International Journal of Mathematics And its Applications
Volume 5, Issue 3–A (2017), 53–55.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Graceful Labeling in the Context of Duplication of Some Graph Elements in $K_{m,n}$

Research Article

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Abstract: In this paper, we obtained graceful labeling or α-labeling for some graphs obtained by duplication of some graph elements in the complete bipartite graph $K_{m,n}$.

MSC: 05C78.

Keywords: Duplication of a vertex by an edge, duplication of an edge by a vertex, graceful labeling, α-labeling.

1. Introduction

Graphs, considered in this paper, are finite, simple and undirected. Let G be a (p, q) graph. Throughout this paper, $K_{m,n}$ will be denoted as complete bipartite graph with m part $M = \{u_1, u_2, \dots, u_m\}$ and n part $N = \{v_1, v_2, \dots, v_n\}$. i.e. $V(K_{m,n}) = M \cup N$. For a graph $G = (V, E)$, a function having domain V or E or $V \cup E$ is known as a graph labeling for G. Graceful labeling (β-valuation) and α-labeling for a graph G are well known concept introduced by Rosa [5]. A graph G which admits an α-labeling, here we call it α-graceful graph. Duplication of a vertex v of a graph G is the graph G' by adding a new vertex v' (duplicant of v) such that $N_{G'}(v') = N_G(v) = N_{G'}(v)$. i.e. v' is adjacent with all vertices of G which are adjacent to v in G. Duplication of a vertex v by a new edge $e = v'v''$ in a graph G produces a new graph $G' = (V(G) \cup \{v', v''\}, E(G) \cup \{v'v'', vv', vv''\})$. Duplication of an edge $e = uv$ by a new vertex w in a graph G produces a new graph $G' = (V(G) \cup \{w\}, E(G) \cup \{uw, vw\})$. i.e. $N_{G'}(w) = \{u, v\}$. Kaneria and Jariya [3] defined smooth graceful labeling and proved $C_n (n \equiv 0 (mod\ 4)), P_n, P_n \times P_m$ and $K_{2,n}$ are smooth graceful graphs. Prajapati and Suthar [4] proved duplication of some graph elements in $K_{2,n}$ are prime graphs. In present work, we obtained graceful labeling or α-graceful labeling for some graphs obtained by duplication of some graph elements in $K_{m,n}$.

2. Main Results

Theorem 2.1. *Duplication of any vertex in $K_{m,n}$ is α-graceful graph.*

Theorem 2.2. *Let G be a graph obtained by duplication of one vertex of $K_{m,n}$. It is obvious that G is either $K_{m+1,n}$ or $K_{m,n+1}$, which both are having α-labeling. So, G is α-graceful graph.*

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Theorem 2.3. *Duplication of all the vertices of m-part or n-part in $K_{m,n}$ is α-graceful graph.*

Proof. Without loss of generality, we assume here G is a graph obtained by duplication of all the vertices of $M = \{u_1, u_2, \ldots, u_m\}$ in $K_{m,n}$. Then $G = K_{2m,n}$, which is an α-graceful graph. □

Theorem 2.4. *Duplication of any vertex v of $K_{m,n}$ by an edge $e = v'v''$ is graceful, but not α-graceful.*

Proof. Without loss of generality, we assume that G is a graph obtained by duplication of the vertex u_1 of $K_{m,n}$ by an edge $u_1'u_1''$. i.e., $V(G) = \{u_1', u_1'', u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n\}$ and $E(G) = \{u_iv_j/1 \leq i \leq m, 1 \leq j \leq n\} \cup \{u_1u_1', u_1u_1'', u_1'u_1''\}$. i.e. $p = |V(G)| = m + n + 2$ and $q = mn + 3$. Define $f : V(G) \to \{0, 1, 2, \ldots, q\}$ as follows:

$$f(u_1') = q, \quad f(u_1'') = q - 1 \ \ and$$
$$f(u_i) = i - 1, \ \ \forall \ \ i = 1, 2, \ldots, m;$$
$$f(v_j) = q - 2 - mj + m, \ \ \forall \ \ j = 1, 2, \ldots, n.$$

