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# Norms of the Composite Convolution Volterra Operators

Research Article

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**Abstract:** In this paper an endeavor has been made to compute the norms of Composite Convolution Volterra operators. An attempt has also been made to obtain the norms of powers of composite convolution operators in general and in specific cases.

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**Keywords:** Composition operator, Composite Convolution Volterra operator, Radon- Nikodym derivative, Expectation operator, Hankel operator.



## 1. Introduction

Let $(X, \Omega, \mu)$ be a $\sigma-$finite measure space. For each $f \in L^p(\mu), 1 \leq p < \infty$, there exists a unique $\phi^{-1}(\Omega)$ measurable function $E(f)$ such that $\int gf d\mu = \int gE(f)d\mu$ for every $\phi^{-1}(\Omega)$ measurable function $g$ for which left integral exists. The function $E(f)$ is called conditional expectation of $f$ with respect to the sub- algebra $\phi^{-1}(\Omega)$. For more details about expectation operator, one can refer to Parthasarthy [11]. Let $\phi : X \to X$ be a non-singular measurable transformation (i.e., $\mu(E) = 0 \Rightarrow \mu\phi^{-1}(E) = 0$). Then a composition transformation , for $1 \leq p < \infty, C_\phi : L^p(\mu) \to L^p(\mu)$ is defined by $C_\phi f = f o \phi$ for every $f \in L^p(\mu)$. In case $C_\phi$ is continuous, we call it a composition operator induced by $\phi$. It is easy to see that $C_\phi$ is a bounded operator if and only if $f_d = \frac{d\mu\phi^{-1}}{d\mu}$, the Radon-Nikodym derivative of the measure $\mu\phi^{-1}$ with respect to the measure $\mu$, is essentially bounded. For more detail about composition operator we refer to Singh and Manhas [14]. Given $f, g \in L^2(\mathbb{R})$, then convolution of $f$ and $g, f * g$ is defined by

$$f * g(x) = \int g(x-y)f(y)d(y),$$

where $g$ is fixed, $k(x,y) = g(x-y)$ is a convolution kernel and the integral operator defined by

$$I_k f(x) = \int k(x-y)f(y)d\mu(y)$$

is known as Convolution operator. Consider the Volterra operator $V$ acting on the Hilbert space $L^2[0,1]$ defined by

$$(Vf)(x) = \int_0^x f(y)d\mu(y)$$

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Thus, the Volterra operator $V$ is an integral operator induced by the kernel $k(x,y)$ defined as

$$k(x,y) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 1 \end{cases}$$

It is well known result that, Volterra operator is quasinilpotent. In 1982, Halmos [6] proved that the operator norm of $V$ is $2/\pi$. Lao and Whitley [9] in 1997 gave the numerical evidence which led them to the conjecture that

$$\lim_{n\rightarrow\ infty} ||n!V^n|| = 1/2.$$

It is shown by Kershaw [8] that

$$1/2n! \leq ||V^n|| \leq \frac{1}{2n!}(1-1/2n)^{-1/2} = ||V^n||_2,$$

where $||.||$ and $||.||_2$ denote the operator norm and Hilbert-Schmidt norm respectively. Thus, both $||V^n||$ and $||V^n||_2$ are asymptotically equal to $1/2n!$ as $n\ rightarrow\infty$. Whitley [15] established the Lyubic's conjecture [10] and generalized it to Volterra composition operators on $L^p[0,1]$. The Volterra composition operator is a composition of Volterra integral operator $V$ and a composition operator $C_\phi$ defined as

$$(V_\phi f)(x) = (C_\phi Vf)(x) = (Vf)o\phi(x)$$
$$(V_\phi f(x) = \int_0^{\phi(x)} f(t)d\mu(t)$$

for every $f \in L^p[0,1]$, where $\phi : [0,1] \rightarrow [0,1]$ is a measurable function. Volterra convolution operator $V_k$ is defined on $L^2[0,1]$ by

$$V_k f(x) = \int_0^x k(x-y)f(y)d\mu(y).$$

Volterra convolution operator $V_k$ is compact, linear, bounded and hence Hilbert-Schimidt operator. $V_k^*$ is the adjoint of $V_k$, given by

$$V_k^* f(x) = \int_x^1 k(t-x)f(t)d\mu(t).$$

The composite convolution Volterra operator (CCVO) $V_{k,\phi}$, is defined as

$$V_{k,\phi}f(x) = \int_0^x k(x-y)f(\phi(y))d\mu(y) = \int_0^x k_\phi(x-y)f(y)d\mu(y)$$

