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Minimizing Property of Two Variable Cubic Spline Interpolation

Research Article

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Abstract: In this paper the minimizing property of two variable natural cubic spline interpolation is derived.

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1. Introduction

Interpolation is the method of finding the value of y for an argument say 'x' from a set of tabular values of y without knowing the expression of $y = f(x)$. There are different types of interpolation techniques. One method is to find an approximate function corresponding to the given function. If the approximated function is a polynomial then the interpolation is known as polynomial interpolation of the different types of polynomial interpolation, a cubic spline interpolation is considered here. Cubic spline interpolation is most widely used due to their smoothness condition. Spline satisfying the conditions $S''(a) = S''(b) = 0$ is called the natural spline. A two variable cubic spline interpolation of a function $z = f(x, y)$ is the fitting of a unique series of cubic splines for a given set of data points (x_i, y_j, z_{ij}). The points (x, y) at which $f(x, y)$ are known lie on a grid in the $x - y$ plane. Two variable spline satisfying the conditions $\frac{\partial^2 S}{\partial x^2}(x_n, y_m) = 0$ and $\frac{\partial^2 S}{\partial x^2}(x_1, y_1) = 0$ is called two variable cubic spline interpolation. In order to derive a two variable natural cubic spline the existence of continuity condition of the spline function and its partial derivatives at the edge of each grid are assumed. [1, 3, 4]. Natural two variable cubic spline interpolation formula [6] is given by

$$\begin{aligned}
S_{ij} = &\frac{M_i(x_{i+1}-x)^3}{6h_i} + \frac{M_{i+1}(x-x_i)^3}{6h_i} + \frac{N_j(y_{j+1}-y)(x-x_i)^2}{2k_j} - \frac{N_{j+1}(y-y_j)(x_{i+1}-x)^2}{2k_j} \\
&+ \left\{\frac{(z_{i+1,j}-z_{ij})}{h_ik_j} - \frac{(z_{i+1,j+1}-z_{i,j+1})}{h_ik_j} + \frac{(N_{j+1}-N_j)h_i}{2k_j}\right\}(y_{j+1}-y)(x-x_i) \\
&+ \left\{\frac{(M_{i+1}-M_i)h_i}{6} + \frac{N_{j+1}h_i}{2} - \frac{(z_{i+1,j+1}-z_{i,j+1})}{h_i}\right\}(x_{i+1}-x) \\
&+ \left\{\frac{(z_{i,j+1}-z_{ij})}{k_j} + \frac{N_{j+1}{h_i}^2}{2k_j}\right\}(y-y_j) + \left\{(z_{i+1,j+1}-z_{i,j+1}) + z_{ij} - \left(\frac{M_{i+1}}{6} + \frac{N_{j+1}}{2}\right){h_i}^2\right\},
\end{aligned}$$

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$\forall\ (x,y)\in[x_i,x_{i+1}]\times[y_j,y_{j+1}]$, $\forall\ i=1,2,\ldots,(n-1)$, $\forall\ j=1,2,\ldots,(m-1)$, where $2M_{i+1}+3N_j+\mu_iM_i+\lambda_iM_{i+2}=d_{ij}$, $i=1,2,\ldots,(n-2)$ and $j=1,2,\ldots,(m-2)$. For two variable natural spline $M_1=M_n=N_1=N_m=0$. Before establishing the minimizing property of two variable natural cubic spline, the minimizing property of one variable natural cubic spline is discussed below.

