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Generalized Doubt Fuzzy Structure of BG-algebra

Research Article

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Abstract: In this paper, we introduced the concept of generalized $(\in, \in \vee q_k)$-doubt fuzzy subalgebra and generalized $(\in, \in \vee q_k)$-doubt fuzzy ideal in BG-algebra by using the combined notion of not quasi coincidence $(\overline{q})$ of a fuzzy point to a fuzzy set and the notion doubt fuzzy ideals in BCK/BCI-algebras. Some characterizations of these generalized doubt fuzzy subalgebra and generalized doubt fuzzy ideal in BG-algebra are derived. We investigated characterizations of $(\in, \in \vee q_k)$-doubt fuzzy subalgebra and $(\in, \in \vee q_k)$-doubt fuzzy ideals by using level sets and $(\in \vee q_k)$-level sets.

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1. Introduction

The concept of fuzzy sets was first proposed by Zadeh([23]) in 1965. Rosenfeld ([18]) was the first who consider the case of a groupoid in terms of fuzzy sets. Since then these ideas have been applied to other algebraic structures such as group, semigroup, ring, field, topology, vector spaces etc. Imai and Iseki ([9]) introduced BCK-algebra as a generalization of notion of the concept of set theoretic difference and propositional calculus and in the same year Iseki ([11]) introduced the notion of BCI-algebra which is a generalization of BCK-algebra. Xi Ougen ([20]) applied the concept of fuzzy set to BCK-algebra. and discussed some properties. Since then B-algebras was introduced in [17] by Neggers and Kim and which is related to several classes of algebras such as BCI/BCK-algebras. In [12] Kim and Kim introduced the notion of BG-algebra which is a generalization of B-algebra.. Fuzzy subalgebras of BG-algebras introduced in [1] by Ahn and Lee and the fuzzification of ideals of BG-algebras were studied in [16] by R. Muthuraj et al. Huang [8] fuzzified BCI-algebras in little different ways. Jun et al. [7, 22] renamed Huang's definition as doubt(anti) fuzzy ideals in BCK/BCI-algebras. Biswas [6] introduced the concept of anti fuzzy subgroup.The concept of doubt fuzzy BF-algebras was introduced by Saeid in [19] and the concept of doubt fuzzy ideal of BF-algebras was introduced by Barbhuiya [3].

Bhakat and Das [4, 5] used the relation of "belongs to" and "quasi coincident with" between fuzzy point and fuzzy set to introduce the concept of $(\in, \in \vee q)$-fuzzy subgroup, $(\in, \in \vee q)$-fuzzy subring and $(\in, \in \vee q)$-level subset. Jun [21]introduced (α, β)-fuzzy ideals of BCK/BCI-algebras. In fact, the$(\in, \in \vee q)$-fuzzy subgroup is an important generalization of Rosenfeld's

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fuzzy subgroup. Further in [13] Larimi generalized $(\in, \in \vee q)$-fuzzy ideals to $(\in, \in \vee q_k)$-fuzzy ideals. Reza Ameri et al [2] introduced the notion of $(\overline{\in}, \overline{\in \wedge q_k})$-fuzzy subalgebras in BCK/BCI-algebras. In this paper, we combined the notion of not quasi coincidence $\overline{q}$ of a fuzzy point to a fuzzy set and the notion doubt(anti) fuzzy ideals in BCK/BCI-algebras, we introduced the concept of generalized doubt fuzzy subalgebra and generalized doubt fuzzy ideal in BG-algebra. Some characterizations of these generalized doubt fuzzy subalgebra and generalized doubt fuzzy ideal in BG-algebra are derived.We investigated characterizations of $(\in, \in \vee q_k)$-doubt fuzzy subalgebra and $(\in, \in \vee q_k)$-doubt fuzzy ideals by using level sets and $(\in \vee q_k)$-level sets.

2. Preliminaries

Definition 2.1 ([12]). *A BG-algebra is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:*

(1). $x * x = 0$

(2). $x * 0 = x$

(3). $(x * y) * (0 * y) = x$ *for all* $x, y \in X$.

For brevity we also call X a BG-algebra. A non empty subset S of BG algebra X is said to be a subalgebra of X if $x * y \in S, \forall x, y \in X$. *A nonempty subset I of a BG-algebra X is called an ideal of X if* (I_1) $0 \in I$ *and* (I_2) $x * y \in I, y \in I \Rightarrow x \in I$ *for all* $x, y \in X$. *A fuzzy subset* μ *of X is called a doubt fuzzy ideal [22] of X if it satisfies the following conditions:* (DF_1) $\mu(0) \leq \mu(x)$ *and* (DF_2) $\mu(x) \leq max\{\mu(x * y), \mu(y)\}$ $\forall x, y \in X$.

Definition 2.2 ([4, 14]). *A fuzzy set* μ *of the form*

$$\mu(y) = \begin{cases} t, & \text{if } y = x,\ t \in (0, 1]; \\ 0, & \text{if } y \neq x. \end{cases}$$

is called a fuzzy point with support x and value t and it is denoted by x_t *[4, 14]. Let* μ *be a fuzzy set in X and* x_t *be a fuzzy point then*

(1). If $\mu(x) \geq t$ *then we say* x_t *belongs to* μ *and write* $x_t \in \mu$.

(2). If $\mu(x) + t > 1$ *then we say* x_t *quasi-coincident with* μ *and write* $x_t q \mu$.

(3). If $x_t \in \vee q\mu \Leftrightarrow x_t \in \mu$ *or* $x_t q\mu$.

(4). If $x_t \in \wedge q\mu \Leftrightarrow x_t \in \mu$ *and* $x_t q\mu$.

The symbol $x_t\overline{\alpha}\mu$ *means* $x_t\alpha\mu$ *does not hold and* $\overline{\in \wedge q}$ *means* $\overline{\in} \vee \overline{q}$. *For a fuzzy point* x_t *and a fuzzy set* μ *in set X, Pu and Liu ([14]) gave meaning to the symbol* $x_t\alpha\mu$ *where* $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$.

Definition 2.3 ([2, 13]). *Let* μ *be a fuzzy set in X and* x_t *be a fuzzy point then*

(1). If $\mu(x) < t$ *then we say* x_t *does not belongs to* μ *and write* $x_t\overline{\in}\mu$.

(2). If $\mu(x) + t \leq 1$ *then we say* x_t *not quasi-coincident with* μ *and write* $x_t\overline{q}\mu$.

(3). If $x_t\overline{\in \vee q}\mu \Leftrightarrow x_t\overline{\in}\mu$ *and* $x_t\overline{q}\mu$.

(4). If $x_t \overline{\in \wedge q} \mu \Leftrightarrow x_t \overline{\in} \mu$ *or* $x_t \overline{q} \mu$.

Definition 2.4 ([2, 13]). *Let* μ *be a fuzzy set in* X *and* x_t *be a fuzzy point then*

(1). If $\mu(x) + t + k > 1$ *then we say* x_t *is k quasi-coincident with* μ *and write* $x_t q_k \mu$ *where* $k \in [01)$.

(2). If $x_t \in \vee q_k \mu \Leftrightarrow x_t \in \mu$ *or* $x_t q_k \mu$.

(3). If $x_t \in \wedge q_k \mu \Leftrightarrow x_t \in \mu$ *and* $x_t q_k \mu$.

Definition 2.5 ([2, 13]). *Let* μ *be a fuzzy set in* X *and* x_t *be a fuzzy point then*

(1). If $\mu(x) + t + k \leq 1$ *then we say* x_t *is not k quasi-coincident with* μ *and write* $x_t \overline{q_k} \mu$ *where* $k \in [01)$.

(2). If $x_t \overline{\in \vee q_k} \mu \Leftrightarrow x_t \overline{\in} \mu$ *and* $x_t \overline{q}_k \mu$.

(3). If $x_t \overline{\in \wedge q_k} \mu \Leftrightarrow x_t \overline{\in} \mu$ *or* $x_t \overline{q}_k \mu$.

Definition 2.6 ([21]). *A fuzzy set* μ *of a BG-algebra* X *is said to be* (α, β)*-fuzzy ideal of* X *if*

(1). $x_t \alpha \mu \Rightarrow 0_t \beta \mu$ *for all* $x \in X$.