where $k_\phi(x-y) = E(f_d(y)k(x-y)\phi^{-1}(y))$. The study of composite convolution Volterra operator has been initiated in the work of Gupta [4]. This paper broaden the approach that was taken in the papers of Gupta ([3],[4],[5]). The integral operators, in particular convolution Volterra operators have already been studied extensively over the last few decades. For more detail about composition operators, integral operators, convolution operators and composite integral operators, we refer to Singh and Manhas [14], Halmos and Sunder [7], Stepanov ([12],[13]), Biswas [1], Eveson [2], Kershaw [8], Gupta and Komal [3] and Gupta ([4],[5]). Here, I recall some basic notion in operator theory. Let $H$ be a Hilbert space and $B(H)$ be the algebra of all bounded linear operators acting on $H$. Let $L^2(\mu)$ consists of all measurable functions $f : X \rightarrow \mathbb{R}(or\mathbb{C})$ such that $(\int |f(x)|^2 d\mu(x))^{1/2} < \infty$. The space $L^2(X,S,\Omega)$ is a Banach space under the norm defined by $||f|| = (\int |f|^2 d\mu)^{1/2}$. Also, $L^2(\mu)$, the space of square-integrable functions of complex numbers is a Hilbert space. The study of Composite Convolution Volterra operators has been introduced in the work of Gupta [4]. This paper addresses the problem of numerically computation of the norm of powers of composite convolution Volterra operators. We calculate the norms of composite convolution Volterra operators for different conditions on kernel function.

# 2. Computation of Norms of the Composite Convolution Volterra Operators (CCVO)

In this section, $L^2[0,1]$ is a Lebesgue space of real-valued functions in $[0,1]$. It has been proved by Gupta [4] that spectrum of composite convolution Volterra operator is equal to zero. In this section we investigate the norm of powers of composite convolution Volterra operators (CCVO).

**Theorem 2.1.** *Let $V_{k,\phi} \in B(L^2(\mu))$. Suppose $k_\phi(x-y) = \delta(x-y)$. Then $|V_{k,\phi}f| \leq \sqrt{x}||f||_2$. Moreover, $||V_{k,\phi}||_2^2 < 1$.*

*Proof.* For $f \in L^2[0,1]$, we have

$$\begin{aligned} |V_{k,\phi}f(x)| &= \left|\int_0^x k_\phi(x-y)f(y)d\mu(y)\right| \\ &= \left|\int_0^1 \chi_{[0,x]}(y)k_\phi(x-y)f(y)d\mu(y)\right| \\ &= \left|\int_0^1 \chi_{[0,x]}(y)\delta(x-y)f(y)d\mu(y)\right| \\ &\leq \left(\int_0^1 |\chi_{[0,x]}(y)d\mu(y)|\right)^{1/2}\left(\int_0^1 |f(y)d\mu(y)|\right)^{1/2} \\ &= \left(\int_0^x |d\mu(y)|\right)^{1/2}\left(\int_0^1 |f(y)d\mu(y)|\right)^{1/2} \\ &= \sqrt{x}||f||_2. \end{aligned}$$

Thus, we have

$$|V_{k,\phi}f| \leq \sqrt{x}||f||_2. \tag{1}$$

Also, we have

$$\begin{aligned} ||V_{k,\phi}f|| &= \int_0^1 |V_{k,\phi}f(x)|^2 d\mu(x) \\ &\leq \int_0^1 x||f||_2^2 d\mu(x), \\ &= |\frac{x^2}{2}|_0^1||f||_2^2 = ||f||_2^2, \end{aligned}$$

and we conclude that

$$||V_{k,\phi}||_2^2 \leq \frac{1}{2} < 1.$$

□

In the next result, we calculate the norm of nth power of CCVO. To evualate the norm of nth power, firstly we prove above theorem.

