left(3+2(4)+2(5)+\cdots+2\left(\frac{m+5}{2}\right)\right)+(4+2(5)\right.\\
&+ \left. 2(6)+\cdots+2\left(\frac{m+7}{2}\right)\right)+\cdots+\left(\left(\frac{m+3}{2}\right)+2\left(\frac{m+5}{2}\right)\right.\\
&+ \left.\left. 2\left(\frac{m+7}{2}\right)+\cdots+2\,(m+1)\right)\right].
\end{aligned}
$$

and

$$\sum_{\substack{u\in V(S_n^{**}),\\ v\in V(C_m)}} d(u,v) = (n-1)\left[1+2(2)+2(3)+\cdots+2\left(\frac{m+1}{2}\right)\right].$$

Thus,

$$\begin{aligned}
W(S_n.C_m) &= (n-1).W(C_m) + {}^{(n-1)}C_2\left[\left(2+2(3)+2(4)+\cdots+2\left(\frac{m+3}{2}\right)\right)\right]\\
&+ 2\left({}^{(n-1)}C_2\right)\left[\left(3+2(4)+2(5)+\cdots+2\left(\frac{m+5}{2}\right)\right)+(4+2(5)\right.\\
&+ 2(6)+\cdots+2\left(\frac{m+7}{2}\right)\Big)+\cdots+\left(\left(\frac{m+3}{2}\right)+2\left(\frac{m+5}{2}\right)\right.\\
&+ \left.\left.2\left(\frac{m+7}{2}\right)+\cdots+2\,(m+1)\right)\right]+(n-1)\left[1+2(2)+2(3)\,+\,\cdots+2\left(\frac{m+1}{2}\right)\right].
\end{aligned}$$

Now using Lemma 2.2 we have,

$$\begin{aligned}
W(S_n.C_m) &= (n-1)\left(\frac{m^3-m}{8}\right) + {}^{(n-1)}C_2\left[\left(2+2(3)+2(4)+\cdots+2\left(\frac{m+3}{2}\right)\right)\right]\\
&+ 2\left({}^{(n-1)}C_2\right)\left[\left(3+2(4)+2(5)+\cdots+2\left(\frac{m+5}{2}\right)\right)+(4+2(5)\right.\\
&+ 2(6)+\cdots+2\left(\frac{m+7}{2}\right)\Big)+\cdots+\left(\left(\frac{m+3}{2}\right)+2\left(\frac{m+5}{2}\right)\right.\\
&+ \left.\left.2\left(\frac{m+7}{2}\right)+\cdots+2\,(m+1)\right)\right]+(n-1)\left[1+2(2)+2(3)\,+\,\cdots+2\left(\frac{m+1}{2}\right)\right]\\
&= \frac{n-1}{8}\left[m^3(2n-3)+2m^2(4n-7)-2\right]-\frac{m}{8}(2n^2-15n+11).
\end{aligned}$$

Hence,

$$W(S_n.C_m)=\begin{cases}\frac{n-1}{8}\left[m^3(2n-3)+2m^2(4n-7)+8m\right]; & \text{if } n \text{ is even}\\ \frac{n-1}{8}\left[m^3(2n-3)+2m^2(4n-7)-2\right]-\frac{m}{8}(2n^2-15n+11); & \text{if } n \text{ is odd.}\end{cases}$$

□

Theorem 2.6. *The Wiener index of the graph $G = T(S_n.C_m)$,*

$$W(G)=\begin{cases}\frac{n-1}{2}\left[m^3(2n-1)+m^2(10n-17)+m(7n-6)+n\right]; & \textit{if } n \textit{ is even}\\ \frac{n-1}{8}\left[m^3(8n-13)+m^2(19n-26)+m(128n-223)-(75n-158)\right]; & \textit{if } n \textit{ is odd.}\end{cases}$$

Proof. Since,

$$\begin{aligned}
W(T(S_n.C_m)) &= \sum_{u,v\in V(T(S_n.C_m))} d(u,v)\\
&= \sum_{u,v\in V(S_n.C_m)} d(u,v) + \sum_{e,f\in E(S_n.C_m)} d(e,f) + \sum_{\substack{u\in V(S_n.C_m),\\ e\in E(S_n.C_m)}} d(u,e)\\
&= W(S_n.C_m) + \sum_{e,f\in E(S_n.C_m)} d(e,f) + \sum_{\substack{u\in V(S_n.C_m),\\ e\in E(S_n.C_m)}} d(u,e).
\end{aligned}$$