(2). $(x * y)_t, y_s \alpha \mu \Rightarrow x_{m(t,s)} \beta \mu$ *for all* $x, y \in X$. *Where* $\alpha \neq \in \wedge q, m\{t, s\} = min\{t, s\}$ *and* $t, s \in (0, 1]$.

3. Generalized Doubt Fuzzy Structure of BG-algebra

Definition 3.1. *A fuzzy subset* μ *of a BG-algebra* X *is an* $(\in, \in \vee q_k)$*-doubt fuzzy subalgebra of* X *if*

$$\mu(x * y) \leq max\left\{\mu(x), \mu(y), \frac{1-k}{2}\right\} \quad for\ all \quad x, y \in X.$$

Remark 3.2. *A fuzzy subset* μ *of a BG-algebra* X *is an* $(\in, \in \vee q)$*-doubt fuzzy subalgebra of* X *iff*

$$\mu(x * y) \leq M\{\mu(x), \mu(y), 0.5\}$$

Definition 3.3. *A fuzzy subset* μ *of a BG-algebra* X *is an* $(\in, \in \vee q_k)$*-doubt fuzzy ideal of* X *if*

(1). $\mu(0) \leq max\{\mu(x), \frac{1-k}{2}\}$ *for all* $x \in X$

(2). $\mu(x) \leq max\{\mu(x * y), \mu(y), \frac{1-k}{2}\}$ *for all* $x, y \in X$.

Remark 3.4. *A fuzzy subset* μ *of a BG-algebra* X *is an* $(\in, \in \vee q)$*-doubt fuzzy ideal of* X *iff*

$$\mu(0) \leq M\{\mu(x), 0.5\}$$
$$\mu(x) \leq M\{\mu(x * y), \mu(y), 0.5\}$$

Theorem 3.5. *A fuzzy subset* μ *of a BG-algebra* X *is an* $(\in, \in \vee q_k)$*-doubt fuzzy ideal of* X *iff*

(1). $x_t \overline{\in} \mu \Rightarrow 0_t \overline{\in \wedge q_k} \mu$ *for all* $x \in X$

(2). $(x * y)_t, y_s \overline{\in} \mu \Rightarrow x_{M(t,s)} \overline{\in \wedge q_k} \mu$ *for all* $x, y \in X$.

where $M\{t, s\} = max\{t, s\}$ *and* $t, s \in (0, 1]$.

Proof. First let μ be an $(\in, \in \vee q_k)$-doubt fuzzy ideal of X. To prove conditions (1) and (2). Since μ is an $(\in, \in \wedge q_k)$-doubt fuzzy ideal of X.

$$\mu(0) \leq M\{\mu(x), \frac{1-k}{2}\} \quad (1)$$

$$\mu(x) \leq M\{\mu(x*y), \mu(y), \frac{1-k}{2}\} \quad \text{for all} \quad x, y \in X. \quad (2)$$

Let $x \in X$ and $t \in [0,1]$ such that $x_t \overline{\in} \mu$ i.e., $\mu(x) < t$. Now

$$\begin{aligned} (1) \Rightarrow \mu(0) &\leq M\{\mu(x), \frac{1-k}{2}\} \\ &\leq M\{t, \frac{1-k}{2}\} = \begin{cases} t & \text{if} \quad t > \frac{1-k}{2} \\ \frac{1-k}{2} & \text{if} \quad t \leq \frac{1-k}{2} \end{cases} \\ &\Rightarrow \mu(0) < t \quad \text{or} \quad \mu(0) < \frac{1-k}{2} \\ &\Rightarrow \mu(0) < t \quad \text{or} \quad \mu(0) + t < \frac{1-k}{2} + \frac{1-k}{2} = 1-k \\ &\Rightarrow \mu(0) < t \quad \text{or} \quad \mu(0) + t + k < 1 \\ &\Rightarrow x_t \overline{\in} \mu \quad \text{or} \quad 0_t \overline{q_k} \mu \\ &\Rightarrow 0_t \overline{\in \wedge q_k} \mu \end{aligned}$$

Therefore $x_t \overline{\in} \mu \Rightarrow 0_t \overline{\in \wedge q_k} \mu$ which proves (1). Again let $x, y \in X$ such that $(x*y)_t \overline{\in} \mu$ and $y_s \overline{\in} \mu$ where $t, s \in (0,1]$ i.e., $\mu(x*y) < t$ and $\mu(y) < s$.

$$\begin{aligned} (2) \Rightarrow \mu(x) &\leq M\{\mu(x*y), \mu(y), \frac{1-k}{2}\} \\ &\leq M\{t, s, \frac{1-k}{2}\} = \begin{cases} M(t,s) & \text{if} \quad M(t,s) > \frac{1-k}{2} \\ \frac{1-k}{2} & \text{if} \quad M(t,s) \leq \frac{1-k}{2} \end{cases} \\ &\Rightarrow \mu(x) < M(t,s) \quad \text{or} \quad \mu(x) < \frac{1-k}{2} \\ &\Rightarrow \mu(x) < M(t,s) \quad \text{or} \quad \mu(x) + M(t,s) < \frac{1-k}{2} + \frac{1-k}{2} = 1-k \\ &\Rightarrow \mu(x) < M(t,s) \quad \text{or} \quad \mu(x) + M(t,s) + k < 1 \\ &\Rightarrow x_{M(t,s)} \overline{\in} \mu \quad \text{or} \quad x_{M(t,s)} \overline{q_k} \mu \\ &\Rightarrow x_{M(t,s)} \overline{\in \wedge q_k} \mu \end{aligned}$$

Therefore $(x*y)_t \overline{\in} \mu, y_s \overline{\in} \mu \Rightarrow x_{M(t,s)} \overline{\in \wedge q_k} \mu$ which is proves (2).

Conversely, Suppose μ satisfies conditions (1) and (2). To prove μ is an $(\in, \in \vee q_k)$-doubt fuzzy ideal of X. If possible μ is not an $(\in, \in \vee q_k)$-doubt fuzzy ideal of X. The at least one of $\mu(0) > M\{\mu(x), \frac{1-k}{2}\}$ or $\mu(x) > M\{\mu(x*y), \mu(y), \frac{1-k}{2}\}$ must hold for some $x, y \in X$. Suppose $\mu(0) > M\{\mu(x), \frac{1-k}{2}\}$ holds. Choose a real number t such that

$$\mu(0) > t > M\{\mu(x), \frac{1-k}{2}\} \quad (3)$$

$\Rightarrow \mu(x) < t \Rightarrow x_t \overline{\in} \mu \Rightarrow 0_t \overline{\in \wedge q_k} \mu$ [By condition (1)] $\Rightarrow 0_t \overline{\in} \mu$ or $0_t \overline{q}_k \mu \Rightarrow \mu(0) < t$ or $\mu(0) + t + k \leq 1$ first part is not true by (3), therefore we have $\mu(0) + t + k \leq 1 \Rightarrow \mu(0) + t \leq 1 - k \Rightarrow 1 - k \geq \mu(0) + t > t + t = 2t$ [Since $\mu(0) > t$ by (3)] $\Rightarrow t \leq \frac{1-k}{2}$, which contradicts (3) again. Hence we must have $\mu(0) \le M\left\{\mu(x), \frac{1-k}{2}\right\}$. Again if $\mu(x) > M\left\{\mu(x*y), \mu(y), \frac{1-k}{2}\right\}$ holds for some $x, y \in X$. Then choose a real number t such that

$$\mu(x) > t > M\left\{\mu(x*y), \mu(y), \frac{1-k}{2}\right\} \tag{4}$$

$\Rightarrow \mu(x*y) < t$ and $\mu(y) < t \Rightarrow (x*y)_t \overline{\in} \mu$ and $(y)_t \overline{\in} \mu \Rightarrow (x)_{M(t,t)} \overline{\in \wedge q_k} \mu$ [By condition (2)] $\Rightarrow (x)_t \overline{\in} \mu$ or $(x)_t \overline{q}_k \mu \Rightarrow \mu(x) < t$ or $\mu(x)+t+k \le 1$ first part is not true by (4), therefore we have $\mu(x)+t+k \le 1 \Rightarrow \mu(x)+t \le 1-k \Rightarrow 1-k \ge \mu(x)+t > t+t = 2t$ [Since $\mu(x) > t$ by (4)] $\Rightarrow t \le \frac{1-k}{2}$ which contradicts (4). Hence we must have $\mu(x) \le M\left\{\mu(x*y), \mu(y), \frac{1-k}{2}\right\}$. Hence μ is an $(\in, \in \vee q_k)$-doubt fuzzy ideal of X. □

Theorem 3.6. *A fuzzy subset μ of a BG-algebra X is an $(\in, \in \vee q_k)$-doubt fuzzy subalgebra of X iff*

$$x_t, y_s \overline{\in} \mu \Rightarrow (x*y)_{M(t,s)} \overline{\in \wedge q_k} \mu \quad \textit{for all} \quad x, y \in X$$

where $M\{t,s\} = \max\{t,s\}$ and $t, s \in (0,1]$.