**Theorem 2.2.** *Let $V_{k,\phi} \in B(L^2(\mu))$. Suppose $k_\phi \in L^2(\mu \times \mu)$. Then $V_{k,\phi}^n = V_{k^n,\phi}$, where $k_\phi^n(x-y) = \int_y^x k_\phi(x-z)k_\phi^{n-1}(z-y)d\mu(z)$.*

*Proof.* Let $f \in L^2[0,1]$. Then, we have

$$\begin{aligned} V_{k,\phi}^2 f(x) &= V_{k,\phi}(V_{k,\phi}f(x)) \\ &= \int_0^x k_\phi(x-y)V_{k,\phi}f(y)d\mu(y) \\ &= \int_0^x \int_0^y k_\phi(x-y)k_\phi(y-z)f(z)d\mu(z)d\mu(y) \\ &= \int_0^x \int_x^z k_\phi(x-y)k_\phi(y-z)f(z)d\mu(y)d\mu(z) \\ &= \int_0^x k_\phi^2(x-z)f(z)d\mu(z), \end{aligned}$$

where

$$k_{\phi}^{2}(x-z)=\int_{x}^{z} k_{\phi}(x-y)k_{\phi}(y-z)d\mu(y).$$

Suppose the result is true for $n=m$. That is,

$$\begin{aligned} V_{k,\phi}^{m} &= V_{k^m,\phi} \\ &= \int_{0}^{x} k_{\phi}^{m}(x-z))f(z)d\mu(z) \end{aligned}$$

where

$$k_{\phi}^{m}(x-z)=\int_{x}^{z} k_{\phi}(x-y)k_{\phi}^{m-1}(y-z)d\mu(y).$$

Now, for $n=m+1$, we have

$$\begin{aligned} V_{k,\phi}^{m+1}f(x) &= V_{k,\phi}V_{k,\phi}^{m}f(x) \\ &= \int_{0}^{x} k_{\phi}(x-y)V_{k,\phi}^{m}f(y)d\mu(y) \\ &= \int_{0}^{x} k_{\phi}^{m+1}(x-z)f(z)d\mu(z, \end{aligned}$$

where

$$k_{\phi}^{m+1}(x-z)=\int_{x}^{z} k_{\phi}(x-y)k_{\phi}^{m}(y-z)d\mu(y).$$

Hence, by using principle of mathematical induction, we conclude that

$$V_{k,\phi}^{n}=V_{k^n,\phi}.$$

□

**Theorem 2.3.** *For $1\le p<\infty$, suppose $V_{k,\phi}\in B(L^p[0,1])$. Then $||V_{k,\phi}^{n}||\le M^n\frac{(x)^n}{n!}||f||$.*

*Proof.* Let $f\in L^p[0,1]$. Then, we have

$$\begin{aligned} |V_{k,\phi}f(x)| &= \left|\int_{0}^{x} k_{\phi}(x-y)f(y)d\mu(y)\right| \\ &\le \int_{0}^{x} |k_{\phi}(x-y)|d\mu(y)\int_{0}^{x}|f(y)|d\mu(y) \\ &< \int_{0}^{x} |k_{\phi}(x-y)|d\mu(y)\int_{0}^{1}|f(y)|d\mu(y) \\ &\le M\int_{0}^{x} d\mu(y)||f|| \\ &= xM||f||, \end{aligned}$$

where $M=\sup\{|k_{\phi}(x-y)|:0\le y\le x\le 1\}$. Thus, the required result is true for $n=1$. Assume that it is also true for $n=m$. Now, consider $n=m+1$, we have

$$\begin{aligned} |V_{k,\phi}^{m+1}f(x)| &= |V_{k,\phi}(V_{k,\phi}^{m})f(x)| \\ &= \left|\int_{0}^{x} k_{\phi}(x-y)(V_{k,\phi}^{m}f)(y)d\mu(y)\right| \\ &\le \left|\int_{0}^{x} k_{\phi}(x-y)M^m||f||\frac{x^m}{m!}d\mu(y)\right| \\ &\le M^{m+1}\frac{x^{m+1}}{m+1}||f||, \end{aligned}$$

where we have use the theorem 2.2. Hence, by using principle of mathematical induction, we conclude that

$$||V_{k,\phi}^{n}||\le M^n\frac{x^n}{n!}||f||.$$

□

In next result, we have obtained the norm of $nth$ power of CCVO under different conditions on kernel function.