To calculate $\sum\limits_{e,f\in E(S_n.C_m)} d(e,f)$ and $\sum\limits_{\substack{u\in V(S_n.C_m),\\ e\in E(S_n.C_m)}} d(u,e)$ we consider two cases:

Case 1: Suppose n is even. Then,

$$\begin{aligned}\sum_{e,f\in E(S_n.C_m)} d(e,f) &= (n-1)W(C_m) + {}^{(n-1)}C_2 + (n-1)\left[1\left(2(1)+2(2)+\dots+2\left(\frac{m}{2}\right)\right)\right.\\
&+ (n-2)\left.\left(2(2)+2(3)+\dots+2\left(\frac{m}{2}+1\right)\right)\right] + [2(n-2)+2(n-3)+\dots\\
&+ 2(n-(n-1))]\left[\left(2(3)+2(4)+\dots+2\left(\frac{m}{2}+2\right)\right) + (2(4)+2(5)+\dots\right.\\
&+ \left.2\left(\frac{m}{2}+3\right)\right) + \dots + \left.\left(2\left(\frac{m}{2}+2\right)+2\left(\frac{m}{2}+3\right)+\dots+2(m+1)\right)\right].\end{aligned}$$

and

$$\begin{aligned}\sum_{\substack{u\in V(S_n.C_m),\\ e\in E(S_n.C_m)}} d(u,e) &= (n-1)+(n-1)\left(2(2)+2(3)+\dots+2\left(\frac{m}{2}+1\right)\right) + m(n-1)\,(2(1)\\
&+ 2(2)+\dots+2\left(\frac{m}{2}\right)\Big) + (n-1)\,(1+2(n-2)+(n-2)\,(2(3)+2(4)\\
&+ \dots+2\left(\frac{m}{2}+2\right)\Big)\Big) + 2(n-1)\,[(2+3(n-2)+(n-2)\,(2(4)+2(5)\\
&+ \dots+2\left(\frac{m}{2}+3\right)\Big)\Big) + (3+4(n-2)+(n-2)\,(2(5)+2(6)+\dots\\
&+ 2\left(\frac{m}{2}+4\right)\Big)\Big) + \dots + \left(\frac{m}{2}+(n-2)\left(\frac{m}{2}+1\right)+(n-2)\left(2\left(\frac{m}{2}+2\right)\right.\right.\\
&+ 2\left(\frac{m}{2}+3\right)+\dots+2\left(\frac{m}{2}+\frac{m}{2}+1\right)\Big)\Big)\Big] + (n-1)\left[\left(\frac{m}{2}+1\right)\right.\\
&+ (n-2)\left(\frac{m}{2}+2\right)+(n-2)\left(2\left(\frac{m}{2}+3\right)+2\left(\frac{m}{2}+4\right)+\dots\ +\ 2\left(\frac{m}{2}+\frac{m}{2}+2\right)\right)\Big].\end{aligned}$$

Thus,

$$\begin{aligned}W(T(S_n.C_m)) &= W(S_n.C_m) + (n-1)W(C_m) + {}^{(n-1)}C_2 + (n-1)\left[1\left(2(1)+2(2)+\dots+2\left(\frac{m}{2}\right)\right)\right.\\
&+ (n-2)\left.\left(2(2)+2(3)+\dots+2\left(\frac{m}{2}+1\right)\right)\right] + [2(n-2)+2(n-3)+\dots\\
&+ 2(n-(n-1))]\left[\left(2(3)+2(4)+\dots+2\left(\frac{m}{2}+2\right)\right) + (2(4)+2(5)+\dots\right.\\
&+ \left.2\left(\frac{m}{2}+3\right)\right) + \dots + \left.\left(2\left(\frac{m}{2}+2\right)+2\left(\frac{m}{2}+3\right)+\dots+2(m+1)\right)\right]\\
&+ (n-1)+(n-1)\left(2(2)+2(3)+\dots+2\left(\frac{m}{2}+1\right)\right) + m(n-1)\,(2(1)\\
&+ 2(2)+\dots+2\left(\frac{m}{2}\right)\Big) + (n-1)\,(1+2(n-2)+(n-2)\,(2(3)+2(4)\\
&+ \dots+2\left(\frac{m}{2}+2\right)\Big)\Big) + 2(n-1)\,[(2+3(n-2)+(n-2)\,(2(4)+2(5)\\
&+ \dots+2\left(\frac{m}{2}+3\right)\Big)\Big) + (3+4(n-2)+(n-2)\,(2(5)+2(6)+\dots\\
&+ 2\left(\frac{m}{2}+4\right)\Big)\Big) + \dots + \left(\frac{m}{2}+(n-2)\left(\frac{m}{2}+1\right)+(n-2)\left(2\left(\frac{m}{2}+2\right)\right.\right.\\
&+ 2\left(\frac{m}{2}+3\right)+\dots+2\left(\frac{m}{2}+\frac{m}{2}+1\right)\Big)\Big)\Big] + (n-1)\left[\left(\frac{m}{2}+1\right)\right.\\
&+ (n-2)\left(\frac{m}{2}+2\right)+(n-2)\left(2\left(\frac{m}{2}+3\right)+2\left(\frac{m}{2}+4\right)+\dots\ +\ 2\left(\frac{m}{2}+\frac{m}{2}+2\right)\right)\Big].\end{aligned}$$