Theorem 3.7. *A fuzzy subset μ of a BG-algebra X is a doubt fuzzy ideal if and only if μ is an $(\in, \in)$-doubt fuzzy ideal.*

Proof. Let μ be a doubt fuzzy ideal of X, to prove that μ is an $(\in, \in)$-doubt fuzzy ideal. It is enough to show that

(i). $x_t \overline{\in} \mu \Rightarrow 0_t \overline{\in} \mu$ for all $x \in X$

(ii). $(x*y)_t, y_s \overline{\in} \mu \Rightarrow x_{M(t,s)} \overline{\in} \mu$ for all $x, y \in X$.

Where $M\{t,s\} = \max\{t,s\}$ and $t, s \in (0,1]$. Let $x \in$ X, such that $x_t \overline{\in} \mu$ where t $\in (0,1)$, then $\mu(x) < t$. Now $\mu(0) \le \mu(x) < t$ [Since μ is a doubt fuzzy ideal] $\Rightarrow 0_t \overline{\in} \mu$. Therefore $x_t \overline{\in} \mu \Rightarrow 0_t \overline{\in} \mu$. Let $x, y \in X$, such that $(x*y)_t, y_s \overline{\in} \mu$, where $t, s \in (0,1)$, then $\mu(x*y) < t, \mu(y) < s$. Now $\mu(x) \le \max\{\mu(x*y, \mu(y)\} < \max\{t,s\} = M(t,s)$ [Since μ is a doubt fuzzy ideal] $\Rightarrow x_{M(t,s)} \overline{\in} \mu$. Therefore $(x*y)_t, y_s \overline{\in} \mu \Rightarrow x_{M(t,s)} \overline{\in} \mu$. Hence μ is an $(\in, \in \vee q_k)$-doubt fuzzy ideal of X.
Conversely, let μ be an $(\in, \in)$-doubt fuzzy ideal of X. Let $x \in X$ and $\mu(x) = t$, where $t, s \in [0,1]$. Then $\mu(x) < t + \delta$ where δ is arbitrary small positive number. Therefore $(x)_{t+\delta} \overline{\in} \mu \Rightarrow (0)_{t+\delta} \overline{\in} \mu$ [Since μ is an $(\in, \in)$-doubt fuzzy ideal of X] $\Rightarrow \mu(0) < (t+\delta) \Rightarrow \mu(0) \le t = \mu(x)$, Since δ is arbitrary. Again let $x, y \in X$ and $\mu(x*y) = t, \mu(y) = s$ where $t, s \in [0,1]$ then $\mu(x*y) < t+\delta, \mu(y) < s+\delta$ Where δ is arbitrary small positive number. Therefore $(x*y)_{t+\delta}, (y)_{s+\delta} \overline{\in} \mu \Rightarrow (x)_{M(t+\delta, s+\delta)} \overline{\in} \mu$ [Since μ is an $(\in, \in)$-doubt fuzzy ideal of X] $\Rightarrow \mu(x) < M(t+\delta, s+\delta) \Rightarrow \mu(x) \le M(t,s) = M\{\mu(x*y, \mu(y)\}$, Since δ is arbitrary. Hence μ is a doubt fuzzy ideal of X. □

Theorem 3.8. *A fuzzy subset μ of a BG-algebra X is a doubt fuzzy subalgebra if and only if μ is an $(\in, \in)$-doubt fuzzy subalgebra.*

Theorem 3.9. *If μ is a (q,q)-doubt fuzzy ideal of a BG-algebra X, then it is also an $(\in, \in)$-doubt fuzzy ideal of X.*

Proof. Let μ is an (q,q)-doubt fuzzy ideal of a *BG*-algebra X. Let x, y $\in X$ such that $(x*y)_t, y_s \overline{\in} \mu \Rightarrow \mu(x*y) < t$ and $\mu(y) < s \Rightarrow \mu(x*y) - t + 1 < 1$ and $\mu(y) - s + 1 < 1 \Rightarrow \mu(x*y) + \delta - t + 1 \le 1$ and $\mu(y) + \delta - s + 1 \le 1 \Rightarrow (x*y)_{\delta - t+1} \overline{q} \mu$ and $(y)_{\delta - s+1} \overline{q} \mu$. Since μ is a (q,q)-doubt fuzzy ideal X. Therefore we have $x_{M(\delta-t+1, \delta-s+1)} \overline{q} \mu \Rightarrow \mu(x) + M(\delta - t + 1, \delta - s + 1) \le 1 \Rightarrow \mu(x) + \delta + 1 - min(t,s) \le 1 \Rightarrow \mu(x) \le min(t,s) - \delta \Rightarrow \mu(x) < m(t,s) < M(t,s)$. Since δ is arbitrary $\Rightarrow x_{M(t,s)} \overline{\in} \mu$. Therefore $(x*y)_t, y_s \overline{\in} \mu \Rightarrow x_{M(t,s)} \overline{\in} \mu$. Hence μ is an $(\in, \in \vee q_k)$-doubt fuzzy ideal of X. □

Remark 3.10. *Converse of Theorem 3.9 is not true as seen from the following example.*

Example 3.11. *Consider BG-algebra $X = \{0, 1, 2, 3\}$ with the following cayley table.*

*	0	1	2	3
0	0	1	2	3
1	1	0	1	1
2	2	2	0	2
3	3	3	3	0

Define a map $\mu : X \to [0, 1]$ by $\mu(0) = \mu(1) = 0.37, \mu(2) = \mu(3) = 0.46$. Then it is easy to verify that μ is an $(\in, \in)$-doubt fuzzy ideal X, but not an (q, q)-doubt fuzzy ideal of X because if $x = 2, y = 1, t = 0.4, s = 0.6$ then $(x * y)_t \overline{q} \mu, y_s \overline{q} \mu$ but $\mu(x) + M(t, s) = \mu(2) + M(0.4, 0.6) = 0.46 + 0.6 = 1.06 > 1$

Theorem 3.12. *If μ is a (q, q)-doubt fuzzy subalgebra of a BG-algebra X, then it is also an $(\in, \in)$-doubt fuzzy subalgebra of X.*

Theorem 3.13. *Let μ be an $(\in, \in \vee q_k)$-doubt fuzzy ideal of X.*

(1). $\mu(0) > \frac{1-k}{2}$ for some $x \in X$, then μ is also an $(\in, \in)$-doubt fuzzy ideal of X.

(2). $\mu(x) \leq \frac{1-k}{2}$ $\forall x, y \in X$, then $\mu(0) \leq \frac{1-k}{2}$.

Proof.

(1). Let μ be an $(\in, \in \vee q_k)$-doubt fuzzy ideal of X and $\mu(x) > \frac{1-k}{2}$ $\forall x \in X$. Let $x_t \overline{\in} \mu \Rightarrow \mu(x) < t$. Therefore $\frac{1-k}{2} < \mu(x) < t$ also $\mu(0) > \frac{1-k}{2}$. Therefore $\mu(0) + t > \frac{1-k}{2} + \frac{1-k}{2} = 1 - k \Rightarrow \mu(0) + t + k > 1$ that is $0_t q_k \mu$. Since μ is an $(\in, \in \vee q_k)$-doubt fuzzy ideal so we must have $0_t \overline{\in} \mu$. Hence $x_t \overline{\in} \mu \Rightarrow 0_t \overline{\in} \mu$.

