



Bio-Mathematical Model on Competition between Two Species

Research Article

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Abstract: In this paper we shall formulate a mathematical model to study the competition between two species in a given region and at a given interval of time. We shall discuss the nature of competition on the basis of their ability to consume resources in a given period of time that directly depends on population size of a species. Further we shall study and investigate the nature of competition between two species in a long run *i.e.*, when time tends to infinity and to do that we have introduced competition parameter to predict competition and coexistence between both the species when time tends to infinity together with new terminologies like asymptotically superior and asymptotically inferior to define the nature of competition at infinity.

Keywords: Mathematical model, Competition, Superiority Factor, Competition parameter, Competitively superior, Competitively inferior, Competitively analogous, Asymptotically superior, Asymptotically inferior.

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1. Introduction

In an ecological system members of same or different species interact with each other. These interactions depend upon climatic conditions, amount of resources present in the given region, population density of both the species, ability to consume resources and reproduce. These biological interactions are broadly classified into the following categories *viz.* *predation, competition, mutualism, commensalism* and *parasitism*. A biological interaction may have positive, negative or no impact on interacting species. Competition is a biological interaction which has a negative impact on both the species. In competition, two or more different species compete for the same resource in a given geographical region. Many research work have been done in this area to study competition between both related and unrelated species some of the work we can find in [1], [2], [3]. In this paper, we shall consider two different animal species competing for same resources in the geographical area under consideration. We shall study and investigate competition on the basis of relative efficiency of consuming resources and reproducing to leave more progeny by a particular species over the other, further immigration and emigration of members to and from the considered region respectively are also acknowledged.

In this paper, we shall formulate a mathematical model to discuss competition between two species. The main objective of this model is to study the nature of species in the competition. Further we design two different test models to discuss our abstract formulated model. The importance of this model is that with the known constants we can easily predict the nature of species in competition. Moreover, we can predict the asymptotic nature of a species, *i.e.*, the nature of species as time tends to infinity which may be helpful to decide the existence of a species in a long run. In this mathematical model we are

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neglecting intra-specific competition and existence of any other species competing for the same resources other than those two species considered for the study that clearly indicates the scope of extension of this mathematical model where we can remove these assumptions and possibly include them in our future studies. Further we can use the solution of dynamical system defined in [5] to discuss the competition parameter although in that case we have to reduce the factors involved to represent intra-specific competition.

1.1. Nature of Species in Competition

In competition, participating species can be classified into three categories on the basis of their ability to consume resources and reproductive ability (for details see [4]);

- (1). **Competitively Superior Species:** A species in competition is said to be competitively superior if it can consume resources more efficiently than other species competing for the same resource and can leave more progeny in the region in a given interval of time.
- (2). **Competitively Inferior Species:** A species in competition is said to be competitively inferior species if it can consume resource least efficiently than other species competing for the same resource and is reproductively inferior.
- (3). **Competitively Analogous Species:** Two species are said to be competitively analogous if both of them can consume resource and reproduce with comparable efficiency.

1.2. Assumptions

Let us assume the following;

- (1). There are only two species competing for the same resource. And if there is any other species competing for the same resource it should be neglected.
- (2). At a given period of competition, resources are enough so as to avoid intra-specific competition.
- (3). There will be no resource and other ecological adjustments partitioning between the species.
- (4). Every individual in a given species can consume resource with equal efficiency.
- (5). At every instant of time total consumption of resources must not exceed to total resource.
- (6). Competition is based on efficiency of individuals of a species to consume resources and their population density with respect to the other species.
- (7). There is no affect in competition due to climatic change.

2. Formulation of Mathematical Model on Competition

2.1. Mathematical Parameters

We shall define some mathematical parameters with which we can describe the situation of competition between two species in a given region R and in a given period of time.

Notation 2.1. $T =$ period of competition

$|R| =$ total amount of resources available in the region in the time interval $[0, T]$

i_P = number of individual of species P enters the region in unit time

i_Q = number of individual of species Q enters the region in unit time

e_P = number of individual of species P migrates from the region in unit time

e_Q = number of individual of species Q migrates from the region in unit time

B_P = number of individual of species P born in unit time

B_Q = number of individual of species Q born in unit time

D_P = number of individual of species P died in unit time

D_Q = number of individual of species Q died in unit time

N_1 = initial population of species P at $t=0$

N_2 = initial population of species Q at $t=0$

$x(t)$ = population density of species P at time t

$y(t)$ = population density of species Q at time t

Now we shall define the following mathematical parameters with which we could describe the competition between two species.

Natural Growth factor(NG): The change in population of a given species in a given region per unit time due to natural breeding and death is defined as Natural Growth factor. Mathematically NG is defined as follows; $A(t) = B(t) - D(t)$, where $B(t)$ and $D(t)$ are Birth Function and Death Function respectively.

λ -Factor: The change in population size due to immigration and emigration is said to be λ -factor and is defined as $\lambda(t) = i(t) - e(t)$ which is a function of time, where $i(t)$ and $e(t)$ are function of time representing immigration and emigration.

Total Population (TP): Total number of population of a species participated in competition is said to be total population TP and is defined as $TP(T) = \text{initial population} + \int_0^T [A(t) + \lambda(t)]dt$, where $\lambda(t) = i(t) - e(t)$ and $A(t) = B(t) - D(t)$,

Consumption Factor: The amount of resources consumed by an individual of a given species in unit time is denoted by μ .

Total Consumption(TC): The amount of resources consumed by total population of a given species in a given interval of time. It is defined by, $TC = T\mu TP(T)$, and T is time upto which resource has been consumed.