Now using Theorem 2.5 and Lemma 2.2, we have

$$\begin{aligned}W(T(S_n.C_m)) &= \frac{n-1}{8}\left[m^3(2n-3)+2m^2(4n-7)+8m\right]\\
&+ (n-1)\left(\frac{m^3}{8}\right) + {}^{(n-1)}C_2 + (n-1)\left[1\left(2(1)+2(2)+\dots+2\left(\frac{m}{2}\right)\right)\right.\\
&+ (n-2)\left.\left(2(2)+2(3)+\dots+2\left(\frac{m}{2}+1\right)\right)\right] + [2(n-2)+2(n-3)+\dots\\
&+ 2(n-(n-1))]\left[\left(2(3)+2(4)+\dots+2\left(\frac{m}{2}+2\right)\right) + (2(4)+2(5)+\dots\right.\end{aligned}$$

$$
\begin{aligned}
&+ \quad 2\left(\frac{m}{2}+3\right)\Big) + \cdots + \left(2\left(\frac{m}{2}+2\right) + 2\left(\frac{m}{2}+3\right) + \cdots + 2\,(m+1)\right)\Big] \\
&+ \quad (n-1) + (n-1)\left(2(2) + 2(3) + \cdots + 2\left(\frac{m}{2}+1\right)\right) + m(n-1)\,(2(1) \\
&+ \quad 2(2) + \cdots + 2\left(\frac{m}{2}\right)\Big) + (n-1)\,(1 + 2(n-2) + (n-2)\,(2(3) + 2(4) \\
&+ \quad \cdots + 2\left(\frac{m}{2}+2\right)\Big)\Big) + 2(n-1)\,[(2 + 3(n-2) + (n-2)\,(2(4) + 2(5) \\
&+ \quad \cdots + 2\left(\frac{m}{2}+3\right)\Big)\Big) + (3 + 4(n-2) + (n-2)\,(2(5) + 2(6) + \ldots \\
&+ \quad 2\left(\frac{m}{2}+4\right)\Big)\Big) + \cdots + \left(\frac{m}{2} + (n-2)\left(\frac{m}{2}+1\right) + (n-2)\left(2\left(\frac{m}{2}+2\right)\right.\right. \\
&+ \quad 2\left(\frac{m}{2}+3\right) + \cdots + 2\left(\frac{m}{2}+\frac{m}{2}+1\right)\Big)\Big)\Big] + (n-1)\left[\left(\frac{m}{2}+1\right)\right. \\
&+ \quad (n-2)\left(\frac{m}{2}+2\right) + (n-2)\left(2\left(\frac{m}{2}+3\right) + 2\left(\frac{m}{2}+4\right) + \ldots \; + \; 2\left(\frac{m}{2}+\frac{m}{2}+2\right)\right)\Big] \\
&= \quad \frac{n-1}{2}\left[m^3(2n-1) + m^2(10n-17) + m(7n-6) + n\right].
\end{aligned}
$$