Again let $(x * y)_t \overline{\in} \mu, y_s \overline{\in} \mu. \Rightarrow \mu(x * y) < t$ and $\mu(y) < s$ Therefore $\frac{1-k}{2} < \mu(x * y) < t$ and $\frac{1-k}{2} < \mu(y) < s \Rightarrow M\{t, s, \} > \frac{1-k}{2}$ Also $\mu(x) > \frac{1-k}{2}$. Therefore $\mu(x) + M\{t, s, \} > \frac{1-k}{2} + \frac{1-k}{2} = 1 - k \Rightarrow \mu(x) + M\{t, s, \} + k > 1 \Rightarrow x_{M(t,s)} q_k \mu$. Since μ is an $(\in, \in \vee q_k)$-doubt fuzzy ideal so we must have $\Rightarrow x_{M(t,s)} \overline{\in} \mu$. Hence $(x * y)_t \overline{\in} \mu, y_s \overline{\in} \mu \Rightarrow x_{M(t,s)} \overline{\in} \mu$. Therefore μ is an $(\in, \in)$-doubt fuzzy ideal of X.

(2). Let μ be an $(\in, \in \vee q_k)$-doubt fuzzy ideal of X and $\mu(x) \leq \frac{1-k}{2}$ $\forall x \in X$. Now $\mu(0) \leq M\{\mu(x), \frac{1-k}{2}\} = M\{\frac{1-k}{2}, \frac{1-k}{2}\} = \frac{1-k}{2}$. □

Corollary 3.14. *Let μ be an $(\in, \in \vee q)$-doubt fuzzy ideal of X.*

(1). $\mu(0) > 0.5$ for some $x \in X$, then μ is also an $(\in, \in)$-doubt fuzzy ideal of X.

(2). $\mu(x) \leq 0.5$ for some $x \in X$, then $\mu(0) \leq 0.5$.

Theorem 3.15. *Let μ be an $(\in, \in \vee q_k)$-doubt fuzzy subalgebra of X.*

(1). $\mu(0) > \frac{1-k}{2}$ for some $x \in X$, then μ is also an $(\in, \in)$-doubt fuzzy subalgebra of X.

(2). $\mu(x) \leq \frac{1-k}{2}$ $\forall x, y \in X$, then $\mu(0) \leq \frac{1-k}{2}$.

Proof.

(1). Same as Theorem 3.13 (1).

(2). Since $\mu(0) = \mu(x * x) \leq M\{\mu(x), \mu(x), \frac{1-k}{2}\}$ $\forall x, y \in X$. □

Theorem 3.16. *A fuzzy set μ in X is an $(\in, \in \vee q_k)$-doubt fuzzy ideal of X if and only if the set $\overline{\mu}_t = \{x \in X | \mu(x) < t\}$ is an ideal of X for all $t \in (\frac{1-k}{2}, 1]$.*

Proof. Assume that μ be an $(\in, \in \vee q_k)$-doubt fuzzy ideal of X. Let $t \in (\frac{1-k}{2}, 1]$ and $x \in \overline{\mu}_t$, therefore $\mu(x) < t$. It follows that

$$\mu(0) \leq M\left\{\mu(x), \frac{1-k}{2}\right\} < M\left\{t, \frac{1-k}{2}\right\} = t$$

Therefore $\mu(0) < t \Rightarrow 0_t \overline{\in} \mu$, that is $x_t \overline{\in} \mu \Rightarrow 0_t \overline{\in} \mu$. Again let $x * y, y \in \overline{\mu}_t$. Therefore $\mu(x*y) < t$ and $\mu(y) < t$. It follows that

$$\mu(x) \leq M\left\{\mu(x*y), \mu(y), \frac{1-k}{2}\right\} < M\left\{t, \frac{1-k}{2}\right\} = t$$

Which implies $x \in \overline{\mu}_t$. Therefore $x * y, y \in \overline{\mu}_t \Rightarrow x \in \overline{\mu}_t$. Hence $\overline{\mu}_t$ is an ideal of X.

Conversely, suppose that $\overline{\mu}_t$ is an ideal of X for all $t \in (\frac{1-k}{2}, 1]$ and let

$$\mu(0) \leq M\left\{\mu(x), \frac{1-k}{2}\right\}$$

is not valid, then their exists some $a \in X$ such that

$$\mu(0) > M\left\{\mu(a), \frac{1-k}{2}\right\}$$

Hence we can take $t \in (\frac{1-k}{2} 1]$ such that

$$\mu(0) \geq t > M\left\{\mu(a), \frac{1-k}{2}\right\}$$

Which shows that $0 \notin \overline{\mu}_t$ which is a contradiction. Since $\overline{\mu}_t$ is an ideal of X. Therefore we must have

$$\mu(0) \leq M\left\{\mu(x), \frac{1-k}{2}\right\}$$

Again let

$$\mu(x) \leq M\left\{\mu(x*y), \mu(y), \frac{1-k}{2}\right\}$$

is not valid, then their exists some $a, b \in X$ such that

$$\mu(a) > M\left\{\mu(a*b), \mu(b), \frac{1-k}{2}\right\}$$

hence we can take $t \in (\frac{1-k}{2}, 1]$ such that

$$\mu(a) \geq t > M\left\{\mu(a*b), \mu(b), \frac{1-k}{2}\right\}$$

Which implies $a * b, b \in \overline{\mu}_t$. Since $\overline{\mu}_t$ is an ideal of X, it follows that $a \in \overline{\mu}_t$, so that $\mu(a) < t$. This is again a contradiction, therefore

$$\mu(x) \leq M\left\{\mu(x*y), \mu(y), \frac{1-k}{2}\right\}$$

is valid. Consequently μ is an $(\in, \in \vee q_k)$-doubt fuzzy ideal of X. □

Corollary 3.17. *A fuzzy set μ in X is an $(\in, \in \vee q)$-doubt fuzzy ideal of X if and only if the set $\overline{\mu}_t = \{x \in X | \mu(x) < t\}$ is an ideal of X for all $t \in (0.5, 1]$.*

Theorem 3.18. *A fuzzy set μ in X is an $(\in, \in \vee q_k)$-doubt fuzzy subalgebra of X if and only if the set $\overline{\mu}_t = \{x \in X | \mu(x) < t\}$ is an subalgebra of X for all $t \in (\frac{1-k}{2}, 1]$.*

Theorem 3.19. *Let A be a non empty subset of a BG- algebra X. Consider the fuzzy set μ_A in X defined by*

$$\mu_A(x) = \begin{cases} 0 & if \quad x \in A \\ 1 & otherwise \end{cases}$$

Then A is an ideal of X iff μ_A is an $(\in, \in \vee q_k)$-doubt fuzzy ideal of X.

Proof. Let A be an ideal of X, then $\overline{(\mu_A)}_t = \{x \in X | \mu_A(x) < t\}$ $\forall t \in (\frac{1-k}{2}, 1] = A$, which is an ideal. Hence by above theorem μ_A is an $(\in, \in \vee q_k)$-doubt fuzzy ideal of X.

Conversely, assume that μ_A is an $(\in, \in \vee q_k)$-doubt fuzzy ideal of X. Let $x \in A$, then

$$\mu_A(0) \leq M\left\{\mu_A(x), \frac{1-k}{2}\right\} = M\left\{0, \frac{1-k}{2}\right\} = \frac{1-k}{2} < 1 \quad \forall\ k \in [0\ 1)$$

Therefore $\mu_A(0) < 1 \Rightarrow \mu_A(0) = 0 \Rightarrow 0 \in A$. Again let $x * y, y \in A$, then

$$\mu_A(x) \leq M\left\{\mu(x*y), \mu(y), \frac{1-k}{2}\right\} = M\left\{0, 0, \frac{1-k}{2}\right\} = \frac{1-k}{2} < 1 \quad \forall\ k \in [0\ 1)$$

Therefore $\mu_A(x) < 1 \Rightarrow \mu_A(x) = 0 \Rightarrow x \in A$. Hence A is an ideal of X. □

Theorem 3.20. *Let A be a non empty subset of a BG- algebra X. Consider the fuzzy set μ_A in X defined by*

$$\mu_A(x) = \begin{cases} 0 & if \quad x \in A \\ 1 & otherwise \end{cases}$$

Then A is an subalgebra of X iff μ_A is an $(\in, \in \vee q_k)$-doubt fuzzy subalgebra of X.