2.2. Mathematical Model on Competition

Let $x(0)$ and $y(0)$ be the population density of two species P and Q respectively at time $t = 0$. The growth in population due to NG is defined by the function $A_k(t) = [B_k(t) - D_k(t)]$ for $k = P, Q$ at time t . The change in population due to immigration and emigration is defined by $\lambda_k(t) = [i_k(t) - e_k(t)]$ for $k = P, Q$. The Population density of both the species at time t is defined as follows;

$$x(t) = x(0) + A_P(t) + \lambda_P(t)$$

$$y(t) = y(0) + A_Q(t) + \lambda_Q(t)$$

The functions of t representing birth rate B , death rate D , immigration i and emigration e for both the species are assumed to be a continuous in the interval $[0, T]$. Then Total consumption made by both the species is in the time interval $[0, T]$.

$$TC(P) = \mu_x T [x(0) + \int_0^T (A_P(t) + \lambda_P(t))dt]$$

$$TC(Q) = \mu_y T [y(0) + \int_0^T (A_Q(t) + \lambda_Q(t))dt]$$

where μ_x and μ_y are the consumption factors for species P and Q respectively.

2.3. Superiority Factor(SF):

The superiority factor is defined by the ratio of total consumption of one species to the total consumption of other in a given interval $[0, T]$ of competition and is denoted by η .

Remark 2.2. We define η as follows; $\eta = \frac{TC(P)}{TC(Q)}$, where $TC(P)$ and $TC(Q)$ are the total consumption of resources by species P and Q respectively. There are three possible values of η

- (1). If $TC(P) > TC(Q)$ then $\eta > 1$ and in this case species P is competitively superior than species Q.
- (2). If $TC(P) < TC(Q)$ then $\eta < 1$ and in this case species P is competitively inferior than species Q.
- (3). If $TC(P) = TC(Q)$ then $\eta = 1$ and in this case we say both the species are competitively analogous.

2.4. Relation between TC, η and $|R|$:

Since $|R|$ is the total resource available in the region then the total consumption made by both the species must not exceed $|R|$. That is $TC(P) + TC(Q) \leq |R|$ by above definition of η (see remark 2.2) $TC(P) = \eta TC(Q)$. And so $TC(Q) \leq \frac{|R|}{(\eta + 1)}$. Similarly, $TC(P) \leq \frac{\eta |R|}{(\eta + 1)}$. And the sum of total population TP of both the species in the region R that participates in the competition must not exceed the limit $\frac{|R|(\mu_x + \eta\mu_y)}{\mu_x\mu_y T(\eta + 1)}$.

3. Competition Parameter

3.1. Asymptotic Nature of Species in a Competition

As discussed in section 1.1 for a period of competition between two species either one of them is superior or they are analogous. But if the resource is plenty and competition between these two species continues as time increases, *i.e.*, if competition lasts for a long time, *i.e.*, as time tends to infinity the superiority factor defined in section 2.3 is unable to predict the nature of competition between the species defined in section 1.1. To overcome the problem we introduce a parameter that describes the relation between the species at infinity. Further the nature of competition defined in section 1.1 is for finite period of time so we define a new set of terminology to indicate the nature of competition at infinity. There are two levels of asymptotic nature;

- (1). **Asymptotically Superior:** A species is said to be asymptotically superior if it can consume resources more efficiently than the other species participated in the competition for every $T \in [0, \infty]$
- (2). **Asymptotically Inferior:** A species which is not competitively superior but it exists in the competition for every period of competition $T \in [0, \infty]$ is said to be asymptotically inferior

3.2. Competition Parameter

A competition parameter compares the competition between two species and is defined by a positive real number $\zeta \in [0, \infty]$ such that $\zeta = \lim_{t \rightarrow \infty} \frac{\mu_x x(t)}{\mu_y y(t)}$, where μ_x and μ_y are consumption factor of population P and Q respectively

Theorem 3.1. If ζ is the competition parameter then;

- (1). If $\zeta > 1$, then population P is asymptotically superior than Q and $\eta \geq 1$ and $TC(P) > TC(Q)$. Further both the species exist in the region throughout the period of competition.

- (2). If $\zeta < 1$, then population Q is asymptotically superior and $\eta \leq 1$ and $TC(Q) > TC(P)$. Further both the species exist in the region throughout the period of competition.
- (3). If $\zeta = 1$, then either the species are competitively analogous for some time T or the case is doubtful and needs further investigation.
- (4). If $\zeta = 0$, then no more competition between species P and Q . Species Q is asymptotically superior than species P . Moreover, Population of P get diminished after a time t and might get extinct from region R .
- (5). If $\zeta = \infty$, then no more competition between species P and Q . Species P is asymptotically superior than species Q . Moreover, Population of Q get diminished after a time t and might get extinct from region R .

Proof. There are several cases to be considered (for details of real analysis involved in this proof we suggest [6]).

Case I: Let $\epsilon > 0$ be such that, $\frac{\mu_x x(t)}{\mu_y y(t)} > \zeta - \epsilon$, where $\epsilon \rightarrow 0$ as $t \rightarrow \infty$. Let $\zeta > 1$, then $\frac{\mu_x x(t)}{\mu_y y(t)} > 1 - \epsilon$ it follows that $\int_0^T \mu_x x(t) dt > (1 - \epsilon) \int_0^T \mu_y y(t) dt$ for each $T \in (0, \infty)$. And so $\eta > 1 - \epsilon$ for all $T \in (0, \infty)$. Since, $\epsilon > 0$ is arbitrary small positive number and therefore $\eta \geq 1$. Thus the species P is asymptotically superior to species Q .

Case II: Let $\delta > 0$ be such that $\frac{\mu_x x(t)}{\mu_y y(t)} < \zeta + \delta$, where $\delta \rightarrow 0$ as $t \rightarrow \infty$. Let $\zeta < 1$ then as Case I we see that $\eta < 1 + \delta$ for all $T \in [0, \infty)$. Since, δ is any small positive number it follows that $\eta \leq 1$. Thus, species Q is asymptotically superior than species P .

Case III: Let $\zeta = 1$ then there exist $\epsilon > 0$ such that $\frac{\mu_x x(t)}{\mu_y y(t)} < 1 + \epsilon$ and $\frac{\mu_x x(t)}{\mu_y y(t)} > 1 - \epsilon$. Then by Case I and Case II it follows that $\eta \geq 1$ and $\eta \leq 1$. If $\eta = 1$ for some time T then species are competitively analogous at time T . And if $\eta \neq 1$ then we have $\eta > 1$ and $\eta < 1$ which is impossible and so the case is doubtful and needs further investigation that is we shall consider the other factors.