Since, $f(u_i) \in \{0, 1, \ldots, m - 1\}$ and $f(v_j) \in \{m + 1, m + 2, \ldots, q - 2\}$, f is an injective map. Further $f^*(u_1u_1') = q$, $f^*(u_1u_1'') = q - 1$, $(u_1'u_1'') = 1$ and $f^*(u_iv_j) = m(n - j + 1) - i + 2, \forall \ \ i = 1, 2, ..., m; \forall \ \ j = 1, 2, ..., n$. i.e. $\{f^*(u_iv_j)/1 \leq i \leq m, 1 \leq j \leq n\} = \{2, 3, ..., mn + 1\} = \{2, 3, ..., q - 2\}$ as $q = mn + 3$. Therefore $f^* : E(G) \to \{1, 2, \ldots, q\}$ defined by $f^*(uv) = |f(u) - f(v)|, \ \ \forall \ \ uv \in E(G)$ is a bijection. Thus, G admits above graceful labeling. Since, G is not a bipartite graph as it contains triangle $u_1u_1', u_1'u_1'', u_1''u_1$, it can not admit any α-graceful labeling. Thus, G is graceful, but not α-graceful. □

Theorem 2.5. *Duplication of any edge e of $K_{m,n}$ by a new vertex w is graceful, but not α- graceful.*

Proof. Without loss of generality, we assume that G is a graph obtained by duplication of the edge $u_1v_1 \in E(K_{m,n})$ by a vertex w. i.e. $V(G) = V(K_{m,n}) \cup \{w\}$ and $E(G) = E(K_{m,n}) \cup \{u_1w, wv_1\}$. Observe that $p = m + n + 1$ and $q = mn + 2$. Define $f : V(G) \to \{0, 1, 2, \ldots, q\}$ as follows :

$$f(w) = q,$$
$$f(u_i) = i - 1, \ \ \forall \ \ i = 1, 2, ..., m \ \ and$$
$$f(v_j) = q - 1 - mj + m, \ \ \forall \ \ j = 1, 2, ..., n.$$

Note that f is an injective function and $f^*(wu_1) = q, f^*(wv_1) = 1$ and

$$\begin{aligned} f^*(u_iv_j) &= q - jm + m - i \\ &= m(n - j + 1) - i + 2, \ \ \forall \ \ i = 1, 2, ..., m; \ \ \forall \ \ j = 1, 2, ..., n. \end{aligned}$$

i.e. $\{f^*(u_iv_j)/1 \leq i \leq m, 1 \leq j \leq n\} = \{2, 3, ..., mn + 1\}$. Therefore, $f^* : E(G) \to \{1, 2, ..., mn + 2\}$ is a bijection. Thus, G admits graceful labeling. Since, G contains triangle u_1w, wv_1, u_1v_1, it is not a bipartite graph. So, it can not admit any α-graceful labeling. Thus, G is graceful graph, but it is not α-graceful graph. □

Theorem 2.6. *The graph obtained by duplication of both the vertices u_1, u_2 from 2-part in $K_{2,n}$ by edges is graceful, but it is not α-graceful, where n is an odd integer.*

Proof. Let G be a graph obtained by duplication of both the vertices u_1, u_2 from 2-part in $K_{2,n}$ (where n is odd) by edges $e_1 = u_1'u_1'', e_2 = u_2'u_2''$. It is obvious that

$$V(G) = V(K_{2,n}) \cup \{u_1', u_1'', u_2', u_2''\},$$
$$E(G) = E(K_{2,n}) \cup \{u_1u_1', u_1'u_1'', u_1''u_1, u_2u_2', u_2'u_2'', u_2''u_2\}$$

i.e. $p = n + 6$ and $q = 2n + 6$. Define $f : V(G) \to \{0, 1, 2, ..., q\}$ as follows :

$$f(u_1') = q,\ f(u_1'') = q - 1,\ f(u_1) = 0,\ f(u_2) = 2,\ f(u_2') = 4,\ f(u_2'') = 7\ \textit{and}$$
$$f(v_i) = \begin{cases} 2 + 4\lceil \frac{i}{2} \rceil, & \text{when i is an odd number} \\ 5 + 2i, & \text{when i is an even number.} \end{cases}$$

Note that $\{f(v_i)/1 \leq i \leq n\} = \{6, 9, 10, 13, 14, ..., q - 3, q - 2\}$ and so, f is an injective function. Moreover $f^*(u_1u_1') = q$, $f^*(u_1'u_1'') = 1$, $f^*(u_1''u_1) = q - 1$, $f^*(u_2u_2') = 2$, $f^*(u_2'u_2'') = 3$, $f^*(u_2''u_2) = 5$ and $\{f^*(u_iv_j/1 \leq i \leq 2, 1 \leq j \leq n\} = \{4, 6, 7, 8, ..., q - 2\}$. Therefore, $f^* : E(G) \to \{1, 2, ..., 2n + 6\}$ is a bijective function and so, it becomes a graceful labeling for G. Since, G is not a bipartite graph, it is graceful graph, but it is not an α-graceful graph. □

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