Theorem 3.21. *Let A be an ideal of X, then for every $t \in (\frac{1-k}{2}, 1]$, their exists an $(\in, \in \vee q_k)$-doubt fuzzy ideal μ of X, such that $\overline{\mu_t} = A$.*

Proof. Let μ be a fuzzy set in X defined by

$$\mu(x) = \begin{cases} 0 & \text{if} \quad x \in A \\ t & \text{otherwise} \end{cases}$$

for all $x \in X$, where $t \in (\frac{1-k}{2}, 1]$, $\overline{(\mu)}_t = \{x \in X | \mu(x) < t\} = A$. Hence $\overline{(\mu)}_t$ is an ideal. Now if μ is not an $(\in, \in \vee q_k)$-doubt fuzzy ideal of X then at least one of condition(1) or condition (2) in Theorem 3.5 may not hold, suppose condition (1) does not holds then there exists some $a \in X$ such that $\mu(0) > M\left\{\mu(a), \frac{1-k}{2}\right\}$ choose $t = [\mu(0) + M\left\{\mu(a), \frac{1-k}{2}\right\}]/2$ then $\mu(0) > t > M\left\{\mu(a), \frac{1-k}{2}\right\}$. Since A is an ideal of X, therefore $0 \in A$. Hence $\mu(0) < t$ $\forall t \in (0\ 1)$ which is a contradiction. Therefore we must have $\mu(0) \leq M\{\mu(x), \frac{1-k}{2}\}$ for all $x, y \in X$.

Again if condition (2) does not holds then there exists some $a, b \in X$ such that $\mu(a) > M\{\mu(a*b), \mu(b), \frac{1-k}{2}\}$. Choose $t = [\mu(a) + M\left\{\mu(a*b), \mu(b), \frac{1-k}{2}\right\}]/2$ then $\mu(a) > t > M\left\{\mu(a*b), \mu(b), \frac{1-k}{2}\right\}$. Hence $\mu(a*b) < t, \mu(b) < t$ and so $a*b, b \in \overline{(\mu)}_t = A$. Since A is an ideal of X, therefore $a \in A$ hence $\mu(a) = 0 < t$ $\forall t \in (0\ 1)$ which is again a contradiction. Therefore $\mu(x) \leq M\left\{\mu(x*y), \mu(y), \frac{1-k}{2}\right\}$ for all $x, y \in X$. Hence μ is an $(\in, \in \vee q_k)$-doubt fuzzy ideal of X. □

Corollary 3.22. *Let A be an ideal of X, then for every $t \in (0.5, 1]$, their exists an $(\in, \in \vee q_k)$-doubt fuzzy ideal μ of X, such that $\overline{\mu_t} = A$.*

Theorem 3.23. *Let A be an ideal of X, then for every $t \in (\frac{1-k}{2}, 1]$, their exists an $(\in, \in \vee q_k)$-doubt fuzzy subalgebra μ of X, such that $\overline{\mu_t} = A$.*

Definition 3.24. *Let μ be a fuzzy set in BG-algebra X and $t \in (0\ 1]$, let*

$$\mu_t = \{x \in X | x_t \in \mu\} = \{x \in X | \mu(x) \geq t\}$$
$$< \mu >_t = \{x \in X | x_t q\mu\} = \{x \in X | \mu(x) + t > 1\}$$
$$[\mu]_t = \{x \in X | x_t \in \wedge q\mu\} = \{x \in X | \mu(x) \geq t \, or \, \mu(x) + t > 1\}$$
$$\overline{(\mu)}_t = \{x \in X | x_t \overline{\in} \mu\} = \{x \in X | \mu(x) < t\}$$
$$\overline{< \mu >}_t^k = \{x \in X | x_t \overline{q}_k \mu\} = \{x \in X | \mu(x) + t + k \leq 1\}$$
$$\overline{[\mu]}_t^k = \{x \in X | x_t \overline{\in \wedge q_k} \mu\} = \{x \in X | \mu(x) < t \, or \, \mu(x) + t + k \leq 1\}$$

Here $\overline{(\mu)}_t$ is called t level set of μ, $\overline{< \mu >}_t^k$ is called $\overline{q}_k$ level set of μ and $\overline{[\mu]}_t^k$ is called $(\overline{\in \wedge q_k})$ level set of μ. Clearly $[\mu]_t =< \mu >_t \cup \mu_t$ and $\overline{[\mu]}_t^k = \overline{< \mu >}_t^k \cup \overline{(\mu)}_t$.

Theorem 3.25. *Let μ be a fuzzy set in BG-algebra X.Then μ is an $(\in, \in \wedge q_k)$-doubt fuzzy ideal of X iff $\overline{[\mu]}_t^k$ is an ideal of X for all $t \in (0\ 1]$. We call $\overline{[\mu]}_t^k$ as $(\overline{\in \wedge q_k})$ level ideal of μ.*

Proof. Assume that μ is an $(\in, \in \vee q_k)$-doubt fuzzy ideal of X, to prove $\overline{[\mu]}_t^k$ is an ideal of X. Let $x \in \overline{[\mu]}_t^k$ for $t \in (0\ 1]$ then $x_t \overline{\in \wedge q_k} \mu$ then $\mu(x) < t$ or $\mu(x) + t + k \leq 1$. Since μ is an $(\in, \in \vee q_k)$-doubt fuzzy ideal of X, therefore$\mu(0) \leq M\{\mu(x), \frac{1-k}{2}\}$ for all $x, y \in X$.

Case I: $\mu(x) < t$

$$\begin{aligned} \mu(0) &\leq M\{\mu(0), \frac{1-k}{2}\} \\ &\leq M\{t, \frac{1-k}{2}\} = \begin{cases} t, & \text{if } t > \frac{1-k}{2} \\ \frac{1-k}{2}, & \text{if } t \leq \frac{1-k}{2}. \end{cases} \\ &\Rightarrow \mu(0) < t \quad \text{or} \quad \mu(0) < \frac{1-k}{2} \\ &\Rightarrow \mu(0) < t \quad \text{or} \quad \mu(0) + t < \frac{1-k}{2} + \frac{1-k}{2} = 1 - k \\ &\Rightarrow \mu(0) < t \quad \text{or} \quad \mu(0) + t + k < 1 \\ &\Rightarrow x_t \overline{\in} \mu \quad \text{or} \quad 0_t \overline{q_k} \mu \Rightarrow 0_t \overline{\in \wedge q_k} \mu \end{aligned}$$

Therefore $0_t \overline{\in \wedge q_k} \mu$ i.e., $0 \in \overline{[\mu]}_t^k$.

Case II: $\mu(x) + t + k \leq 1$

$$\begin{aligned} \mu(0) &\leq M\{\mu(0), \frac{1-k}{2}\} \\ &\leq M\{1 - t - k, \frac{1-k}{2}\} = \begin{cases} \frac{1-k}{2} & \text{if} \quad t > \frac{1-k}{2} \\ 1 - t - k & \text{if} \quad t \leq \frac{1-k}{2} \end{cases} \\ &\Rightarrow \mu(0) < \frac{1-k}{2} < t \quad \text{or} \quad \mu(0) < 1 - t - k \\ &\Rightarrow \mu(0) < t \quad \text{or} \quad \mu(0) + t + k < 1 \\ &\Rightarrow x_t \overline{\in} \mu \quad \text{or} \quad 0_t \overline{q_k} \mu \Rightarrow 0_t \overline{\in \wedge q_k} \mu \end{aligned}$$

Therefore in both cases $0_t\overline{\in \wedge q_k}\mu$ i.e., $0 \in \overline{[\mu]}_t^k$. Again let $x * y, y \in \overline{[\mu]}_t^k$ for $t \in (0\ 1]$ then $(x*y)_t\overline{\in \wedge q_k}\mu$ and $(y)_t\overline{\in \wedge q_k}\mu$. Then $\mu(x*y) < t$ or $\mu(x*y)+t \leq 1$ and $\mu(y) < t$ or $\mu(y)+t \leq 1$. Since μ is an $(\in, \in \vee q_k)$-doubt fuzzy ideal of X. Therefore$\mu(x) \leq M\{\mu(x*y), \mu(y), \frac{1-k}{2}\}$ for all $x, y \in X$. Therefore we have the following cases:

Case I: Let $\mu(x*y) < t$ and $\mu(y) < t$

$$\begin{aligned}
\mu(x) &\leq M\{\mu(x*y), \mu(y), \frac{1-k}{2}\} \\
&\leq M\{t, t, \frac{1-k}{2}\} = \begin{cases} t & \text{if} \quad t > \frac{1-k}{2} \\ \frac{1-k}{2} & \text{if} \quad t \leq \frac{1-k}{2} \end{cases} \\
&\Rightarrow \mu(x) < t \quad \text{or} \quad \mu(x) < \frac{1-k}{2} \\
&\Rightarrow \mu(x) < t \quad \text{or} \quad \mu(x) + t < \frac{1-k}{2} + \frac{1-k}{2} = 1-k \\
&\Rightarrow \mu(x) < t \quad \text{or} \quad \mu(x) + t + k < 1 \\
&\Rightarrow x_t\overline{\in}\mu \quad \text{or} \quad x_t\overline{q_k}\mu \\
&\Rightarrow x_t\overline{\in \wedge q_k}\mu
\end{aligned}$$

Therefore $x_t\overline{\in \wedge q_k}\mu$ i.e.,$x \in \overline{[\mu]}_t^k$.