2. Main Result

Theorem 2.1 (Minimizing property of one variable natural cubic spline). *Let $I_l=[a,b]$ be an interval with $a=x_1<x_2<\cdots<x_n=b$. Let $z^{''}(x)$ be continuous in I_l and if $S(x)$ is the natural cubic spline interpolating $z(x)$ at the knots then [2, 5],*

$$\int_a^b\left[z^{''}(x)\right]^2dx\geq\int_a^b\left[S^{''}(x)\right]^2dx$$

Proof. Given $a=x_1<x_2<\cdots<x_n=b$. Let $z(x_i)=y_i\ \forall\ i$ and $z(x)$ $z^{'}(x)$ $z^{''}(x)$ are continuous in $I_l=[a,b]$.

$$\begin{aligned}\int_a^b\left[z^{''}(x)\right]^2dx&=\int_a^b\left[S^{''}(x)+z^{''}(x)-S^{''}(x)\right]^2dx\\&=\int_a^b\left[S^{''}(x)\right]^2dx+2\int_a^bS^{''}(x)\left[z^{''}(x)-S^{''}(x)\right]dx+\int_a^b\left[z^{''}(x)-S^{''}(x)\right]^2dx\end{aligned}\tag{1}$$

Consider

$$\begin{aligned}\int_a^bS^{''}(x)\left[z^{''}(x)-S^{''}(x)\right]dx&=\sum_{i=1}^{n-1}\int_{x_i}^{x_{i+1}}S^{''}(x)\left[z^{''}(x)-S^{''}(x)\right]dx\\&=\sum_{i=1}^{n-1}S^{''}(x)\left(\left[z^{'}(x)-S^{'}(x)\right]\right)_{x_i}^{x_{i+1}}-\sum_{i=1}^{n-1}\int_{x_i}^{x_{i+1}}S^{'''}(x)\left[z^{'}(x)-S^{'}(x)\right]dx\end{aligned}\tag{2}$$

Consider the first term of equation (2)

$$S^{''}(x_n)\left[z^{'}(x_n)-S^{'}(x_n)\right]-S^{''}(x_0)\left[z^{'}(x_0)-S^{'}(x_0)\right]=0,$$

since $S^{''}(x_n)=S^{''}(x_0)=0$. Hence (2) reduces to

$$\begin{aligned}\int_a^bS^{''}(x)\left[z^{''}(x)-S^{''}(x)\right]dx&=-\sum_{i=1}^{n-1}\int_{x_i}^{x_{i+1}}S^{'''}(x)\left[z^{'}(x)-S^{'}(x)\right]dx\\&=-\sum_{i=1}^{n-1}S^{'''}(x)\int_{x_i}^{x_{i+1}}\left[z^{'}(x)-S^{'}(x)\right]dx&&(\text{Since } S^{'''}\text{ is a constant})\\&=-\sum_{i=1}^{n-1}S^{'''}(x)\left[z(x)-S(x)\right]_{x_i}^{x_{i+1}}\\&=0&&(\text{Since } z(x)-S(x)=0,\ \forall\, i=1,2,\ldots,n)\end{aligned}$$

Hence (1) reduces to

$$\begin{aligned}\int_a^b\left[z^{''}(x)\right]^2dx&=\int_a^b\left[S^{''}(x)\right]^2dx+\int_a^b\left[z^{''}(x)-S^{''}(x)\right]^2dx\\\int_a^b\left[z^{''}(x)\right]^2dx&\geq\int_a^b\left[S^{''}(x)\right]^2dx\end{aligned}$$

Hence $\int_a^b[z^{''}(x)]^2dx$ will be minimum iff $\int_a^b[z^{''}(x)-S^{''}(x)]^2dx=0$. Hence $z^{''}(x)=S^{''}(x)$. Hence $z(x)-S(x)$ is a polynomial in $[a,b]$ with degree at most three. For $i=1,2,\ldots,n$, the difference $z(x)-S(x)$ vanishes. Hence $z(x)=S(x)$, $a\leq x\leq b$. □

Theorem 2.2 (Minimizing property of two variable natural cubic spline). *Let $I_g = [a,b] \times [c,d]$ be the rectangular grid with $a = x_1 < x_2 < \cdots < x_n = b$ and $c = y_1 < y_2 < \cdots < y_m = d$. Let $\frac{\partial^2 u}{\partial x^2}(x,y)$ be continuous in I_g and if $S(x,y)$ is the two variable natural cubic spline interpolating $u(x,y)$ at the knots then,*

$$\int_a^b \left[\frac{\partial^2 u}{\partial x^2}(x_i,y_j)\right]^2 dx \geq \int_a^b \left[\frac{\partial^2 S}{\partial x^2}(x_i,y_j)\right]^2 dx$$