**Theorem 2.4.** *Let $V_{k,\phi} \in B(L^p[0,1])$ and $1 \leq p < \infty$. Suppose $V^n_{k,\phi}$ is CCVO induced by kernel function $k^n_\phi$. Then*

*(i). $||V^n_{k,\phi}|| \leq 1$, whenever $k_\phi(x-y) = \delta(x-y)$,*

*(ii). $||V^n_{k,\phi}|| \leq \frac{1}{\sqrt[p]{np+1}}$, whenever $k_\phi(x-y) = x$.*

*Proof.* Case (i): For $f \in L^p[a,b]$, we have

$$\begin{aligned}\left\|V^n_{k,\phi}f\right\|^p_p &= \left\|\int_0^x k^n_\phi(x-y)f(y)d\mu(y)\right\|^p_p \\ &= \int_0^1 \left|\int_0^x k^n_\phi(x-y)f(y)d\mu(y)\right|^p d\mu(x) \\ &\leq \int_0^1 \left|\int_0^1 k^n_\phi(x-y)f(y)d\mu(y)\right|^p d\mu(x) \\ &\leq \int_0^1\int_0^1 |k^n_\phi(x-y)|^p d\mu(y)d\mu(x)\int_0^1 |f(y)|^p d\mu(y) \\ &= ||f||^p_p,\end{aligned}$$

by using the given condition, $k_\phi(x-y) = \delta(x-y)$ and Holder's inequality, we obtained the required result, $||V^n_{k,\phi}|| \leq 1$.
Case (ii): Given $k_\phi(x-y) = x$, this implies that $k^n_\phi(x-y) = x^n$. Then, from case (i), we have

$$\begin{aligned}||V^n_{k,\phi}f||^p_p &= \int_0^1 \left|\int_o^x k^n_\phi(x-y)f(y)d\mu(y)\right|^p \\ &= \int_0^1 \left|x^n\int_0^x f(y)d\mu(y)\right|^p d\mu(x) \\ &\leq \int_0^1 x^{np}\left(\int_0^x |f(y)|d\mu(y)\right)^p d\mu(x) \\ &< \left(\int_0^1 |f(y)|d\mu(y)\right)^p \int_0^1 x^{np}d\mu(x) \\ &= \frac{1}{np+1}\left(\int_0^1 |f(y)|d\mu(y)\right)^p \\ &= \frac{1}{np+1}||f||,\end{aligned}$$

which shows that

$$||V^n_{k,\phi}|| \leq \frac{1}{\sqrt[p]{np+1}}.$$

□

**Theorem 2.5.** *Let $V_{k,\phi} \in B(L^2[0,1])$ and $V^n_{k,\phi}$ is CCVO induced by kernel function $k^n_\phi$. Suppose $k_\phi$ is continuous kernel on $[0,1]$. Then $||k^n_\phi||_2 \leq ||k_\phi||^n_2$. Moreover, $||V^n_{k,\phi}|| \leq ||V_{k,\phi}||^n$.*

*Proof.* For $f \in L^2[0,1]$, we have

$$\begin{aligned}\int_0^x [k^n_\phi(x-t) - k^n_\phi(x_0-t_0)]d\mu(t) &= \int_0^x [k_\phi(x-s)k^{n-1}_\phi(s-t)d\mu(t) - k^n_\phi(x_0-s)k^{n-1}_\phi(s,t)]d\mu(t) \\ &+ \int_0^1 [k_\phi(x_0-s)k^{n-1}_\phi(s-t) - k_\phi(x_0-s)k^{n-1}_\phi(s,t_0)]d\mu(t) \\ &= \int_0^1 [k_\phi(x-s) - k_\phi(x_0-s)]k^{n-1}_\phi(s-t)d\mu(t) \\ &+ \int_0^1 k_\phi(x_0-s)[k^{n-1}_\phi(s-t) - k^{n-1}_\phi(s-t_0)]d\mu(t)\end{aligned}$$

For $\epsilon > 0$, there exist a $\delta > 0$, such that

$$|k_\phi(x-s) - k(x_0, s)| < \epsilon \text{ for } |x - x_0| < 0, \ \ \forall \ s \in [0,1] \text{ and}$$
$$|k_\phi^{n-1}(s,t) - k_\phi^{n-1}(s,t_0)| < \epsilon \text{ for } |t - t_0| < 0, \forall \ \ s \in [0,1]$$

If $|x - x_0| < 0, \forall s \in [0,1]$ and $|t - t_0| < 0, \forall s \in [0,1]$, then we get

$$\begin{aligned} |k_\phi^n(x-t) - k_\phi^n(x_0 - t_0)| &\leq \int_0^1 \epsilon ||k_\phi^{n-1}|| d\mu(s) + \epsilon \int_0^1 ||k_\phi|| d\mu(s) \\ &= ||k_\phi|| + ||k_\phi^{n-1}|| \end{aligned}$$