Case IV: Let $\zeta = 0$. Let $0 < \epsilon < 1$ be given, then there exist $M > 0$ such that $\left| \frac{\mu_x x(t)}{\mu_y y(t)} \right| < \epsilon$ for all $t > M$ and by Case II, $\zeta < \epsilon < 1$ follows that population Q is competitively superior than population P . Also $x(t) < \frac{\mu_y}{\mu_x} y(t) \epsilon$ and so $\int_0^T x(t) dt < \frac{\mu_y \epsilon}{\mu_x} \int_0^T y(t) dt$ for each $T \in [0, \infty)$. It follows that, $\eta < \epsilon$ where, $\epsilon \rightarrow 0$ as $T \rightarrow \infty$. Thus, $\eta \rightarrow 0$ for sufficiently large T , i.e., $TC(Q) \gg \gg TC(P)$ for sufficiently large T . Hence, the consumption of resources by the species Q is increasing with time on the other hand consumption of resources by the species P is either constant or decreases with the increase of time and so this leads to a situation in which species P no longer remain in competition with species Q .

Case V: Let $\zeta = \infty$ then for every positive real number M however large there exist $G > 0$ such that, $\left| \frac{\mu_x x(t)}{\mu_y y(t)} \right| > M$ for every $t > G$. And for every such M there exist $\epsilon > 0$ such that $\frac{1}{M} < \epsilon$ whenever $t > G$. Thus, $\left| \frac{\mu_y y(t)}{\mu_x x(t)} \right| < \epsilon$ whenever $t > G$ and so by Case IV we conclude that species Q no longer remain in competition with species P .

Hence, we see that competition parameter ζ alone is sufficient to discuss all the parameters TC , TP , η of the interaction between two species due to competition. Further, analysis of ζ can predict the asymptotic nature of competition that we have defined in definition- □

Now we shall discuss two model of the biological interaction between two species due to competition.

4. Mathematical Model on Competition-I

4.1. Introduction

Let us consider two species P and Q with initial population N_1 and N_2 respectively at time $t = 0$. Suppose the change in population of species P and Q due to NG is given by;

$$A_P(t) = at^2 - bt \text{ and}$$

$$A_Q(t) = ht^2 - kt.$$

Again the change in population due to λ ;

$$\lambda_P(t) = [i(P) - e(P)]t \text{ and}$$

$$\lambda_Q(t) = [i(Q) - e(Q)]t.$$

At any time t , the population density of both the species is given by;

$$x(t) = N_1 + at^2 - bt + [i(P) - e(P)]t$$

$$y(t) = N_2 + ht^2 - kt + [i(Q) - e(Q)]t$$

Where $N_1, N_2, a, b, h, k, i(P), i(Q), e(P), e(Q)$ are all constants and not all zero. Let T be the period of competition and consumption factor $\mu_x = p$ and $\mu_y = q$ units per individual per unit time, then TP for both the species is given by;

$$TP(T)_P = N_1 + \int_0^T [at^2 + (i(P) - e(P) - b)t] dt$$

$$TP(T)_P = N_1 + \left[\frac{aT^3}{3} + \frac{(i(P) - e(P) - b)T^2}{2} \right]$$

$$TP(T)_P = \frac{1}{6} [2aT^3 + (i(P) - e(P) - b)T^2 + 6N_1]$$

Similarly,

$$TP(T)_Q = \frac{1}{6} [2hT^3 + (i(Q) - e(Q) - k)T^2 + 6N_2]$$

And Total Consumption is defined as;

$$TC(P) = \frac{1}{6} pT [2aT^3 + (i(P) - e(P) - b)T^2 + 6N_1]$$

Similarly,

$$TC(Q) = \frac{1}{6} qT [2hT^3 + (i(Q) - e(Q) - k)T^2 + 6N_2]$$

Thus, the superiority factor SF is given by;

$$\eta = \frac{TC(P)}{TC(Q)} = \frac{p[2aT^3 + 3(i(P) - e(P) - b)T^2 + 6N_1]}{q[2hT^3 + 3(i(Q) - e(Q) - k)T^2 + 6N_2]}$$

Theorem 4.1. *If η is the SF defined above between two species P and Q under competition then;*

$$(1). \eta > 1, \text{ if } \frac{a}{h} < \frac{q}{p} < \frac{i(P) - e(P) - b}{i(Q) - e(Q) - k} \text{ and } \frac{q}{p} \geq \frac{N_1}{N_2}$$

(2). $\eta < 1$, if $\frac{q}{p} < \min \left\{ \frac{a}{h}, \frac{i(P) - e(P) - b}{i(Q) - e(Q) - k} \right\}$ and $\frac{q}{p} \leq \frac{N_1}{N_2}$

Proof. Let $T \in [0, \infty)$ and let us consider the function,

$$F(T) = 2(ap - hq)T^3 + 3[p(i(P) - e(P) - b) - q(i(Q) - e(Q) - k)]T^2 + 6(pN_1 - qN_2)$$

Clearly $F(T)$ is a cubic polynomial in T . It is sufficient to show the conditions under which either $F(T) > 0$ or $F(T) < 0$.

Let $F(T) = LT^3 + MT^2 + N$, where $L = 2(ap - hq)$, $M = 3[p(i(P) - e(P) - b) - q(i(Q) - e(Q) - k)]$ and $N = 6(pN_1 - qN_2)$

are all constants. $F'(T) = 3LT^2 + 2MT$ and so $T = -\frac{2M}{3L}$. Since $T > 0$ so either $M > 0$ and $L < 0$ or $M < 0$ and $L > 0$.

For $M > 0$ and $L < 0$ we have $\frac{i(P) - e(P) - b}{i(Q) - e(Q) - k} < \frac{q}{p} < \frac{a}{h}$. And for $M < 0$ and $L > 0$ we have $\frac{a}{h} < \frac{q}{p} < \frac{i(P) - e(P) - b}{i(Q) - e(Q) - k}$.