Case 2: Suppose n is odd. Now,

$$
\begin{aligned}
\sum_{e,f \in E(S_n.C_m)} d(e,f) \;=\;& {}^{(n-1)}C_2 + (n-1)W(C_m) + (n-1)\left[\left(2(1) + 2(2) + \cdots + 2\left(\frac{m-1}{2}\right)\right.\right. \\
+&\quad \frac{m+1}{2}\Big) + (n-2)\left(2(2) + 2(3) + \cdots + 2\left(\frac{m+1}{2}\right) + \frac{m+3}{2}\right)\Big] \\
+&\quad (2(n-2) + 2(n-3) + \cdots + 2(n-(n-1)))\,[(2(3) + 2(4) + \ldots \\
+&\quad 2\left(\frac{m+3}{2}\right) + \frac{m+5}{2}\Big) + \left(2(4) + 2(5) + \cdots + 2\left(\frac{m+5}{2}\right) + \frac{m+7}{2}\right) \\
+&\quad \cdots + \left(2\left(\frac{m+3}{2}\right) + 2\left(\frac{m+5}{2}\right) + \cdots + 2\,(m) + (m+1)\right)\Big] \\
+&\quad ((n-2) + (n-3) + \cdots + (n-(n-1)))\left(2\left(\frac{m+5}{2}\right) + 2\left(\frac{m+7}{2}\right)\right. \\
+&\quad \cdots + 2\,(m+1) + (m+2)\Big).
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{\substack{u \in V(S_n.C_m),\\ e \in E(S_n.C_m)}} d(u,e) \;=\;& (n-1) + (n-1)\left(2(2) + 2(3) + \cdots + 2\left(\frac{m+1}{2}\right) + \frac{m+3}{2}\right) \\
+&\quad m(n-1)\left(2(1) + 2(2) + \cdots + 2\left(\frac{m-1}{2}\right) + \frac{m+1}{2}\right) + (n-1)\,(1 \\
+&\quad 2(n-2) + (n-2)\left(2(3) + 2(4) + \cdots + 2\left(\frac{m+3}{2}\right) + \frac{m+5}{2}\right)\Big) \\
+&\quad 2(n-1)\left[\left(2 + 3(n-2) + (n-2)\left(2(4) + 2(5) + \cdots + 2\left(\frac{m+5}{2}\right)\right.\right.\right. \\
+&\quad \frac{m+7}{2}\Big)\Big) + \left(3 + 4(n-2) + (n-2)\left(2(5) + 2(6) + \cdots + 2\left(\frac{m+7}{2}\right)\right.\right. \\
+&\quad \frac{m+9}{2}\Big)\Big) + \cdots + \left(\frac{m+1}{2} + (n-2)\left(\frac{m+3}{2}\right) + (n-2)\left(2\left(\frac{m+5}{2}\right)\right.\right. \\
+&\quad 2\left(\frac{m+7}{2}\right) + \cdots + 2\,(m+1) + (m+2)\Big)\Big)\Big].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
W(T(S_n.C_m)) \;=\;& W(S_n.C_m) + {}^{(n-1)}C_2 + (n-1)W(C_m) + (n-1)\left[\left(2(1) + 2(2) + \cdots + 2\left(\frac{m-1}{2}\right)\right.\right. \\
+&\quad \frac{m+1}{2}\Big) + (n-2)\left(2(2) + 2(3) + \cdots + 2\left(\frac{m+1}{2}\right) + \frac{m+3}{2}\right)\Big] + (2(n-2) \\
+&\quad 2(n-3) + \cdots + 2(n-(n-1)))\left[\left(2(3) + 2(4) + \cdots + 2\left(\frac{m+3}{2}\right) + \frac{m+5}{2}\right)\right. \\
+&\quad \left(2(4) + 2(5) + \cdots + 2\left(\frac{m+5}{2}\right) + \frac{m+7}{2}\right) + \cdots + \left(2\left(\frac{m+3}{2}\right) + 2\left(\frac{m+5}{2}\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \dots + 2\,(m) + (m+1))] + ((n-2) + (n-3) + \dots + (n-(n-1)))\left(2\left(\frac{m+5}{2}\right)\right.\\
&+ \quad 2\left(\frac{m+7}{2}\right) + \dots + 2\,(m+1) + (m+2)\Big) + (n-1) + (n-1)\,(2(2) + 2(3) + \dots\\
&+ \quad 2\left(\frac{m+1}{2}\right) + \frac{m+3}{2}\Big) + m(n-1)\left(2(1) + 2(2) + \dots + 2\left(\frac{m-1}{2}\right) + \frac{m+1}{2}\right)\\
&+ \quad (n-1)\left(1 + 2(n-2) + (n-2)\left(2(3) + 2(4) + \dots + 2\left(\frac{m+3}{2}\right) + \frac{m+5}{2}\right)\right)\\
&+ \quad 2(n-1)\left[\left(2 + 3(n-2) + (n-2)\left(2(4) + 2(5) + \dots + 2\left(\frac{m+5}{2}\right) + \frac{m+7}{2}\right)\right)\right.\\
&+ \quad \left(3 + 4(n-2) + (n-2)\left(2(5) + 2(6) + \dots + 2\left(\frac{m+7}{2}\right) + \frac{m+9}{2}\right)\right) + \dots\\
&+ \quad \left(\frac{m+1}{2} + (n-2)\left(\frac{m+3}{2}\right) + (n-2)\left(2\left(\frac{m+5}{2}\right) + 2\left(\frac{m+7}{2}\right) + \dots\right.\right.\\
&+ \quad 2\,(m+1) + (m+2)))]\,.
\end{aligned}
$$