Case II: $\mu(x*y) < t$ and $\mu(y) + t + k \leq 1$

$$\begin{aligned}
\mu(x) &\leq M\{\mu(x*y), \mu(y), \frac{1-k}{2}\} \\
&\leq M\{t, 1-t-k, \frac{1-k}{2}\} = \begin{cases} t & \text{if} \quad t > \frac{1-k}{2} \\ 1-t-k & \text{if} \quad t \leq \frac{1-k}{2} \end{cases} \\
&\Rightarrow \mu(x) < t \quad \text{or} \quad \mu(x) < 1-t-k \\
&\Rightarrow \mu(x) < t \quad \text{or} \quad \mu(x) + t + k < 1 \\
&\Rightarrow x_t\overline{\in}\mu \quad \text{or} \quad x_t\overline{q_k}\mu \\
&\Rightarrow x_t\overline{\in \wedge q_k}\mu
\end{aligned}$$

Therefore $x_t\overline{\in \wedge q_k}\mu$ i.e.,$x \in \overline{[\mu]}_t^k$.

Case III: $\mu(x*y) + t + k \leq 1$ and $\mu(y) < t$. This is similar to Case II

Case IV: $\mu(x*y) + t + k \leq 1$ and $\mu(y) + t + k \leq 1$

$$\begin{aligned}
\mu(x) &\leq M\{\mu(x*y), \mu(y), \frac{1-k}{2}\} \\
&\leq M\{1-t-k, 1-t-k, \frac{1-k}{2}\} = \begin{cases} \frac{1-k}{2} & \text{if} \quad t > \frac{1-k}{2} \\ 1-t-k & \text{if} \quad t \leq \frac{1-k}{2} \end{cases} \\
&\Rightarrow \mu(x) < \frac{1-k}{2} < t \quad \text{or} \quad \mu(x) < 1-t-k \\
&\Rightarrow \mu(x) < t \quad \text{or} \quad \mu(x) + t + k < 1 \\
&\Rightarrow x_t\overline{\in}\mu \quad \text{or} \quad x_t\overline{q_k}\mu \\
&\Rightarrow x_t\overline{\in \wedge q_k}\mu
\end{aligned}$$

Therefore for all cases $x_t\overline{\in \wedge q_k}\mu$ i.e.,$x \in \overline{[\mu]}_t^k$. Hence $x * y, y \in \overline{[\mu]}_t^k \Rightarrow x \in \overline{[\mu]}_t^k$. That is $\overline{[\mu]}_t^k$ is an ideal of X.

Conversely, let μ be a fuzzy set in X and $t \in (0\ 1]$ such that $\overline{[\mu]}_t^k$ is an ideal of X. To prove μ is an $(\in, \in \vee q_k)$-doubt fuzzy ideal of X. If μ is not an $(\in, \in \vee q_k)$-doubt fuzzy ideal of X, then at least one of the conditions of Definition 3.3 may not be hold, suppose condition (i) is not true, then there exists some $a \in X$ such that $\mu(0) > M\{\mu(a), \frac{1-k}{2}\}$ holds. Choose $t \in (0\ 1]$ such that $\mu(0) > t > M\{\mu(a), \frac{1-k}{2}\} \Rightarrow \mu(0) \not> t \Rightarrow 0 \notin \overline{\mu}_t \subseteq \overline{[\mu]}_t^k$,which is a contradiction since $\overline{[\mu]}_t^k$ is an ideal. Thus $\mu(0) \leq M\{\mu(x), \frac{1-k}{2}\}$ for all $x, y \in X$. Again if condition (ii) is not true, there exists some $a, b \in X$ such that $\mu(a) > M\{\mu(a * b), \mu(b), \frac{1-k}{2}\}$ holds. Choose $t \in (0\ 1]$ such that $\mu(a) > t > M\{\mu(a * b), \mu(b), \frac{1-k}{2}\}$ then $a * b, b \in \overline{\mu}_t \subseteq \overline{[\mu]}_t^k$, which implies $a \in \overline{[\mu]}_t^k$. Hence $\mu(a) < t$ or $\mu(a * b) + t + k \leq 1$ a contradiction. Thus $\mu(x) \leq M\{\mu(x * y), \mu(y), \frac{1-k}{2}\}$ for all $x, y \in X$. Hence μ is an $(\in, \in \vee q_k)$-doubt fuzzy ideal of X. □

Theorem 3.26. *Let μ be a fuzzy set in BG-algebra X. Then μ is an $(\in, \in \vee q_k)$-doubt fuzzy subalgebra of X iff $\overline{[\mu]}_t^k$ is an subalgebra of X for all $t \in (0\ 1]$. We call $\overline{[\mu]}_t^k$ as $(\overline{\in \wedge q_k})$ level subalgebra of μ.*

Theorem 3.27. *Every an $(\in, \in)$-doubt fuzzy ideal of X is an $(\in, \in \vee q_k)$-doubt fuzzy ideal of X.*

Theorem 3.28. *Every an $(\in, \in)$-doubt fuzzy subalgebra of X is an $(\in, \in \vee q_k)$-doubt fuzzy subalgebra of X.*

Remark 3.29. *The converse of above Theorem 3.27 is not true as seen from following example.*

Example 3.30. *Consider BG-algebra $X = \{0, 1, 2, 3\}$ with the following cayley table.*

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Define a map $\mu : X \to [0, 1]$ by $\mu(0) = \mu(1) = 0.2, \mu(2) = 0.4, \mu(3) = 0.48$ then by Definition 3.3 it is easy to verify that μ is an $(\in, \in \vee q_{0.5})$-doubt fuzzy ideal X. But not an $(\in, \in)$-doubt fuzzy ideal of X because if $x = 3, y = 1$ $\mu(x*y) = \mu(3*1) = \mu(2) = 0.4$ $\Rightarrow (3 * 1)_{0.45} = 2_{0.45} \overline{\in} \mu$, also $1_{0.45} \overline{\in} \mu$ But $3_{0.45} \in \mu$. Therefore μ is not an $(\in, \in)$-doubt fuzzy ideal of X.