Proof. Let $S_{ij}(x,y)$ be the two variable cubic spline. Given $a = x_1 < x_2 < \cdots < x_n = b$ and $c = y_1 < y_2 < \cdots < y_m = d$. We have $S_{ij}(x_i,y_j) = z_{ij}$. Assume $u(x,y)$ be function such that $u(x_i,y_j) = z_{ij}\ \forall\ i,j$ and $u(x_i,y_j)$, $\frac{\partial u}{\partial x}(x_i,y_j)$, $\frac{\partial^2 u}{\partial x^2}(x_i,y_j)$ are continuous in I_g. Then

$$\begin{aligned}\int_a^b \left[\frac{\partial^2 u}{\partial x^2}(x_i,y_j)\right]^2 dx &= \int_a^b \left[\frac{\partial^2 S}{\partial x^2}(x_i,y_j) + \frac{\partial^2 u}{\partial x^2}(x_i,y_j) - \frac{\partial^2 S}{\partial x^2}(x_i,y_j)\right]^2 dx \\ &= \int_a^b \left[\frac{\partial^2 S}{\partial x^2}(x_i,y_j)\right]^2 dx + 2\int_a^b \frac{\partial^2 S}{\partial x^2}(x_i,y_j)\left[\frac{\partial^2 u}{\partial x^2}(x_i,y_j) - \frac{\partial^2 S}{\partial x^2}(x_i,y_j)\right] dx \\ &+ \int_a^b \left[\frac{\partial^2 u}{\partial x^2}(x_i,y_j) - \frac{\partial^2 S}{\partial x^2}(x_i,y_j)\right]^2 dx \qquad (3)\end{aligned}$$

Consider

$$\begin{aligned}\int_a^b \frac{\partial^2 S}{\partial x^2}(x_i,y_j)\left[\frac{\partial^2 u}{\partial x^2}(x_i,y_j) - \frac{\partial^2 S}{\partial x^2}(x_i,y_j)\right] dx &= \sum_{i=1}^{n-1}\sum_{j=1}^{m}\int_{x_i}^{x_{i+1}} \frac{\partial^2 S}{\partial x^2}(x_i,y_j)\left[\frac{\partial^2 u}{\partial x^2}(x_i,y_j) - \frac{\partial^2 S}{\partial x^2}(x_i,y_j)\right] dx \\ &= \sum_{i=1}^{n-1}\sum_{j=1}^{m}\left[\begin{array}{l}\left\{\frac{\partial^2 S}{\partial x^2}(x_i,y_j)\left(\frac{\partial u}{\partial x}(x_i,y_j) - \frac{\partial S}{\partial x}(x_i,y_j)\right)\right\}_{x_i}^{x_{i+1}} - \\ \int\limits_{x_i}^{x_{i+1}} \frac{\partial^3 S}{\partial x^3}(x_i,y_j)\left(\frac{\partial u}{\partial x}(x_i,y_j) - \frac{\partial S}{\partial x}(x_i,y_j)\right) dx\end{array}\right] \qquad (4)\end{aligned}$$

The first term of (4)

$$\begin{aligned}&= \sum_{i=1}^{n-1}\sum_{j=1}^{m} \frac{\partial^2 S}{\partial x^2}(x_{i+1},y_j)\left[\frac{\partial u}{\partial x}(x_{i+1},y_j) - \frac{\partial S}{\partial x}(x_{i+1},y_j)\right] - \sum_{i=1}^{n-1}\sum_{j=1}^{m} \frac{\partial^2 S}{\partial x^2}(x_i,y_j)\left[\frac{\partial u}{\partial x}(x_i,y_j) - \frac{\partial S}{\partial x}(x_i,y_j)\right] \\ &= \frac{\partial^2 S}{\partial x^2}(x_n,y_m)\left[\frac{\partial u}{\partial x}(x_n,y_m) - \frac{\partial S}{\partial x}(x_n,y_m)\right] - \frac{\partial^2 S}{\partial x^2}(x_1,y_1)\left[\frac{\partial u}{\partial x}(x_1,y_1) - \frac{\partial S}{\partial x}(x_1,y_1)\right] \\ &= 0 \quad \left(\text{Since } \frac{\partial^2 S}{\partial x^2}(x_n,y_m) = 0 \text{ and } \frac{\partial^2 S}{\partial x^2}(x_1,y_1) = 0\right)\end{aligned}$$