Thus for the existence of monotonicity of function $F(T)$ we must have either

$$\frac{i(P) - e(P) - b}{i(Q) - e(Q) - k} < \frac{q}{p} < \frac{a}{h} \quad \text{or} \quad \frac{a}{h} < \frac{q}{p} < \frac{i(P) - e(P) - b}{i(Q) - e(Q) - k}$$

Suppose the constants satisfies the above condition. Now, $F'(T) > 0$ if $M < 0$ and $L > 0$ i.e., if $\frac{a}{h} < \frac{q}{p} < \frac{i(P) - e(P) - b}{i(Q) - e(Q) - k}$.

Therefore, F is strictly increasing if $\frac{a}{h} < \frac{q}{p} < \frac{i(P) - e(P) - b}{i(Q) - e(Q) - k}$ for $T > 0$. Also $F(0) = N \geq 0$ if $N \geq 0$ i.e., $\frac{q}{p} \geq \frac{N_1}{N_2}$.

Thus $F(T) > 0$ for all $T > 0$ if $\frac{a}{h} < \frac{q}{p} < \frac{i(P) - e(P) - b}{i(Q) - e(Q) - k}$ and $\frac{q}{p} \geq \frac{N_1}{N_2}$. But $F(T) > 0$ implies $\eta > 1$. Hence, $\eta > 1$ if

$\frac{a}{h} < \frac{q}{p} < \frac{i(P) - e(P) - b}{i(Q) - e(Q) - k}$ and $\frac{q}{p} \geq \frac{N_1}{N_2}$. Similarly, we can prove the other part. □

Lemma 4.2. *The condition for which the cubic polynomial $F(T) = LT^3 + MT^2 + N$ has a positive root is that*

$$\left(\frac{L}{M} \right)^4 > \left[\frac{2(M-1)}{N} \right]^2.$$

Proof. Consider the cubic equation $LT^3 + MT^2 + N = 0$ then by Cardan's method the nature nature of roots for this cubic equation is real positive, if

$$\left(\frac{L^2N + M^2}{L^2} \right)^2 + 4 \left(-\frac{M^2}{L^2} \right)^3 < 0 \quad \text{and} \quad -\frac{M^2}{L^2} < 0.$$

$$\frac{(L^2N + 2M^2)^2 - 4M^6}{L^6} < 0 \quad \text{and} \quad -\frac{M^2}{L^2} < 0$$

$$(L^2N + 2M^2)^2 - 4M^6 > 0$$

$$(L^2N + 2M^2)^2 > 4M^6$$

$$L^2N > 2M^2(M-1) \quad \text{or} \quad L^2N < -2M^2(M-1)$$

$$L^4N^2 > 4M^4(M-1)^2$$

$$\left(\frac{L}{M} \right)^4 > \left[\frac{2(M-1)}{N} \right]^2.$$

Hence the required condition. □

Theorem 4.3. *If $\left(\frac{2(ap - hq)}{3[p(i(P) - e(P) - b) - q(i(Q) - e(Q) - k)]} \right)^4 > \left[\frac{2(3[p(i(P) - e(P) - b) - q(i(Q) - e(Q) - k)] - 1)}{6(pN_1 - qN_2)} \right]^2$ then there exists $T_0 \in (0, \infty)$ such that $\eta = 1$.*

Proof. Consider $F(T) = 2(ap - hq)T^3 + 3[p(i(P) - e(P) - b) - q(i(Q) - e(Q) - k)]T^2 + 6(pN_1 - qN_2)$ for all $T \in [0, \infty)$

which is a cubic polynomial function in T such that the constants satisfies the given condition then by Lemma 4.2 there

exist $T_0 \in [0, \infty)$ such that $F(T_0) = 0$ or $\eta = 1$ for $T = T_0$ □

4.2. Analysis of Competition Parameter

The value of competition parameter for this model is given by;

$$\zeta = \lim_{t \rightarrow \infty} \frac{px(t)}{qy(y)} = \lim_{t \rightarrow \infty} \frac{p[N_1 + at^2 - bt + [i(P) - e(P)]t]}{q[N_2 + ht^2 - kt + [i(Q) - e(Q)]t]} = \frac{pa}{qh}$$

Given that a, h, p and q are all positive constants and so we have several cases to discuss regarding the competition between the species P and Q .

Case I: The value ζ exceeds 1, when $pa > qh$ as $t \rightarrow \infty$. It follows that birth rate and consumption factor of species P are sufficiently large than birth rate and consumption factor of species Q . Although both the species exist throughout the period of competition but species P is considered to be asymptotically superior than species Q i.e., species P is competitively superior than species Q and there may be a time when both the species shows analogous nature of competition provided the total resource $|R| \rightarrow 0^+$ as $t \rightarrow \infty$.

Case II: The value of ζ is less than 1, when $pa < qh$ and similarly as Case I we can see that in this case species Q is asymptotically superior than species P .

Case III: If $\zeta = 1$ then $pa = qh$ i.e., $\frac{a}{h} = \frac{q}{p}$ or $\frac{a}{h} = \frac{1}{\left(\frac{p}{q}\right)}$. Thus ratio of birth rate is inversely proportional to ratio of their consumption factor that is, the species under competition are competitively analogous if the species with greater birth rate have smaller consumption factor and vice versa. But in nature every species does not satisfies this condition so the case remains doubtful and need further investigation to predict the nature species under competition for a finite time T by considering the other factors like death rate, immigration rate and migration rate.

Case IV: If $\zeta = 0$ then $pa = 0$ but in a competition one has to consume resources and so $p \neq 0$ it follows that $a = 0$. So we can conclude that there will be no contribution to the population P due to breeding between two individuals of same species and therefore population of species P exists in the region R at any time t , whenever $i(P) > e(P) + b$. Further, $TC(Q) \gg TC(P)$ as $t \rightarrow \infty$ which clearly follows that species Q is asymptotically competitively superior species and as t increases $\eta \rightarrow 0$ i.e., there will be no competition left between these two species for the same resource.

Case V: IF $\zeta = \infty$, then $\frac{qh}{pa} = 0$ or $qh = 0$ and similarly as Case IV, we can conclude that species P is asymptotically competitively superior than species Q . Further, as t increases there will be no competition left between these two species for the same resources.

