Unique Fixed Point Theorems for Non-Self Contraction Mapping in Banach Space

Research Article

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Abstract: In this paper we prove a unique fixed point theorem for weakly inward condition for non-self contraction mapping of nonempty convex subset of Banach Space.

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1.   Introduction

We know that the Banach contraction principle was extended by Nadler for multi-valued contraction mappings in complete metric space [9]. In 1965 Halpern study some fixed points of non-self mapping which was a generalization of schauder-Tychonof theorem. Kirk in 1971 worked on Non-self non-expansive mapping over S, where S is a convex subset of vector space, this work is nothing but the extension of his previous result for self-mapping done by him in 1971 with some boundary conditions. Pseudo-contractive Non-self mapping over a normed linear space was derived by kirk and Assad in 1972 satisfying some conditions called as opial's condition. In 1974 the mathematician Reich [10] had shown that for pseudo-contractive only inwardness conditions are to be used for Non-self mappings. Many researchers have been working with different types of boundary conditions and inwardness conditions involving Non-self mappings. Also in 1967 Petryshyn and Browder have given different type of fixed point theorems of Non-self mapping also they invented a easy process to obtain a fixed point by using approximation mapping. Kannan also studied fixed point property for a class of self-mappings which are not necessarily fixed point for Non-self mappings, recently H. K. Xu [12] has extended the kirk and Massa theorem to Non-self mapping which satisfying an inwardness condition. The purpose of this paper is nothing but by taking weakly inward condition of contraction mapping we have to show that the fixed point will be exist and our main result is the extension of Banach contraction principle for Non-self contraction mapping.

2.   Preliminaries and Hypothesis

We recall some useful definitions and properties are given below

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In a Banach Space X, let S be a nonempty closed subset denoted by $SB(S)$ [6], the family of all nonempty closed bounded subset of S and $f(S)$ be the family of all nonempty compact subset of S. The inwardness condition for Non-self mappings used to prove some theorem to show the fixed point is as given below. Let X be a Banach Space and S be a nonempty subset of it then for any t of S and $P : S \to X$ be a mapping then P is said to be inward mapping such that $P(t) \in I_S(t)$ for all $t \in S$, P is said weakly inward mapping [10]. Where $I_S(t) = \{t + q(u - t); u \in S \ and \ q \geq 0\}$.

Definition 2.1 (Non-expansive Mapping [8])**.** *A mapping $P : S \to X$ is said to be non-expansive if it satisfies the inequality* $\|k - l\| \geq \|P(k) - P(l)\|$ *for all k, l belongs to S.*

Example 2.2. *In Normed space X, let S be a nonempty subset then each pair k, l in X we have $\|k - l\| \geq |d(k, s) - d(l, s)|$. Hence we say that the function is non-expansive.*

The comparison between inwardness and weakly inwardness condition with other conditions set

(1). Rothe's Condition: $P(\mu) \subseteq S$.

(2). Inwardness Condition: $P_k \in I_s(k) \ \forall \ k \in S$.

(3). Weakly Inwardness Condition: $P_k \in I_s(k) \ \forall \ k \in S$.

(4). Leray-Schauder condition: i.e if the interior of S is nonempty there exist a point m in $int(s)$ such that $P_k - m \neq \partial(k - m) \ \forall \ k \in \mu$ and $\partial > 1$.

Theorem 2.3. *Let S be a closed convex subset of a Banach Space X and a mapping $P : S \to K(t)$ is contraction mapping if $P(t)$ is subset of weakly inward mapping $I_s(t)$ for all t in S, then P has a fixed Point.*

Lemma 2.4. *Let X be a Banach Space and S be a nonempty closed convex subset of X, then will be weakly inward mapping if and only if*

$$\lim_{a \to a^+} \frac{1}{q} d[(1 - q)k + qP_k, s] = 0 \ \ forall \ \ k \in S \tag{1}$$

Proof. Let us assume that the given condition has satisfies. Consider $k \in S$ for any $\gamma > 0$ assume that $t \in (0, 1)$ and $l \in S$ such that following inequality holds $\|(1 - t)k + tPk - l\| \leq d[(1 - t)k + tPk, s)] + t\gamma$. Then $\|Pk - [(1 - t^{-1})k + t^{-1}l\| \leq t^{-1}d[(1 - t)k + tPk, s)] + \gamma \Rightarrow Pk \in I_S(k)$ such that $\|l - Pk\| \leq \gamma$.

Conversely, Assume that P is weakly inward mapping which implies that $Pk \in I_S(k)$ for all k belongs to S. For $\varepsilon > 0$ there exist $y \in I_S(k)$ such that $\|l - Pk\| \leq \gamma$. Hence S is convex there exist $q_0 > 0$ such that $(1 - q)k + qPk \in S$ for $0 < q \leq q_0$, therefore

$$\frac{1}{q} d[(1 - q)k + qPk, S] \leq \frac{1}{q} \|(1 - q)k + qPk - [(1 - q)k + ql\| \leq \gamma$$

Satisfies the given condition. □

3.   The Main Result

Our main result is the extension of Banach Contraction Principle for Non-self contraction mapping in which the weakly inward mapping is used and we are in the position to apply the above lemma.

Theorem 3.1. *Let X is a Banach Space and S is a nonempty closed convex subset of a Banach Space and a mapping $P : S \to X$ is a weakly inward contraction mapping, then it has a unique fixed point in S.*

Proof. Let r be the Lipschit'z constant in S and consider $\gamma > 0$ such that $(1 \pm \gamma) > r$. By Lemma 2.4. We say P satisfies the above condition in 2.1 then for k in S with $k \neq Pk$ there exist q in $(0, 1)$ such that, $\|k - Pk\| q\gamma > d[(1 - q)k + qPk, s)]$. Let $l \in S$ then by definition of distance

$$\|k - Pk\| q\gamma > \|(1 - q)k + qPk - l)\| \tag{2}$$

$$\|k - Pk\| q\gamma > \|k - l - q(k - Pk)\|$$

$$\geq \|k - l\| - q \|(k - Pk)\|$$

Hence,

$$\|k - l\| < \|k - Pk\| q(1 + \gamma) \tag{3}$$

Then by using the above inequality (2), we have

$$\|l - Pl\| \leq \|l - [(1 - q)k + qPk]\| + \|(1 - q)k + qPk - Pk\| + \|Pk - Pl\|$$

$$\leq qr \|k - pk\| + (1 - q) \|k - pk\| + r \|k - l\|$$

$$= \|k - pk\| + q(\gamma - 1) \|k - Pk\| + r \|k - l\|$$

$$= \|k - pk\| + q(\gamma - 1) \|k - Pk\| + \frac{1 - \gamma}{1 + \gamma} \|k - l\| - \left[\frac{1 - \gamma}{1 + \gamma} - r\right] \|k - l\|$$

$$< \|k - pk\| - \left[\frac{1 - \gamma}{1 + \gamma} - r\right] \|k - l\| \quad \text{Since by (3)}$$

If $k, l \in S$ and $k \neq Pk$ and $f(k)$ where f is a self-mapping on S.

$$\beta(k) = \left[\frac{1 - \gamma}{1 + \gamma} - r\right] \|k - Pk\| \tag{4}$$

Where $\beta : S \to \mathrm{R}^+$ is a continuous function and

$$\beta(k) - \beta(fk) > \|k - fk\| \tag{5}$$

Then by Caristi's Theorem [6] f has a fixed point. This is a contradiction of our assumption. □

4. Conclusion

In this paper we have presented a unique fixed point theorem which is the extension of Banach Contraction mapping by using weakly inwardness mapping (1) given in Lemma 2.4.

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