and we conclude that $k_\phi^n(x-t)$ is continuous. Furthermore,

$$\begin{aligned} ||k_\phi^n||_2^2 &= \int_0^1 \int_0^1 |k_\phi^n(x-t)|^2 d\mu(x) d\mu(t) \\ &= \int_0^1 \int_0^1 \left( \left| \int_0^1 k_\phi(x-s) k_\phi^{n-1}(s-t) |d\mu(s) \right| \right) \left( \left| \int_0^1 k_\phi(x-r) k_\phi^{n-1}(r-t) |d\mu(s) \right| \right) d\mu(x) d\mu(t) \\ &\leq \int_0^1 \int_0^1 \int_0^1 \int_0^1 |k_\phi(x-s)||k_\phi^{n-1}(s-t)| d\mu(s) \int_0^1 |k_\phi(x-r)||k_\phi^{n-1}(r-t)| d\mu(r) d\mu(x) d\mu(t) \\ &\leq \int_0^1 \int_0^1 \int_0^1 \int_0^1 \left( |k_\phi(x-s)|^2 |k_\phi^{n-1}(r-t)|^2 + |k_\phi^{n-1}(s-t)|^2 |k_\phi(x,r)|^2 \right) d\mu(s) d\mu(r) |d\mu(x) d\mu(t) \\ &= \frac{1}{2} [||k_\phi||_2^2 ||k_\phi^{n-1}||_2^2 + ||k_\phi^{n-1}||_2^2 ||k_\phi||_2^2] \\ &= ||k_\phi||_2^2 ||k_\phi^{n-1}||_2^2. \end{aligned}$$

Hence,

$$||k_\phi^n||_2^2 \leq ||k_\phi||_2^2 ||k_\phi^{n-1}||_2^2.$$

Therefore, for $n = 2$, we have

$$||k_\phi^2|| \leq ||k_\phi||_2^2.$$

Assume that $||k_\phi^{n-1}|| \leq ||k_\phi||_2^{n-1}$. Thus, by principle of mathematical induction , we have

$$\begin{aligned} ||k_\phi^n||_2 &\leq ||k_\phi||_2 ||k_\phi^{n-1}||_2 \\ &\leq ||k_\phi||_2 ||k_\phi||_2^{n-1} \\ &= ||k_\phi||_2^n. \end{aligned}$$

Again, we have

$$||V_{k,\phi}^n f(x)|| = \left\| \int_0^1 k_\phi^n(x-y) f(y) d\mu(y) \right\|.$$

$$||V_{k,\phi} f(x)||^n = \left\| \int_0^1 k_\phi(x-y) f(y) d\mu(y) \right\|^n.$$

Hence, we have

$$||V_{k,\phi}^n|| \leq ||V_{k,\phi}||^n.$$

□

Consider a translation operator $U : L^2[0,1] \to L^2[0,1]$ defined as $Uf(x) = f(1-x)$, where $U$ is also unitary operator.

**Theorem 2.6.** *Let* $V_{k,\phi} \in B(L^2[0,1])$ *and* $k_\phi$ *be a non-negative kernel. Then*

$$||UV_{k,\phi}||^2 \leq \int_0^1 \int_x^1 k_\phi^2(t-x)d\mu(t)d\mu(x).$$

*Proof.* Given $f \in L^2[0,1]$, we define

$$UV_{k,\phi}f(x) = V_{k,\phi}f(1-x) = \int_0^{1-x} k_\phi(1-x-y)f(y)d\mu(y),$$

where $UV_{k,\phi}$ is a linear operator from $L^2[0,1]$ into $L^2[0,1]$ with kernel $k_\phi(1-x-y)$ and which is of the type Hankel operator. Suppose $A = UV_{k,\phi}$. Then

$$\begin{aligned} |Af(x)|^2 &= \left| \int_0^{1-x} k_\phi(1-x-y)f(y)d\mu(y) \right|^2 \\ &\leq \int_0^{1-x} k_\phi^2(1-x-y)d\mu(y) \int_0^{1-x} |f(y)|^2 d\mu(y) \end{aligned}$$

by using Cauchy's Schwartz inequality. Again, we have

$$\begin{aligned} \int_0^1 |Af(x)|^2 d\mu(x) &\leq \int_0^{1-x} k_\phi^2(1-x-y)d\mu(y)d\mu(x) \int_0^{1-x} |f(y)|^2 d\mu(y) \\ &= \int_0^1 \int_0^{1-x} k_\phi^2(1-x-y)d\mu(y)d\mu(x) \int_0^1 |f(y)|^2 d\mu(y) \end{aligned}$$

Thus, we have

$$\begin{aligned} ||A|| &\leq \int_0^1 \int_0^{1-x} k_\phi^2(1-x-y)d\mu(y)d\mu(x) \\ &= \int_0^1 \int_x^1 k_\phi^2(t-x)d\mu(t)d\mu(x) \end{aligned}$$

Hence, the desired result follows. □

In the next result, we have obtained thr adjoint of $UV_{k,\phi}$.