The analysis of competition parameter provides a clear picture of the nature of competition between two species as t increases indefinitely except the case when $\zeta = 1$.

4.3. Conclusion

In this bio-mathematical model we have study and investigate the competition between two different species in a region competing for the same resource may be light, water, space or food. It should be noted that, the functions $B_P(t) = at^2$ and $B(t)_Q = ht^2$ representing number of individuals born in time t is quadratic in nature and choice of this function to represent the number of individuals introduced to the population of both the species due to interbreeding has considerable impact to the level of competition i.e., in the long run only the comparative study of birth rate of these two species is enough to predict the asymptotic nature of competition. Further the consumption factor also plays an important role to study the asymptotic nature of competition. In theorem 4.1, it should be noted that either $i(P) > e(P) + b$ and $i(Q) > e(Q) - k$ or $i(P) < e(P) + b$

and $i(Q) < e(Q) - k$ as consumption factor is a positive quantity and it is clear that if ratio of consumption factor satisfies the following relation then we can conclude the nature of competition. In theorem 4.3 we have stated a condition under which there exist a period T_0 at which both the species shows analogous nature of competition.

5. Mathematical Model on Competition-II

5.1. Introduction

Let us consider a natural ecosystem covering a geographical region R consisting of two species P and Q with population N_1 and N_2 at time $t = 0$. We shall consider all the assumptions made in section 1.2. Suppose population density of P and Q increases exponentially due to interbreeding and decreases arithmetically due to death of individuals. Then birth function and death function are defined by;

$$\begin{aligned} B_P(t) &= N_1 a^t, \quad a > 0 & B_Q(t) &= N_2 b^t, \quad b > 0 \text{ and} \\ D_P(t) &= N_1 + td_1 & D_Q(t) &= N_2 + td_2 \end{aligned}$$

where d_1 and d_2 may be positive, negative or zero. Thus, the natural growth factor NG for this competition model is defined by; $A_P = N_1 a^t - (N_1 + td_1)$ and $A_Q = N_2 b^t - (N_2 + td_2)$. The immigration and migration of individuals in a given species differs by a constant and so we define the λ -factor as follows; $\lambda_P(t) = \alpha e_1^t$ and $\lambda_Q(t) = \beta e_2^t$, where $\alpha = i_P - e_P$ and $\beta = i_Q - e_Q$, the numerical values of α and β are integers may be positive, negative or zero. And e_1 and e_2 are the geometric ratios that represents the rate at which population size of a species changes with respect to immigration and emigration rate. The total population of species P is given by;

$$\begin{aligned} TP_P(T) &= N_1 + \int_0^T [A_P(t) + \lambda_Q(t)] dt \\ TP_P(T) &= N_1 + \int_0^T [N_1 a^t - (N_1 + td_1) + \alpha e_1^t] dt \\ TP_P(T) &= N_1 + \left[\frac{N_1 a^t}{\log a} - N_1 t - \frac{d_1 t^2}{2} + \frac{\alpha e_1^t}{\log e_1} \right]_0^T \\ TP_P(T) &= N_1 + \left[\frac{N_1 a^T}{\log a} - N_1 T - \frac{d_1 T^2}{2} + \frac{\alpha e_1^T}{\log e_1} \right] - \left[\frac{N_1}{\log a} + \frac{\alpha}{\log e_1} \right] \end{aligned}$$

Similarly the total population of species Q is given by;

$$TP_Q(T) = N_2 + \left[\frac{N_2 b^T}{\log b} - N_2 T - \frac{d_2 T^2}{2} + \frac{\beta e_2^T}{\log e_2} \right] - \left[\frac{N_2}{\log b} + \frac{\beta}{\log e_2} \right]$$

5.2. Superiority Factor SF

Let p and q be the consumption factor of the population P and Q respectively. In this section we shall discuss the superiority factor SF, though we have consider the similar NG and λ -factor for both the species so the superiority factor SF is perfectly depends on the constants involved viz, a , b , α , β and d_i ($i = 1, 2$). In a natural ecosystem we can observe that, species with similar NG and λ -factors participating in the competition for consuming resources placed in different hierarchy level of nature of competition defined in the section 1.1. This is due to the variation of breeding procedure, support from other environmental factors, natural and physical constraints which either support or hinders the immigration or emigration from one place to another. Thus, instead of having similar functional structure, the constants involved have a considerable role in deciding the final gradation or degradation of population size of a species. Thus, we cannot neglect the superiority factor

SF that decides the hierarchy of nature of competition for a species on the basis of constants. The superiority factor for this model is given by;

$$\eta = \frac{p \left[N_1 + \left[\frac{N_1 a^T}{\log a} - N_1 T - \frac{d_1 T^2}{2} + \frac{\alpha e_1^T}{\log e_1} \right] - \left[\frac{N_1}{\log a} + \frac{\alpha}{\log e_1} \right] \right]}{q \left[N_2 + \left[\frac{N_2 b^T}{\log b} - N_2 T - \frac{d_2 T^2}{2} + \frac{\beta e_2^T}{\log e_2} \right] - \left[\frac{N_2}{\log b} + \frac{\beta}{\log e_2} \right] \right]}$$

$$\eta = \frac{p \left[N_1 + \frac{N_1}{\log a} (a^T - 1) + \frac{\alpha}{\log e_1} (e_1^T - 1) - N_1 T - \frac{d_1 T^2}{2} \right]}{q \left[N_2 + \frac{N_2}{\log b} (b^T - 1) + \frac{\beta}{\log e_2} (e_2^T - 1) - N_2 T - \frac{d_2 T^2}{2} \right]} \quad (1)$$

Theorem 5.1. If η is the SF as defined above for a given time such that constants satisfies the following conditions;

- (1). $\frac{p}{q} = \frac{N_1}{N_2} = \frac{\beta}{\alpha}$
- (2). $a > b$ and $0 < e_2 < e_1$
- (3). the constants are related by; $a^{pN_1} e_1^{p\alpha} = b_{qN_2} e_2^{q\beta} \exp(pd_1 - qd_2)$