Theorem 3.31. *Every doubt fuzzy ideal is an $(\in, \in \vee q_k)$-doubt fuzzy ideal of X.*

Theorem 3.32. *Every doubt fuzzy subalgebra is a $(\in, \in \vee q_k)$-doubt fuzzy subalgebra of X.*

Remark 3.33. *The converse of above Theorem 3.31 is not true which can be easily seen from Theorem 3.7 and Example 3.30.*

Remark 3.34. *The converse of above Theorem 3.32 is also not true.*

Theorem 3.35. *If μ is an $(\in, \in \vee q_k)$-doubt fuzzy ideal of X. Then $\overline{< \mu >}_t^k$ is an ideal of X, for all $t \in [0, \frac{1-k}{2})$.*

Proof. Assume μ is an $(\in, \in \vee q_k)$-doubt fuzzy ideal of X, and let $t \in (0, \frac{1-k}{2})$. Let $x \in X$ such that $x \in \overline{< \mu >}_t^k \Rightarrow x_t \overline{q}_k \mu \Rightarrow \mu(x) + t + k \leq 1$. Now $\mu(0) \leq M\{\mu(x), \frac{1-k}{2}\}$ [Since μ is an $(\in, \in \wedge q_k)$-doubt fuzzy ideal of X] $\leq M\{1 - t - k, \frac{1-k}{2}\}$ [Since $t < \frac{1-k}{2}$] $= 1 - t - k \Rightarrow \mu(0) + t + k \leq 1 \Rightarrow 0_t \overline{q}_k \mu \Rightarrow 0 \in \overline{< \mu >}_t^k$. Let $x, y \in X$ such that $x * y, y \in \overline{< \mu >}_t^k \Rightarrow (x * y)_t \overline{q}_k \mu$ and $(y)_t \overline{q}_k \mu \Rightarrow \mu(x * y) + t + k \leq 1$ and $\mu(y) + t + k \leq 1$. Now $\mu(x) \leq M\{\mu(x * y), \mu(y), \frac{1-k}{2}\}$ [Since μ is an $(\in, \in \vee q_k)$-doubt fuzzy ideal of X] $\leq M\{1 - t - k, 1 - t - k, \frac{1-k}{2}\}$ [Since $t < \frac{1-k}{2}$] $= 1 - t - k \Rightarrow \mu(x) + t + k \leq 1 \Rightarrow x_t \overline{q}_k \mu \Rightarrow x \in \overline{< \mu >}_t^k$. Hence $\overline{< \mu >}_t^k$ is an ideal of X. □

Corollary 3.36. *If μ is an $(\in, \in \vee q_k)$-doubt fuzzy ideal of X. Then $\overline{< \mu >}_t$ is an ideal of X, for all $t \in [0, 0.5)$*

Theorem 3.37. *If μ is an $(\in, \in \vee q_k)$-doubt fuzzy subalgebra of X. Then $\overline{< \mu >}_t^k$ is an subalgebra of X, for all $t \in [0, \frac{1-k}{2})$.*

Proposition 3.38. *If $k_1, k_2 \in (0,1]$ such that $k_1 < k_2$, then every $(\in, \in \vee q_{k_2})$-doubt fuzzy ideal of X is an an $(\in, \in \vee q_{k_1})$-doubt fuzzy ideal.*

Proof. Here $k_1, k_2 \in (0,1]$ such that $k_1 < k_2$ and let μ be an $(\in, \in \vee q_k)$-doubt fuzzy ideal of X. Therefore

$$\mu(0) \leq M\left\{\mu(x), \frac{1-k_2}{2}\right\} \quad \text{for all } x, y \in X.$$
$$\leq M\left\{\mu(x), \frac{1-k_1}{2}\right\} \quad \left[\text{Since } k_1 < k_2 \Rightarrow \frac{1-k_2}{2} \leq \frac{1-k_1}{2}\right]$$

also $\mu(x) \leq M\{\mu(x*y), \mu(y), \frac{1-k_2}{2}\}$ for all $x, y \in X$

$$\leq M\left\{\mu(x*y), \mu(y), \frac{1-k_1}{2}\right\}$$

Hence by Definition 3.3 μ is an $(\in, \in \vee q_k)$-doubt fuzzy ideal of X. □

Proposition 3.39. *If $k_1, k_2 \in (0,1]$ such that $k_1 < k_2$, then every $(\in, \in \vee q_{k_2})$-doubt fuzzy subalgebra of X is an an $(\in, \in \vee q_{k_1})$-doubt fuzzy ideal.*

Remark 3.40. *The converse of above Proposition 3.38 is not true as seen from following example.*

Example 3.41. *Let X and μ as in Example 3.30. If $k_1 = 0.1, k_2 = 0.4$, then μ ia an $(\in, \in \vee q_{0.1})$-doubt fuzzy ideal of X by Definition 3.3, butμ is not an $(\in, \in \vee q_{0.4})$-doubt fuzzy ideal of X. Since $(3*1)_{0.45} = 2_{0.45} \overline{\in} \mu, 1_{0.45} \overline{\in} \mu$ but $3_{0.45} q_{0.4} \mu$.*

Corollary 3.42. *If $k_1, k_2 \in (0,1]$ such that $k_1 < k_2$. If μ be an $(\in, \in \vee q_{k_2})$-doubt fuzzy ideal of X, then $\overline{<\mu>}_t^{k_1}$ is an ideal of X for all $t \in (0, \frac{1-k_1}{2})$.*

Corollary 3.43. *If $k_1, k_2 \in (0,1]$ such that $k_1 < k_2$. If μ be an $(\in, \in \vee q_{k_2})$-doubt fuzzy subalgebra of X, then $\overline{<\mu>}_t^{k_1}$ is a subalgebra of X for all $t \in (0, \frac{1-k_1}{2})$.*

Theorem 3.44. *Every $(\in, q_k)$-doubt fuzzy ideal of X is an $(\in, \in \vee q_k)$-doubt fuzzy ideal of X.*

Remark 3.45. *The converse of above Theorem 3.44 is not true as seen from following example.*

Example 3.46. *Let BG-algebra as in Example 3.30 if we take $\mu(0) = \mu(1) = \mu(2) = 0.3, \mu(3) = 0.48, t = 0.5, s = 0.6, k =$ 0.01 then it is easy to verify that μ is an $(\in, \in \vee q_k)$-doubt fuzzy ideal X. But not an $(\in, q_k)$-doubt fuzzy ideal of X. Since $\mu(3*1) = \mu(2) < 0.5 = t$ and $\mu(1) < 0.6 = s$. But $\mu(3) + M(t,s) + k = 0.48 + M(0.5, 0.6) + 0.01 = 0.48 + 0.6 + 0.01 = 1.09 > 1$.*

Theorem 3.47. *Let λ and μ be two $(\in, \in \vee q_k)$-doubt fuzzy ideals of X then $\lambda \cup \mu$ is an $(\in, \in \vee q_k)$-doubt fuzzy ideal of X.*

Proof. Here λ and μ both are $(\overline{\in}, \overline{\in \wedge q_k})$-fuzzy ideals of X. Therefore

$$\begin{aligned}
\lambda(0) &\leq M\left\{\lambda(x), \frac{1-k}{2}\right\} \\
\mu(0) &\leq M\left\{\mu(x), \frac{1-k}{2}\right\} \quad \text{for all} \quad x \in X \\
\lambda(x) &\leq M\left\{\lambda(x*y), \lambda(y), \frac{1-k}{2}\right\} \\
\mu(x) &\leq M\left\{\mu(x*y), \mu(y), \frac{1-k}{2}\right\} \quad \text{for all} \quad x, y \in X \\
\text{Now,} \quad (\lambda \cup \mu)(0) &= M\{\lambda(0), \mu(0)\} \\
&\leq M\left\{M\left\{\lambda(x), \frac{1-k}{2}\right\}, M\left\{\mu(x), \frac{1-k}{2}\right\}\right\} \\
&= M\left\{M(\lambda(x), \mu(x)), \frac{1-k}{2}\right\}
\end{aligned}$$

$$\leq M\left\{(\lambda\cup\mu)(x)), \frac{1-k}{2}\right\}$$

$$\text{And,}\quad (\lambda\cup\mu)(x) = M\{\lambda(x),\mu(x)\}$$
$$\leq M\left\{M\left\{\lambda(x*y),\lambda(y),\frac{1-k}{2}\right\}, M\left\{\mu(x*y),\mu(y),\frac{1-k}{2}\right\}\right\}$$
$$= M\left\{M(\lambda(x*y),\mu(x*y)), M(\lambda(y),\mu(y)),\frac{1-k}{2}\right\}$$
$$\leq M\left\{(\lambda\cup\mu)(x*y)),(\lambda\cup\mu)(y)),\frac{1-k}{2}\right\}$$

Hence $\lambda\cup\mu$ is an $(\in,\in\vee q_k)$-doubt fuzzy ideal of X. □

Theorem 3.48. *Let $\{\mu_i : i\in\wedge\}$ be a family of $(\in,\in\vee q_k)$-doubt fuzzy ideals of X, then $\mu=\cup\{\mu_i : i\in\wedge\}$ is an $(\in,\in\vee q_k)$-doubt fuzzy ideal of X.*