**Theorem 2.7.** *Let* $V_{k,\phi} \in B(L^2[0,1])$ *and* $k_\phi$ *be a non-negative kernel. Then the adjoint of* $UV_{k,\phi}$ *is given by the formula* $(UV_{k,\phi})^* = f_d E(V_k^* o\phi^{-1})$*. Moreover,* $||(UV_{k,\phi})(UV_{k,\phi})^*|| = ||(UV_{k,\phi})^*||^2$*.*

*Proof.* Suppose $A = UV_{k,\phi}$. Given $f, g \in L^2[0,1]$, we have

$$\begin{aligned} \langle f, Ag \rangle &= \int_0^1 f(x)Ag(x)d\mu(x) \\ &= \int_0^1 f(x) \left( \int_0^{1-x} k(1-x-y)(go\phi)(y)d\mu(y) \right) d\mu(x) \\ &= \int_0^1 (go\phi)(y) \left( \int_y^1 k(y-x)f(x)d\mu(x) \right) d\mu(y) \\ &= \int_0^1 (go\phi)(y)(V_k^* f)(y)d\mu(y) \\ &= \langle V_k^* f, C_\phi g \rangle \\ &= \langle C_\phi^* V_k^* f, g \rangle \end{aligned}$$

Hence,

$$A^* f(x) = (UV_{k,\phi})^* f(x) = C_\phi^* V_k^* f(x) = f_d E(V_k^* o\phi^{-1}(x)).$$

Also, for $f \in L^2[0,1]$ and $||f||_2 = 1$, we have

$$\begin{aligned} ||(UV_{k,\phi})(UV_{k,\phi})^*|| &= sup\{|\langle AA^*f, f\rangle\} \\ &= \sup\{|\langle A^*f, A^*f\rangle\} \\ &= \sup ||A^*f||^2 \\ &= ||A^*||^2. \end{aligned}$$

Hence, the result follows. □

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## References

[1] A.Biswas, A.Lambert and S.Petrovic, *Extended eigen values and the Volterra operator*, Glasgow Math J., 44(2002), 521-534.

[2] S.P.Eveson, *Norms of iterates of Volterra operators on $L^2$*, Journal Operator Theory, 50(2003), 369-386.

[3] A.Gupta and B.S.Komal, *Bounded Composite integral operators*, Investigations in Mathematical Sciences, 1(2011), 33-39.

[4] A.Gupta, *Composite Convolution Volterra Operators*, International Journal of Mathematical Research and Sciences, 1(3)(2013), 51-58.

[5] A.Gupta, *Applications of Composite Convolution Operators*, Applications and Applied Mathematics: An international Journal, 11(1)(2016), 397-408.

[6] P.R.Halmos, *A Hilbert space problem Book*, Springer-Verlag, New York, (1982).

[7] P.R.Halmos and V.S.Sunder, *Bounded integral operators on $L^2$ spaces*, Springer- Verlag,New York, (1978).

[8] D.Kershaw, *Operator norms of powers of the Volterra operator*, Journal of Integral Equations and Applications, 11(3)(1999), 351-362.

[9] N.Lao and R.Whitley, Norms of powers of the Volterra operator, Integral Equation and Operator Theory, 27(1997), 419-425.

[10] Lyubic and I.Yu, *Composition of integration and substitution, Linear and complex Analysis, Problem Book*, Springer Lect. Notes in Maths., 1043, Berlin, 249-250, (1984).

[11] K.R.Parthasarathy, *Introduction to probability and measure*, Macmillan Limited, (1977).

[12] V.D.Stepanov, *On boundedness and compactness of a class of convolution operators*, Soviet Math. Dokl., 41(1980), 468-470.

[13] V.D.Stepanov, *On convolution integral operators*, Dokl. Akad. Nauk SSSR, 243(1978), 45-48.

[14] R.K.Singh and J.S.Manhas, *Composition operators on function spaces*, North Holland,Mathematics studies 179, Elsevier sciences publishers Amsterdam, New York, (1993).

[15] R.Whitley, *The spectrum of a Volterra composition operator*, Integral equation and Operator theory, 10(1987), 146-149.