Hence (4) reduces to

$$\begin{aligned}\int_a^b \frac{\partial^2 S}{\partial x^2}(x_i,y_j)\left[\frac{\partial^2 u}{\partial x^2}(x_i,y_j) - \frac{\partial^2 S}{\partial x^2}(x_i,y_j)\right] dx &= -\sum_{i=1}^{n-1}\sum_{j=1}^{m}\int_{x_i}^{x_{i+1}} \frac{\partial^3 S}{\partial x^3}(x_i,y_j)\left(\frac{\partial u}{\partial x}(x_i,y_j) - \frac{\partial S}{\partial x}(x_i,y_j)\right) dx \\ &= -\sum_{i=1}^{n-1}\sum_{j=1}^{m}\frac{\partial^3 S}{\partial x^3}(x_i,y_j)\int_{x_i}^{x_{i+1}} \left(\frac{\partial u}{\partial x}(x_i,y_j) - \frac{\partial S}{\partial x}(x_i,y_j)\right) dx \\ &= -\sum_{i=1}^{n-1}\sum_{j=1}^{m}\frac{\partial^3 S}{\partial x^3}(x_i,y_j)\left[u(x_i,y_j) - S(x_i,y_j)\right]_{x_i}^{x_{i+1}} \\ &= 0\end{aligned}$$

Thus the middle term of (3) is zero and so

$$\begin{aligned}&\int_a^b \left[\frac{\partial^2 u}{\partial x^2}(x_i,y_j)\right]^2 dx = \int_a^b \left[\frac{\partial^2 S}{\partial x^2}(x_i,y_j)\right]^2 dx + \int_a^b \left[\frac{\partial^2 u}{\partial x^2}(x_i,y_j) - \frac{\partial^2 S}{\partial x^2}(x_i,y_j)\right]^2 dx \\ &\int_a^b \left[\frac{\partial^2 u}{\partial x^2}(x_i,y_j)\right]^2 dx \geq \int_a^b \left[\frac{\partial^2 S}{\partial x^2}(x_i,y_j)\right]^2 dx \quad \left(\text{Since } \int_a^b \left[\frac{\partial^2 u}{\partial x^2}(x_i,y_j) - \frac{\partial^2 S}{\partial x^2}(x_i,y_j)\right]^2 dx = 0\right)\end{aligned}$$

Hence $\int_a^b [\frac{\partial^2 u}{\partial x^2}(x_i,y_j)]^2 dx$ will be minimum iff $\int_a^b [\frac{\partial^2 u}{\partial x^2}(x_i,y_j) - \frac{\partial^2 S}{\partial x^2}(x_i,y_j)]^2 dx = 0$ i.e., $\frac{\partial^2 u}{\partial x^2}(x_i,y_j) = \frac{\partial^2 S}{\partial x^2}(x_i,y_j)$. Hence $u(x,y) - S(x,y)$ is a polynomial in x and y of degree atmost 3 in $[a,b] \times [c,d]$. But the difference $u(x,y) - S(x,y)$ vanishes at the points $i = 1,2,\ldots,n$ and $j = 1,2,\ldots,m$. Hence $u(x,y) = S(x,y)$, $a \leq x \leq b$, $c \leq y \leq d$. □

3. Conclusion

As in the case of one variable natural cubic spline, minimizing property exists for two variable natural cubic splines under the smoothness conditions for natural two variable cubic spline interpolation.

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