Theorem 3.49. *Let X, Y be two BG-algebras, then their Cartesian Product $X\times Y=\{(x,y)|x\in X, y\in Y\}$ is also a BG-algebra under the binary operation $*$ defined in $X\times Y$ by $(x,y)*(p,q)=(x*p,y*q)$ for all $(x,y),(p,q)\in X\times Y$.*

Definition 3.50. *Let μ_1 and μ_2 be two $(\in,\in\vee q_k)$-doubt fuzzy ideals of a BG-algebra X. Then their Cartesian product $\mu_1\otimes\mu_2$ is defined by $(\mu_1\otimes\mu_2)(x,y)=max\{\mu_1(x),\mu_2(y),\frac{1-k}{2}\}=M\{\mu_1(x),\mu_2(y),\frac{1-k}{2}\}$. Where $(\mu_1\times\mu_2):X\times X\to[0,1]\quad\forall x,y\in X$.*

Theorem 3.51. *Let μ_1 and μ_2 be two $(\in,\in\vee q_k)$-doubt fuzzy ideals of a BG-algebra X. Then $\mu_1\otimes\mu_2$ is also an $(\in,\in\vee q_k)$-doubt fuzzy ideal of $X\times X$.*

Definition 3.52. *Let X and X' be two BG-algebras. Then a mapping $f:X\to X'$is said to be homomorphism if $f(x*y)=f(x)*f(y)\quad\forall x,y\in X$.*

Theorem 3.53. *Let X and X' be two BG-algebras and $f:X\to X'$ be a homomorphism. Then $f(0)=0'$.*

Proof. Let $x\in X$ therefore $f(x)\in X'$. Now $f(0)=f(x*x)=f(x)*f(x)=0'$. □

Theorem 3.54. *Let X and X' be two BG-algebras and $f:X\to X'$ be homomorphism. If μ be an $(\in,\in\vee q_k)$-doubt fuzzy ideal of X', then $f^{-1}(\mu)$ is an $(\in,\in\vee q_k)$-doubt fuzzy ideal of X.*

Proof. $f^{-1}(\mu)$ is defined as $f^{-1}(\mu)(x)=\mu(f(x))\forall x\in X$. Let μ be an $(\in,\in\vee q_k)$-doubt fuzzy ideal of X'. Let $x\in X$ such that $x_t\overline{\in}f^{-1}(\mu)$ then $f^{-1}(\mu)(x)<t\Rightarrow\mu(f(x))<t\Rightarrow(f(x))_t\overline{\in}\mu\Rightarrow((f(0))_t\overline{\in\wedge q_k}\mu$ [$\because\mu$ be an $(\in,\in\vee q_k)$-doubt fuzzy ideal of X'] $\Rightarrow((f(0))_t\overline{\in}\mu$ or $((f(0))_t\overline{q_k}\mu\Rightarrow\mu(f(0))<t$ or $\mu(f(0))+t+k\leq 1\Rightarrow f^{-1}(\mu)(0)<t$ or $f^{-1}(\mu)(0)+t+k\leq 1\Rightarrow 0_t\overline{\in}f^{-1}(\mu)$ or $0_t\overline{q}_kf^{-1}(\mu)\Rightarrow 0_t\overline{\in\wedge q_k}f^{-1}(\mu)$. Therefore $x_t\overline{\in}f^{-1}(\mu)\Rightarrow 0_t\overline{\in\wedge q_k}f^{-1}(\mu)$.
Again let $x,y\in X$ such that $(x*y)_t,y_s\overline{\in}f^{-1}(\mu)$ then $f^{-1}(\mu)(x*y)<t$ and $f^{-1}(\mu)(y)<s$. $\mu(f(x*y))<t$ and $\mu(f(y))<s\Rightarrow(f(x*y))_t\overline{\in}\mu$ and $f(y)_s\overline{\in}\mu\Rightarrow(f(x)*f(y))_t\overline{\in}\mu$ and $f(y)_s\overline{\in}\mu$ since f is a homomorphism $\Rightarrow((f(x))_{M(t,s)}\overline{\in\wedge q_k}\mu$ [$\because\mu$ be an $(\in,\in\vee q_k)$-doubt fuzzy ideal of X'] $\Rightarrow\mu(f(x))<M(t,s)$ or $\mu(f(x))+M(t,s)+k\leq 1\Rightarrow f^{-1}(\mu)(x)<M(t,s)$ or $f^{-1}(\mu)(x)+M(t,s)+k\leq 1\Rightarrow x_{M(t,s)}\overline{\in}f^{-1}(\mu)$ or $x_{M(t,s)}\overline{q}_kf^{-1}(\mu)\Rightarrow x_{M(t,s)}\overline{\in\wedge q_k}f^{-1}(\mu)$. Therefore$(x*y)_t,y_s\overline{\in}f^{-1}(\mu)\Rightarrow x_{M(t,s)}\overline{\in\wedge q_k}f^{-1}(\mu)$. Hence $f^{-1}(\mu)$ is an $(\in,\in\vee q_k)$-doubt fuzzy ideal of X. □

Theorem 3.55. *Let X and X' be two BG-algebras and $f:X\to X'$ be an onto homomorphism. If μ be a fuzzy subset of X' such that $f^{-1}(\mu)$ is an $(\in,\in\vee q_k)$-doubt fuzzy ideal of X, then μ is also an $(\in,\in\vee q_k)$-doubt fuzzy ideal of X'.*

Proof. Let $x',y'\in X'$ since f is onto so there exists $x,y\in X$. such that $f(x)=x', f(y)=y'$ also f is homomorphism so $f(x*y)=f(x)*f(y)=x'*y'$ such that $x'_t\overline{\in}\mu$ where $t,s\in[0,1]$ then $\mu(x')<t\Rightarrow\mu(f(x))<t\Rightarrow f^{-1}(\mu)(x)<t\Rightarrow(x)_t\overline{\in}f^{-1}(\mu)\Rightarrow(0)_t\overline{\in\wedge q_k}f^{-1}(\mu)$ [Since $f^{-1}(\mu)$ is an $(\in,\in\vee q_k)$-doubt fuzzy ideal of X.] $\Rightarrow f^{-1}(\mu)(0)<t$ or $f^{-1}(\mu)(0)+$

$t+k \leq 1 \Rightarrow \mu(f(0)) < t$ or $\mu(f(0))+t+k \leq 1 \Rightarrow \mu(0') < t$ or $\mu(0')+t+k \leq 1 \Rightarrow 0'_t\overline{\in}\mu$ or $0'_t\overline{q}_k\mu \Rightarrow 0'_t\overline{\in \wedge q_k}\mu$. Therefore $x'_t\overline{\in}\mu \Rightarrow 0'_t\overline{\in \wedge q_k}\mu$.

Again let $(x'*y')_t, y'_s\overline{\in}\mu$ where $t,s \in [01]$ then $\mu(x'*y') < t$ and $\mu(y') < s$. Therefore $\mu(f(x*y)) < t$ and $\mu((f(y)) < s \Rightarrow f^{-1}(\mu)(x*y) < t$ and $f^{-1}(\mu)(y) < s \Rightarrow (x*y)_t\overline{\in}f^{-1}(\mu)$ and $(y)_s\overline{\in}f^{-1}(\mu) \Rightarrow (x)_{M(t,s)}\overline{\in \wedge q_k}f^{-1}(\mu)$ [since $f^{-1}(\mu)$ is an $(\in, \in \vee q_k)$-doubt fuzzy ideal of X.] $\Rightarrow f^{-1}(\mu)(x) < M(t,s)$ or $f^{-1}(\mu)(x)+M(t,s)+k \leq 1 \Rightarrow \mu(f(x)) < M(t,s)$ or $\mu(f(x))+M(t,s)+k \leq 1 \Rightarrow \mu(x') < M(t,s)$ or $\mu(x')+M(t,s)+k \leq 1 \Rightarrow x'_{M(t,s)}\overline{\in}\mu$ or $x'_{M(t,s)}\overline{q}_k\mu \Rightarrow x'_{M(t,s)}\overline{\in \wedge q_k}\mu$. Therefore $(x'*y')_t, y'_s\overline{\in}\mu \Rightarrow x'_{M(t,s)}\overline{\in \wedge q_k}\mu$. Hence μ is an $(\in, \in \vee q_k)$-doubt fuzzy ideal of X'. □

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