Then SF, $\eta > 1$

Proof. Let the constants in the expression of SF, *eta* satisfies all the conditions. Consider the function;

$$F(T) = \left(\frac{pN_1}{\log a} a^T - \frac{qN_2}{\log b} b^T \right) + \left(\frac{p\alpha}{\log e_1} e_1^T - \frac{q\beta}{\log e_2} e_2^T \right) - (pd_1 - qd_2) \frac{T^2}{2} - (pN_1 - qN_2)T + M$$

for all $T \in [0, \infty)$, where $M = p \left(N_1 - \frac{N_1}{\log a} - \frac{\alpha}{\log e_1} \right) - q \left(N_2 - \frac{N_2}{\log b} - \frac{\beta}{\log e_2} \right)$. At $T = 0$, $F(0) = 0$ by condition 1. We shall apply the idea of monotonicity to prove our result.

$$F'(T) = (pN_1 a^T - qN_2 b^T) + (p\alpha e_1^T - q\beta e_2^T) - (pd_1 - qd_2)T - (pN_1 - qN_2)$$

At $T = 0$, $F'(0) = 0$ by condition 1. Again

$$F''(T) = (pN_1 a^T \log a - qN_2 b^T \log b) + (p\alpha e_1^T \log e_1 - q\beta e_2^T \log e_2) - (pd_1 - qd_2)$$

$$\text{At } T = 0, F''(0) = pN_1 \log a - qN_2 \log b + p\alpha \log e_1 - q\beta \log e_2 - (pd_1 - qd_2)$$

Now by condition 3., $a^{pN_1} e_1^{p\alpha} = b_{qN_2} e_2^{q\beta} \exp(pd_1 - qd_2)$. Taking log on both side we get;

$$pN_1 \log a + p\alpha \log e_1 = qN_2 \log b + q\beta \log e_2 + (pd_1 - qd_2)$$

It follows that,

$$pN_1 \log a + p\alpha \log e_1 - qN_2 \log b - q\beta \log e_2 - (pd_1 - qd_2) = 0 \text{ or,}$$

$$pN_1 \log a - qN_2 \log b + p\alpha \log e_1 - q\beta \log e_2 - (pd_1 - qd_2) = 0$$

and so $F''(0) = 0$. Finally,

$$F'''(T) = (pN_1 a^T (\log a)^2 - qN_2 b^T (\log b)^2) + (p\alpha e_1^T (\log e_1)^2 - q\beta e_2^T (\log e_2)^2)$$

And clearly by condition 2. $F'''(T) > 0$. Retracing each step to the beginning we conclude that $F(T)$ is strictly increasing function on $[0, \infty)$ also $F(0) = 0$ and so $F(T) > 0$ for all $T \in (0, \infty)$. Hence, for any $T \in (0, \infty)$, $\eta > 1$. \square

Theorem 5.2. If η is the SF as defined above for a given time such that constants satisfies the following conditions;

- (1). $\frac{p}{q} = \frac{N_1}{N_2} = \frac{\beta}{\alpha}$
- (2). $a < b$ and $0 < e_1 < e_2$
- (3). the constants are related by; $a^{pN_1} e_1^{p\alpha} = b_{qN_2} e_2^{q\beta} \exp(pd_1 - qd_2)$

Then SF, $\eta < 1$

Proof. Proof of the theorem is same as Theorem 5.1 □

5.3. Analysis of Competition Parameter

To study the hierarchal of nature of competition in a long run *i.e.*, to investigate the asymptotic nature of competition as time t tends to infinity we need to define the population density of species at any time t is defined as follows;

$$\begin{cases} x(t) = N_1 + N_1 a^t - (N_1 + td_1) + \alpha e_1^t \\ y(t) = N_2 + N_2 b^t - (N_2 + td_2) + \beta e_2^t \end{cases} . \text{ On simplification, } \begin{cases} x(t) = N_1 a^t - td_1 + \alpha e_1^t \\ y(t) = N_2 b^t - td_2 + \beta e_2^t \end{cases} . \text{ The competition parameter for this model is;}$$

$$\zeta = \lim_{t \rightarrow \infty} \frac{px(t)}{qy(t)}$$

$$\zeta = \lim_{t \rightarrow \infty} \frac{p[N_1 a^t - td_1 + \alpha e_1^t]}{q[N_2 b^t - td_2 + \beta e_2^t]} \quad (2)$$

$$\zeta = \lim_{t \rightarrow \infty} \frac{p(ae_1)^t [N_1 e_1^{-t} - td_1 (ae_1)^{-t} + \alpha a^{-t}]}{q(be_2)^t [N_2 e_2^{-t} - td_2 (be_2)^{-t} + \beta b^{-t}]} \quad (3)$$

There are several possible cases

- (1). If $a = b = 1$ and $e_1 = e_2$ then equation (3) becomes;

$$\zeta = \lim_{t \rightarrow \infty} \frac{p[N_1 e_1^{-t} - te_1^{-t} + \alpha]}{q[N_2 e_2^{-t} - te_2^{-t} + \beta]} = \frac{p\alpha}{q\beta}$$

This case is equivalent to the situation when the growth rate due to interbreeding is a multiple of initial population and the ratio with which population size of both the species changes due to λ -factor are same. For this case the hierarchal of nature of competition depends upon four constants *viz.* p , q , α and β . It is to be noted that α and β are of same sign otherwise ζ becomes negative which is against the definition of competition parameter, *i.e.*, either both α and β are positive or negative *i.e.*, the population density of both the species either increase or decrease due to λ -factor. In reality this situation arise when either in the geographical region R resources are not sufficient to meet the requirement of species and to avoid intra-specific competition individuals of the respective species move on to some other region and due to lack of resources individuals from other region avoid to immigrate in this region or due to plenty of resource in the region R attracts the individuals of respective species immigrate from different to R and there are very rare situation in which individuals of respective species migrate from R . But as per our assumptions resources are enough to avoid intra-specific competition *i.e.*, for any time t , $0 < t < \infty$, there exist positive real numbers W_1 and W_2 with $|TC(P)| > W_1$, $|TC(Q)| > W_1$ and $|TC(P) + TC(Q)| > W_2$ such that $W_1 < |R| < W_2$ it follows that α and β must be positive. Now to discuss this case we need to subdivide it into several sub-cases;

- (i). If $\frac{\alpha}{\beta} > \frac{q}{p}$ then $\zeta > 1$ that is species P is asymptotically superior than species Q . It is to be noted that, species with low consumption rate can be superior in a long run if the population size increases considerably due to λ -factor as in Case 1 NG-factor directly depends on initial population so it does not play any considerable role in a long run except when $d_1 \ll d_2$ and $N_1 \gg N_2$.