Now using Theorem 2.5 and Lemma 2.2, we have

$$
\begin{aligned}
W(T(S_n.C_m)) &= \frac{n-1}{8}\left[m^3(2n-3) + 2m^2(4n-7) - 2\right] - \frac{m}{8}(2n^2 - 15n + 11)\\
&+ \; {}^{(n-1)}C_2 + (n-1)\left(\frac{m^3-m}{8}\right) + (n-1)\left[\left(2(1) + 2(2) + \dots + 2\left(\frac{m-1}{2}\right)\right.\right.\\
&+ \; \frac{m+1}{2}\Big) + (n-2)\left(2(2) + 2(3) + \dots + 2\left(\frac{m+1}{2}\right) + \frac{m+3}{2}\right)\Big] + (2(n-2)\\
&+ \; 2(n-3) + \dots + 2(n-(n-1)))\left[\left(2(3) + 2(4) + \dots + 2\left(\frac{m+3}{2}\right) + \frac{m+5}{2}\right)\right.\\
&+ \; \left(2(4) + 2(5) + \dots + 2\left(\frac{m+5}{2}\right) + \frac{m+7}{2}\right) + \dots + \left(2\left(\frac{m+3}{2}\right) + 2\left(\frac{m+5}{2}\right)\right.\\
&+ \; \dots + 2\,(m) + (m+1))] + ((n-2) + (n-3) + \dots + (n-(n-1)))\left(2\left(\frac{m+5}{2}\right)\right.\\
&+ \; 2\left(\frac{m+7}{2}\right) + \dots + 2\,(m+1) + (m+2)\Big) + (n-1) + (n-1)\,(2(2) + 2(3) + \dots\\
&+ \; 2\left(\frac{m+1}{2}\right) + \frac{m+3}{2}\Big) + m(n-1)\left(2(1) + 2(2) + \dots + 2\left(\frac{m-1}{2}\right) + \frac{m+1}{2}\right)\\
&+ \; (n-1)\left(1 + 2(n-2) + (n-2)\left(2(3) + 2(4) + \dots + 2\left(\frac{m+3}{2}\right) + \frac{m+5}{2}\right)\right)\\
&+ \; 2(n-1)\left[\left(2 + 3(n-2) + (n-2)\left(2(4) + 2(5) + \dots + 2\left(\frac{m+5}{2}\right) + \frac{m+7}{2}\right)\right)\right.\\
&+ \; \left(3 + 4(n-2) + (n-2)\left(2(5) + 2(6) + \dots + 2\left(\frac{m+7}{2}\right) + \frac{m+9}{2}\right)\right) + \dots\\
&+ \; \left(\frac{m+1}{2} + (n-2)\left(\frac{m+3}{2}\right) + (n-2)\left(2\left(\frac{m+5}{2}\right) + 2\left(\frac{m+7}{2}\right) + \dots\right.\right.\\
&+ \; 2\,(m+1) + (m+2)))]\\
&= \frac{n-1}{8}\left[m^3(8n-13) + m^2(19n-26) + m(128n-223) - (75n-158)\right].
\end{aligned}
$$

Hence,

$$
W(T(S_n.C_m)) = \begin{cases} \frac{n-1}{2}\left[m^3(2n-1) + m^2(10n-17) + m(7n-6) + n\right]; & \text{if } n \text{ is even} \\ \frac{n-1}{8}\left[m^3(8n-13) + m^2(19n-26) + m(128n-223) - (75n-158)\right]; & \text{if } n \text{ is odd.} \end{cases}
$$

□

3. Conclusion

In this article, we have investigated the results related to the Wiener index of the total graphs of two particular graphs, namely, $C_n.S_m$ and $S_n.C_m$.

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