- (ii). If $\frac{\alpha}{\beta} < \frac{q}{p}$ then $\zeta < 1$ and similarly as Case (a), species Q is asymptotically superior than species P .
- (iii). If $p = q$ then superiority and inferiority of a species in a long run directly depends on α and β that is, if $0 < \alpha < \beta$ then, species Q is asymptotically superior than species P and if $0 < \beta < \alpha$ then, species P is asymptotically superior than species Q .
- (iv). If $\alpha = \beta$ then a species is asymptotically superior than other if and only if its consumption factor is greater than the other.
- (v). If $p = q$ and $\alpha = \beta$ then $\zeta = 1$ and the case is doubtful and needs further investigation. In this case population of species at any time t is given by;

$$\begin{cases} x(t) = N_1 - td_1 + k(t) \\ y(t) = N_2 - td_2 + k(t) \end{cases} \quad \text{where } k(t) = \alpha e_1^t = \beta e_2^t \text{ is a function of } t. \text{ Now at any time } t > 0;$$

$$x(t) - y(t) = (N_1 - N_2) - t(d_1 - d_2) \quad (4)$$

without loss of generality we assume $N_1 > N_2$ then there exist a time T_0 such that from Equation (4), $x(t) > y(t)$ for all $t > T_0$ if $d_1 < d_2$ i.e., species P is asymptotically superior than species Q or $x(t) < y(t)$ for all $t > T_0$ if $d_1 > d_2$ i.e., that is species Q is asymptotically superior than species P . If $N_1 < N_2$, we proceed similarly. Thus in this case asymptotic nature of competition depends on death rate.

- (vi). If $p \neq q$, $\alpha \neq \beta$ and $p\alpha = q\beta$ then $\zeta = 1$ and again the case is doubtful and needs further investigation.

$$\text{At any time } t; \begin{cases} x(t) = N_1 - td_1 + \alpha e_1^t \\ y(t) = N_2 - td_2 + \beta e_1^t \end{cases} \quad . \text{ Now at any time } t > 0;$$

$$px(t) - qy(t) = (pN_1 - qN_2) - t(pd_1 - qd_2)$$

$$px(t) - qy(t) = U - tV$$

Similarly, as case (e) species P is asymptotically superior than species Q , if $V < 0$ i.e., $\frac{d_1}{d_2} < \frac{q}{p}$ and species Q is asymptotically superior than species P , if $V > 0$ i.e., $\frac{d_1}{d_2} > \frac{q}{p}$. Thus, in this case also death rate plays an important role in deciding the asymptotic nature of competition.

- (2). If $e_1 = e_2 = 1$ and $a = b$ then equation (3) becomes

$$\zeta = \lim_{t \rightarrow \infty} \frac{p[N_1 - td_1a^{-t} + \alpha a^{-t}]}{q[N_2 - td_2b^{-t} + \beta b^{-t}]} = \frac{pN_1}{qN_2}$$

This case is similar to the situation when ratio with which population density changes due to birth is same for the two species P and Q . Also population change due to λ -factor is independent of time further it does not have any considerable role in the investigation of asymptotic nature. There are several case that needs to be investigated.

- (i). If $\frac{N_1}{N_2} > \frac{q}{p}$ then ζ and species P is asymptotically superior than species Q
- (ii). If $\frac{N_1}{N_2} < \frac{q}{p}$ then $\zeta > 1$ and species Q is asymptotically superior than species P
- (iii). If $p = q$, then a species in a long run is asymptotically superior if it has greater population size at $t = 0$
- (iv). If $N_1 = N_2$, then a species in a long run is asymptotically superior if its consumption rate is larger.

(v). If $N_1 = N_2$ and $p = q$ then $\zeta = 1$ and so the case is doubtful and needs further investigation; At any time $t > 0$

$$\begin{cases} x(t) = N_1 a^t - t d_1 + \alpha \\ y(t) = N_1 a^t - t d_2 + \beta \end{cases} . \text{ Now, } x(t) - y(t) = -t(d_1 - d_2) + (\alpha - \beta). \text{ Thus, in this case asymptotic nature of}$$

competition depends on death rate of population *i.e.*, a population with low death rate is asymptotically superior than the other.

(vi). If $N_1 \neq N_2$, $p \neq q$ and $pN_1 = qN_2$ then $\zeta = 1$ and the case is doubtful and needs further investigation. At any time $t > 0$;

$$\begin{cases} x(t) = N_1 a^t - t d_1 + \alpha \\ y(t) = N_2 a^t - t d_2 + \beta \end{cases} . \text{ Now, at any time } t > 0;$$

$$px(t) - qy(t) = (pN_1 - qN_2)a^t - t(pd_1 - qd_2) + (p\alpha - q\beta)$$

$$px(t) - qy(t) = U_1 - tV_1$$

where $U_1 = p\alpha - q\beta$ and $V_1 = pd_1 - qd_2$ are constants. This case is same as Case 1(f).

(3). If $a = b = 1$ and $e_1 = e_2 = 1$ then equation (2) becomes;

$$\zeta = \lim_{t \rightarrow \infty} \frac{p[N_1 - t d_1 + \alpha]}{q[N_2 - t d_2 + \beta]} = \frac{p d_1}{q d_2}$$

This case is equivalent to the situation when birth ratio and the ratio with which λ -factor changes does not play any considerable role in changing population of both the species and in that case death rate plays an important role to investigate asymptotic nature of competition between species P and Q . It is to be noted that, either $d_1 > 0$ and $d_2 > 0$ or $d_1 < 0$ and $d_2 < 0$ otherwise ζ becomes negative. And so we have two possible case for death rate to be considered.

(i). If $d_1 > 0$ and $d_2 > 0$ then there exist a time $T_0 \in [0, \infty)$ such that, $x(t) \leq 0$ and $y(t) \leq 0$ for all $t \geq T_0$ which is impossible as population density of a species at any instant of time is either positive or zero. Thus in this case period of competition is atmost T_0 . Moreover, $\zeta > 1$ if $\frac{d_1}{d_2} > \frac{q}{p}$ and $\zeta < 1$ if $\frac{d_1}{d_2} < \frac{q}{p}$.

Thus by the theory of competition parameter, $\begin{cases} \eta > 1, \text{ if } \frac{d_1}{d_2} > \frac{q}{p} \\ \eta < 1, \text{ if } \frac{d_1}{d_2} < \frac{q}{p} \end{cases}$. But, this case is impossible as it does not satisfy equation (1)

(ii). If $d_1 < 0$ and $d_2 < 0$ then the situation is like that, number of death of individuals in respective species decreases as time increases. So, there exist a time T_0 such that, $D_k(t) = 0$ for all $t \geq T_0$ and $k = P, Q$ that is NG-factor,

$$A_P(t) = N_1 \text{ and } A_Q(t) = N_2 \text{ for all } t \geq T_0 \text{ then } \begin{cases} x(t) = 2N_1 + \alpha \\ y(t) = 2N_2 + \beta \end{cases} \text{ for all } t \geq T_0. \text{ Which is a constant and this}$$

does not provide any satisfactory conclusion regarding asymptotic nature of competition.

Thus, this case does not exist as in a long run population density of a species cannot be independent of time as a whole. Further, change in population density due other reasons are considered to be constant except death rate and in that case both the species will not survive for a long period of time and so the question of superiority in long run for a given competition does not arise.

(4). If $e_1 < 1$ and $e_2 < 1$ then from equation (2),

$$\zeta = \lim_{t \rightarrow \infty} \frac{p[N_1 a^t - t d_1 + \alpha e_1^t]}{q[N_2 b^t - t d_2 + \beta e_2^t]} = \begin{cases} 0, & \text{if } a < b \\ \frac{p N_1}{q N_2}, & \text{if } a = b \\ \infty, & \text{if } a > b \end{cases}$$

Thus we have three possible cases;

- (i). If $a < b$ then $\zeta = 0$ and by the theory of competition parameter species Q is asymptotically superior than species P . Further as $t \rightarrow \infty$ population density of species P tends to zero that is no competition exist between species P and Q for the given resource.
- (ii). If $a = b$, then $\zeta = \frac{p N_1}{q N_2}$ and this case is similar to case 2 except the doubtful cases which we shall discuss here

$$px(t) - qy(t) = -(pd_1 - qd_2)t + (p\alpha e_1^t - q\beta e_2^t)$$

and for that we have two possible cases;

- (a). If $0 < e_1 < e_2 < 1$, then we have;

$$\begin{aligned} px(t) - qy(t) < 0 & \text{ for sufficiently large } t, \text{ if } \frac{d_1}{d_2} > \frac{q}{p} \text{ and} \\ px(t) - qy(t) > 0 & \text{ for sufficiently large } t, \text{ if } \frac{d_1}{d_2} < \frac{q}{p} \end{aligned}$$

Thus, a species is asymptotically inferior if it has greater death rate and low consumption rate and it is clear that this case is feasible as in a natural ecosystem a species having greater death rate and low consumption rate get together with equal birth rate and negligible λ -factor wiped out of the competition in a long run by a competitively superior species. Further, this result satisfies superiority factor η for large t .

- (b). If $0 < e_2 < e_1 < 1$, then we have;

$$\begin{aligned} px(t) - qy(t) < 0 & \text{ for sufficiently large } t, \text{ if } \frac{d_1}{d_2} < \frac{q}{p} \text{ and} \\ px(t) - qy(t) > 0 & \text{ for sufficiently large } t, \text{ if } \frac{d_1}{d_2} > \frac{q}{p} \end{aligned}$$

This case is not feasible and conditions does not satisfies superiority factor SF in equation (1) for sufficiently large t .

- (iii). If $a > b$ then $\zeta = \infty$ and by the theory of competition parameter species P is asymptotically superior than species Q . Further as $t \rightarrow \infty$, population density of species P tends to zero that is no competition exists between species P and Q for the given resource.

In the above discussion we have investigated several cases to study the asymptotic nature of competition and it is found that under continuous time functions $A(t)$ and $\lambda(t)$ the constants involved plays a considerable role to decide the hierarchal of nature of competition. The functions considered here are assumed to be continuous so in that case constants involved in the function plays a very role *e.g.*, $D(t)$ is considered to be linear and $B(t)$ is considered to be exponential and so any point of time t , the sign of $A(t)$ depends on the constants involved as time changes continuously. The discussion follows that, by studying the relation between constants we are able get a glance of the nature of competition between two species in a long run.

5.4. Conclusion

In this bio-mathematical model on competition we have considered a natural ecosystem consisting of two species P and Q in a region R . The functions considered here to represent birth rate and death rate are exponential and arithmetic in nature respectively. Further the population change due to λ -factor is an exponential function of time. In this competition model, our main objective is to get the relation between the two species competing for the same resource in a natural ecosystem in terms of competitive superiority and study the relation we calculate superiority factor SF. In section 5.2, we have proposed Theorems 5.1 and 5.2 to discuss SF and found that under certain conditions and in a given period of competition species attains different level of hierarchal of nature of competition *i.e.*, one species is competitively superior and other one is competitively inferior and on some occasion they may be competitively analogous. The main part of our discussion is the analysis of competition parameter ζ that allows us to see the nature of species in a long run, *i.e.*, under what condition a species remain competitively superior when time tends to infinity. This analysis produces very interesting results such as if the change in population density with time solely depends on death rate and other factors are constant then this situation does not arise in reality for a long run competition. There are many possibilities to improve the model such as to include intra-specific competition and continuous time dynamical system to get the population density at any time as a solution of autonomous dynamical system use it to obtain the value of ζ .

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